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# SPLINES AND THE LOGARITHMIC FUNCTION

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**The paper studies a spline function $S_n(x)\,(0 < x < \infty)$ of degree $n$, with knots at the points of the geometric progression $x_k = q^k$ ($q$ is fixed $> 1$, $k = 0, \pm 1, \pm 2, \cdots$), which in shown to be uniquely defined by the following two properties: 1°. $S_n(x)$ interpolates the function $f(x) = \log x/\log q$ at all the knots $q^k$, 2°. $S_n(x)$ satisfies the functional equation $S_n(qx) = S_n(x) + 1$ for $x > 0$. $S_n(x)$ is explicitly determined and shown to share with $f(x)$ some of its global properties. The main point is the detailed study of the somewhat surprising behavior of $S_n(x)$ as $n \to \infty$.**

**Introduction.** Spline interpolants of exponential functions were discussed in [3], Lecture 3, and were found to be useful in a study of cardinal spline interpolation ([3], Lecture 4). Here we attempt to interpolate by splines the logarithmic function $\log x\,(0 < x < \infty)$. As the domain of definition of $\log x$ is the positive halfline we do not use the usual cardinal polynomial splines, but rather so-called *cardinal q-splines* defined as follows.

Let $n$ be a natural number and $q$ be real, $q > 1$. We denote by

$$\sum_{n,q} = \{S(x)\} \tag{1}$$

the class of functions $S(x)$, defined on the half-axis $0 < x < \infty$, and satisfying the following two conditions:

1. The restriction of $S(x)$ to every interval $(q^\nu, q^{\nu+1})$ ($\nu$ any integer) is a polynomial of degree at most $n$.
2. 

$$S(x) \in C^{n-1}(0, \infty)\ . \tag{2}$$

Such functions $S(x)$ are commonly called *spline functions* (or splines) of degree $n$ having as knots the points $q^\nu(-\infty < \nu < \infty)$. We also call them *q-splines* to remind us of the nature of their knots. Finally, the term *cardinal* $q$-splines is to indicate that the domain of $S(x)$ is $(0, \infty)$ and that *all* knots $q^\nu$ many occur. We now propose the following

*Problem* 1. To find

$$S_n(x) \in \sum_{n,q} \tag{3}$$

such as to interpolate the function

$$f(x) = \frac{\log x}{\log q} \tag{4}$$

at the points $(q^\nu)$, hence such that

$$S_n(q^\nu) = \nu \quad \text{for all integers } \nu\,. \tag{5}$$

As stated this problem admits infinitely many solutions. Indeed, a moment's reflexion (see [3], Lemma 1.1 on p. 33) will show that we obtain all solutions of Problem 1 as follows. The symbol $\pi_n$ denoting the class of polynomials of degree $\leqq n$, let $P(x)$ be an arbitrary element of $\pi_n$ such that

$$P(1) = 0\,, \quad P(q) = 1\,, \tag{6}$$

and let

$$\begin{aligned} S(x) = P(x) &+ a_1(x-q)_+^n + a_2(x-q^2)_+^n + \cdots \\ &+ a_0(1-x)_+^n + a_{-1}(q^{-1}-x)_+^n + \cdots\,, \quad (0 < x < \infty)\,, \end{aligned} \tag{7}$$

where we use the notation $u_+ = \max(u, 0)$. In the interval $[1, q]$ all truncated powers vanish and so $S(x) = P(x)$. In the interval $[q, q^2]$, besides $P(x)$, only the term $a_1(x-q)_+^n$ does not vanish and we can determine $a_1$ uniquely by requiring that $S(q^2) = 2$. Next we determin $a_2$ uniquely from $S(q^3) = 3$, a.s.f. Likewise the coefficients $a_0, a_{-1}, \cdots$ are successively and uniquely determined by the interpolatory condition $S(q^{-1}) = -1$, $S(q^{-2}) = -2, \cdots$. We see that the interpolating $q$-spline $S(x)$ is uniquely defined once we have chosen $P(x) \in \pi_n$ such as to satisfy (6).

Clearly Problem 1 will become meaningful only if we impose on $S_n(x)$ further conditions. This we do as follows. We observe that the function (4) satisfies the functional equation $f(qx) = f(x) + 1$. This suggests that *we require the q-spline $S_n(x)$ to satisfy the functional equation*

$$S_n(qx) = S_n(x) + 1\,, \quad (0 < x < \infty)\,. \tag{8}$$

It should be noticed that if (8) is satisfied and also

$$S_n(1) = 0\,, \tag{9}$$

then all relations (5) evidently follow.

The contents of the seven sections of this paper are briefly as follows. In § 1 we construct the unique $S_n(x)$ satisfying the conditions (5) and (8), and call it *the logarithmic q-spline of degree n*. Unexpectedly, its representation (7) can be explicitly described (Theorem 1). In § 2 we show that the global behavior of $S_n(x)$ is much like that of the function $\log x/\log q$: $S_n'(x)$ is multiply monotone (Theorem 2).

This gives a direct construction of $S_n(x)$ by successive integrations of the explicitly described step-function $S_n^{(n)}(x)$ (Corollary 1 of § 2).

These results led us naturally to expect that the logarithmic $q$-spline $S_n(x)$ will converge to $\log x/\log q$ as $n \to \infty$. This, however, is not the case and the §§ 4, 5, 6, 7 describe the peculiar behavior of $S_n(x)$ as $n \to \infty$. Our main results are as follows. We define

$$F(x, \theta) = \sum_{s=-\infty}^{\infty} (e^{-q^{\theta+s}} - e^{-xq^{\theta+s}}) \,, \quad (x > 0) \,. \tag{10}$$

This is a periodic function of $\theta$ of period 1, whose Fourier expansion (5.2) is described in § 5. *In terms of* $F(x, \theta)$ *we find that*

$$S_n(x) = F\Big(x, \frac{\log n}{\log q}\Big) + o(1) \text{ as } n \to \infty \,, \tag{11}$$

*where the error term* $o(1)$ *is uniform in* $x$ for $x > 0$ (Theorem 4 of § 4). The average value of $F(x, \theta)$ over a period in $\theta$ is $a_0(x) = \log x/\log q$, while its total variation in a period is of the order of

$$\left|\Gamma\left(-\frac{2\pi i}{\log q}\right)\right| = \left.\frac{\log q}{e^{2\pi^2/\log q} - e^{-2\pi^2/\log q}}\right)^{1/2} \tag{12}$$

which is small if $q$ is not too large (for $q = 2$ the quantity (12) is of the order of $10^{-6}$). Figure 1 of Section 6 shows graphically all the terms of the relation (11) for $q = 2$, $x = \sqrt{2}$, and $n = 16, 17, \cdots, 64$.

In the last section 7 it is shown that the logarithmic means of the sequence $S_n(x)$ do converge to the "correct" limit $\log x/\log q$ (Theorem 5). In a way this paper may be regarded as a contribution to the study of the functional equation $f(qx) = f(x) + 1$.

**1. Construction of the cardinal $q$-spline $S_n(x)$ satisfying (5) and (8).** The representation (7) of our solution is explicitly described by the following

THEOREM 1 *There is a unique* $S_n(x)$ *satisfying the conditions* (5) *and* (8) *and it is given by*

$$S_n(x) = P_n(x) + (-1)^n \sum_{\nu=1}^{\infty} q^{-n\nu}(x - q^\nu)_+^n - \sum_{\nu=0}^{\infty} q^{n\nu}(q^{-\nu} - x)_+^n \,, \quad (0 < x < \infty) \,, \tag{1.1}$$

*where*

$$P_n(x) = \binom{n}{1}\frac{x-1}{q-1} - \binom{n}{2}\frac{x^2-1}{q^2-1} + \cdots + (-1)^{n-1}\frac{x^n-1}{q^n-1} \,. \tag{1.2}$$

*We call* $S_n(x)$ *the logarithmic q-spline of degree* $n$.

*Proof.* We begin by determining the polynomial $P_n(x) = P(x)$ representing $S_n(x) = S(x)$ in the interval $(1, q)$. From the conditions (2) and (8) we obtain by differentiation

$$q^\nu S^{(\nu)}(qx) = S^{(\nu)}(x) \, , \quad (\nu = 1, \cdots, n-1) \, , \tag{1.3}$$

and setting $x = 1$ we obtain

$$q^\nu P^{(\nu)}(q) - P^{(\nu)}(1) = 0 \, , \quad (\nu = 1, \cdots, n-1) \, . \tag{1.4}$$

Clearly $P(1) = 0$, $P(q) = 1$, and therefore

$$P(q) - P(1) = 1 \, . \tag{1.5}$$

We may summarize the relations (1.4) and (1.5) by saying that the polynomial $P(qx) - P(x)$ assumes for $x = 1$ the value 1 to the $n$th order. This means that for an appropriate constant $c$ we have the identity

$$P(qx) - P(x) = 1 + c(x-1)^n \, . \tag{1.6}$$

The left side vanishing if $x = 0$, we must have $c = (-1)^{n-1}$ and (1.6) becomes

$$P(qx) - P(x) = 1 - (1-x)^n \, . \tag{1.7}$$

All $P(x) \in \pi_n$ satisfying (1.7) are easily determined. Writing

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n \, , \tag{1.8}$$

substituting into (1.7) and comparing coefficients on both sides, we obtain that

$$a_1(q-1) = \binom{n}{1}, \; a_2(q^2-1) = -\binom{n}{2}, \; \cdots, \; a_n(q^n - 1) = (-1)^{n-1} \, .$$

Therefore all solutions of (1.7) are of the form.

$$P(x) = a_0 + \binom{n}{1}\frac{x}{q-1} - \binom{n}{2}\frac{x^2}{q^2-1} + \cdots + (-1)^{n-1}\frac{x^n}{q^n-1} \, .$$

Finally, the condition $P(1) = 0$ shows that $P_n(x)$ indeed has the form (1.2). We may retrace our steps and see that the polynomial (1.2) satisfies (1.7), hence (1.4) and that $P(1) = 0$, $P(q) = 1$.

The polynomial $P_n(x)$ representing $S_n(x)$ in the interval $(1, q)$ having been determined, we could now extend the definition of $S_n(x)$ to all of $(0, \infty)$ by the representation (7) and the procedure described for determining the coefficients $a_\nu$. However, it is better to proceed as follows: *We extend the definition of $S_n(x)$ from the interval $[1, q]$, where $S_n(x) = P_n(x)$, to all positive values of $x$ by means of* *the functional equation*

$$S_n(qx) = S_n(x) + 1 \,. \tag{1.9}$$

However, now we must verify that the condition $S(x) \in C^{n-1}(0, \infty)$ is satisfied. By iteration of (1.9), i.e. replacing $x$ by $qx, q^2x, \cdots$ and adding the results, we find that

$$S(x) = P\left(\frac{x}{q^{r-1}}\right) + r - 1 \quad \text{if } q^{r-1} < x < q^r \,, \tag{1.10}$$

$$S(x) = P\left(\frac{x}{q^r}\right) + r \qquad \text{if } q^r < x < q^{r+1} \,. \tag{1.11}$$

From (1.11) and letting $x$ decrease to $q^r$ we obtain that $S(q^r + 0) = P(1) + r = r$, while (1.10) on letting $x$ increase to $q^r$ we find $S(q^r - 0) = P(q) + r - 1 = 1 + r - 1 = r$. Therefore $S(q^r + 0) = S(q^r - 0)$ and $S(x) \in C(0, \infty)$. There remains to show that

$$S^{(\nu)}(q^r + 0) = S^{(\nu)}(q^r - 0) \,, \quad (\nu = 1, \cdots, n - 1) \,. \tag{1.12}$$

However, on differentiating (1.10) and (1.11), we obtain

$$S^{(\nu)}(q^r - 0) = \frac{1}{q^{\nu(r-1)}} P^{(\nu)}(q) \text{ and } S^{(\nu)}(q^r + 0) = \frac{1}{q^{\nu r}} P^{(\nu)}(1) \,, \tag{1.13}$$

respectively. Now the relations (1.12) amount to $q^\nu P^{(\nu)}(q) = P^{(\nu)}(1)$, and these are precisely the boundary conditions (1.4) satisfied by our polynomial (1.2).

We are still to determine the explicit values of the coefficient $a_\nu$ of (7), as described by (1.1). If $x \neq q^r$, for all integers $r$, then we may differentiate the relation (1.9) even $n$ times when (1.13) becomes, in view of (1.2),

$$S^{(n)}(q^r - 0) = \frac{1}{q^{n(r-1)}} P^{(n)}(q) = \frac{q^n}{q^{nr}} (-1)^{n-1} \frac{n!}{q^n - 1} \,, \tag{1.14}$$

$$S^{(n)}(q^r + 0) = \frac{1}{q^{nr}} P^{(n)}(1) = \frac{1}{q^{nr}} (-1)^{n-1} \frac{n!}{q^n - 1} \,. \tag{1.15}$$

These values show that the coefficients $a_r$ of (7) are equal to

$$a_r = \frac{1}{n!}\{S^{(n)}(q^r + 0) - S^{(n)}(q^r - 0)\} = \frac{1}{n!\, q^{nr}} (-1)^n n! \text{ if } r \geqq 1 \,,$$

and

$$a_r = (-1)^{n-1} \frac{1}{n!}\{S^{(n)}(q^r + 0) - S^{(n)}(q^r - 0)\} = \frac{(-1)^{n-1}}{n! q^{nr}} (-1)^n n!$$
$$\text{if } r \leqq 0 \,.$$

Therefore

$$a_r = \begin{cases} (-1)^n q^{-nr} & \text{if } r \geqq 1 \,, \\ -q^{-nr} & \text{if } r \leqq 0 \,. \end{cases}$$

This establishes (1.1) and also Theorem 1.

**2. The nice global behavior of $S_n(x)$.** The function

$$f(x) = \frac{\log x}{\log q} \tag{2.1}$$

that we are interpolating by $S_n(x)$ has the derivative $(x \log q)^{-1}$ which is *completely monotone in* $(0, \infty)$, because

$$f^{(\nu)}(x) = \frac{(-1)^{\nu-1}}{\log q} \frac{(\nu-1)!}{x^\nu} \,, \quad (\nu = 1, 2, \cdots) \,. \tag{2.2}$$

*To what extent does $S_n(x)$ imitate this regular behavior of $f(x)$?* That it does so as best as it can (remember that $S_n(x)$ has only $n$ derivatives, the $n$th being a step-function) is shown by

THEOREM 2. *The logarithmic $q$-spline $S_n(x)$ has the properties*

$$(-1)^{\nu-1} S_n^{(\nu)}(x) > 0 \ \textit{in}\ 0 < x < \infty \,, \tag{2.3}$$

$$S_n^{(\nu)}(x) = O(x^{-\nu}) \ \textit{for}\ \nu = 1, 2, \cdots, n \,. \tag{2.4}$$

*Proof.* We first establish (2.4). Differentiation of (1.9) gives $q^\nu S^{(\nu)}(qx) = S^{(\nu)}(x)$, where $x \neq q^r$ for all $r$, if $\nu = n$. Iterating this we obtain that

$$S^{(\nu)}(q^r x) = q^{-\nu r} S^{(\nu)}(x) \quad (r \text{ integer}) \,. \tag{2.5}$$

Let us write

$$M_\nu = \sup_{1<x<q} | S^{(\nu)}(x) | \,. \tag{2.6}$$

Assuming that

$$q^r < x < q^{r+1} \,, \tag{2.7}$$

then (2.5) and (2.6) show, on replacing $x$ by $xq^{-r}$, that

$$| S^{(\nu)}(x) | \leqq M_\nu q^{-\nu r} = M q^{-\nu(r+1)} q^\nu < M_\nu x^{-\nu} q^\nu$$

and therefore

$$| S^{(\nu)}(x) | < (M_\nu q^\nu) x^{-\nu} \,. \tag{2.8}$$

As the right side does not depend on $r$, the condition (2.7) becomes irrelevant and (2.8) holds for all positive $x$. This establishes (2.4).

We now establish (2.3) by induction for decreasing values of $\nu$. (2.3) *is true for* $\nu = n$: This is shown by (1.15) which gives

$$S_n^{(n)}(x) = (-1)^{n-1}\frac{n!}{q^n - 1}(q^r)^{-n} \quad \text{if } q^r < x < q^{r+1} . \tag{2.9}$$

Therefore (2.3) holds for $\nu = n$. Assuming that (2.3) holds for a value of $\nu(\nu > 1)$, let us establish (2.3) for $\nu - 1$ in place of $\nu$. By (2.4) we can write

$$S_n^{(\nu-1)}(x) = -\int_x^\infty S_n^{(\nu)}(t)dt \quad (x > 0) , \tag{2.10}$$

and this already shows that $S_n^{(\nu-1)}(x)$ has the correct sign. This completes a proof of Theorem 2.

We may summarize our findings by stating

COROLLARY 1. *We may think of the logarithmic q-spline* $S_n(x)$ *as generated as follows: We define* $S_n^{(n)}(x)$ *as a step-function on* $(0, \infty)$ *by the relations* (2.9), *for all integral values of* $r$. *We integrate successively* $S_n^{(n)}(x)$ *by means of the relations* (2.10) *for* $\nu = n, n - 1, \cdots, 3, 2$, *obtaining finally the decreasing positive function* $S_n'(x)$. *Finally, we define* $S_n(x)$ *by*

$$S_n(x) = \int_1^x S_n'(t)dt , \quad (x > 0) . \tag{2.11}$$

That the resulting function $S_n(x)$ is an element of $\sum_{n,q}$, i.e. a cardinal $q$-spline, is clear because we integrate a step-function $S_n^{(n)}(x)$ $n$ times. What does seem remarkable in that the resulting function should solve the interpolation problem

$$S_n(q^\nu) = \nu , \quad \text{for all } \nu , \tag{2.12}$$

and, what is more, even satisfy the functional equation

$$S_n(qx) = S_n(x) + 1 . \tag{2.13}$$

**3. Direct reconstruction of $S_n(x)$ by Corollary 1.** Actually the reconstruction of $S_n(x)$ as described by Corollary 1 is easily carried out explicitly yielding our previous results. We start from (2.9) and abserve that this step function may certainly by written in the form

$$S_n^{(n)}(x) = \sum_{s=-\infty}^{\infty} c_s(q^s - x)_+^0 , \quad (x > 0) . \tag{3.1}$$

If $q^r < x < q^{r+1}$ then (3.1) becomes $S_n^{(n)}(x) = \sum_{s-r} c_s$ and this must equal the right side of (2.9). By differencing we obtain

$$c_r = (-1)^{n-1}\frac{n!}{q^n - 1}q^{-rn}(q^n - 1) = (-1)^{n-1}n!\,q^{-rn}\,,$$

and substituting into (3.1) we obtain that

$$S_n^{(n)}(x) = (-1)^{n-1}n!\sum_{r=-\infty}^{\infty} q^{-rn}(q^r - x)_+^0 \tag{3.2}$$

On performing $\nu$ times the operation (2.10) we obtain

$$S_n^{(n-\nu)}(x) = (-1)^{\nu}(-1)^{n-1}n!\sum_{n} q^{-rn}(q^r - x)_+^{\nu}/\nu!$$

and in particular that

$$S_n'(x) = \sum_{-\infty}^{\infty} q^{-r}n(1 - xq^{-r})_+^{n-1}. \tag{3.3}$$

The last integration (2.11) yields

$$S_n(x) = \sum_{-\infty}^{\infty}\{(1 - q^{-r})_+^n - (1 - xy^{-r})_+^n\}\,, \quad (x > 0)\,. \tag{3.4}$$

If we assume that $1 < x < q$, then (3.4) reduces to

$$\begin{aligned} S_n(x) &= \sum_{r=1}^{\infty}\{(1 - q^{-r})^n - (1 - xq^{-r})^n\} \\ &= \sum_{r=1}^{\infty}\left\{\left(1 - \binom{n}{1}q^{-r} + \cdots + (-1)^n q^{-nr}\right)\right. \\ &\qquad \left. - \left(1 - \binom{n}{1}xq^{-r} + \cdots + (-1)^n x^n q^{-rn}\right)\right\}. \end{aligned}$$

If we cancel within the brackets the two unit terms, then the resulting series may be summed termwise and we obtain that

$$\begin{aligned} S_n(x) &= \binom{n}{1}(x - 1)\frac{q^{-1}}{1 - q^{-1}} - \binom{n}{2}(x^2 - 1)\frac{q^{-2}}{1 - q^{-2}} \\ &\quad + \cdots + (-1)^{n-1}(x^n - 1)\frac{q^{-n}}{1 - q^{-n}} \\ &= \binom{n}{1}\frac{x - 1}{q - 1} - \binom{n}{2}\frac{x^2 - 1}{q^2 - 1} + \cdots + (-)^{n-1}\frac{x^n - 1}{q^n - 1} \\ &\quad \text{in } 1 < x < q\,, \end{aligned}$$

which agrees with the result of Theorem 1. The expression (3.4) will be found useful in our next section.

**4. The peculiar behavior of $S_n(x)$ as $n \to \infty$.** We have derived in (3.4) the expression

$$(4.1) \qquad S_n(x) = \sum_{-\infty}^{\infty} \{(1 - q^{-r})_+^n - (1 - xq^{-r})_+^n\}, \quad (0 < x < \infty).$$

This expression will show against all expectations that the $q$-spline $S_n(x)$ does not converge to $f(x) = \log x/\log q$ as $n \to \infty$. The key to our discussion is the following way of writing the series (4.1) Using the square brackets to denote the integral part we define

$$(4.2) \qquad \theta_n = \frac{\log n}{\log q} - \left[\frac{\log n}{\log q}\right] = \frac{\log n}{\log q} - k_n,$$

hence $n = q^{\theta_n + k_n}$ and therefore $1 = q^{\theta_n + k_n}/n$. We may therefore rewrite (4.1) as

$$S_n(x) = \sum_r \left\{\left(1 - \frac{1}{n} q^{\theta_n + k_n - r}\right)_+^n - \left(1 - \frac{1}{n} xq^{\theta_n + k_n - r}\right)_+^n\right\}$$

and finally as

$$(4.3) \qquad S_n(x) = \sum_{s=-\infty}^{\infty} \left\{\left(1 - \frac{1}{n} q^{\theta_n + s}\right)_+^n - \left(1 - \frac{1}{n} xq^{\theta_n + s}\right)_+^n\right\}.$$

We also define the function

$$(4.4) \qquad F(x, \theta) = \sum_{s=-\infty}^{\infty} \{(e^{-q^{\theta+s}} - e^{-xq^{\theta+s}}), \quad (x > 0, -\infty < \theta < \infty)$$

which is evidently periodic in $\theta$ of period 1. For each fixed positive $x$ the function $F(x, \theta)$ of $\theta$ has an interval of variability

$$(4.5) \qquad I(x) = \left[\min_\theta F(x, \theta), \max_\theta F(x, \theta)\right]$$

in terms of which we can state the following

THEOREM 3. *For each fixed positive $x$ the set of limit points of the sequence $(S_n(x))_{n=1,2,\ldots}$ is identical with the interval $I(x)$ defined by* (4.5).

Our discussion will simplify if we consider the derivative with respect to $x$ of the functions (4.3) and (4.4), hence

$$(4.6) \qquad S_n'(x) = \sum_s q^{\theta_n + s}\left(1 - \frac{1}{n} xq^{\theta_n + s}\right)_+^{n-1}$$

and

$$(4.7) \qquad F_x(x, \theta) = \sum_s q^{\theta+s} e^{-xq^{\theta+s}}.$$

At this point we need some lemmas.

LEMMA 1. *If $a_n$ and $a$ are real, $a_n \to a$, then*

$$\lim_{n\to\infty}\left(1 - \frac{a_n}{n}\right)_+^{n-1} = e^{-a} . \tag{4.8}$$

We may omit the elementary proof.

LEMMA 2. *The inequality*

$$\left(1 - \frac{x}{n}\right)_+^{n-1} < 2e^{-x} \text{ holds for } x \geqq 0,\ n \geqq 2 . \tag{4.9}$$

*Proof.* (4.9) is trivial if $x > n$. If $0 \leqq x \leqq n$ we consider

$$g(x) = \left(1 - \frac{x}{n}\right)^{n-1} e^x$$

and observe that

$$\begin{aligned} g'(x) &= \left(1 - \frac{x}{n}\right)^{n-1} e^x - \frac{n-1}{n}\left(1 - \frac{x}{n}\right)^{n-2} e^x \\ &= e^x\left(1 - \frac{x}{n}\right)^{n-2}\left\{1 - \frac{x}{n} - \frac{n-1}{n}\right\} \\ &= \frac{1}{n}e^x\left(1 - \frac{x}{n}\right)^{n-2}(1 - x) . \end{aligned}$$

This shows that in $[0, n]$ we have

$$\max g(x) = g(1) = \left(1 - \frac{1}{n}\right)^{n-1} \cdot e \to e^{-1} \cdot e = 1 \text{ as } n \to \infty .$$

Actually $(1 - n^{-1})^{n-1}e < 2$ if $n \geqq 2$ which proves the lemma.

LEMMA 3. *The function $F(x, \theta)$, defined by* (4.4), *satisfies the functional equation*

$$F(qx, \theta) = F(x, \theta) + 1 , \quad (x > 0) . \tag{4.10}$$

*Proof.* By (4.4)

$$\begin{aligned} F(qx, \theta) &= \sum_s (e^{-q^{\theta+s}} - e^{-xq^{\theta+s+1}}) \\ &= \sum_s (e^{-q^{\theta+s}} - e^{-xq^{\theta+s}} + e^{-xq^{\theta+s}} - e^{xq^{\theta+s+1}}) , \end{aligned}$$

hence

$$F(qx, \theta) = F(x, \theta) + \sum_s (e^{-xq^{\theta+s}} - e^{-xq^{\theta+s+1}}) .$$

Writing $xq^\theta = \alpha$, the last series becomes $\sum_s (e^{-\alpha q^s} - e^{-\alpha q^{s+1}})$ and this is easily shown to converge to the sum 1 as follows: Letting $A \to -\infty$ and $B \to +\infty$ we have

$$\begin{aligned}\sum_s (e^{-\alpha q^s} - e^{-\alpha q^{s+1}}) &= \lim \sum_{s=A}^{B-1} (e^{-\alpha q^s} - e^{-\alpha q^{s+1}}) \\ &= \lim (e^{-\alpha q^A} - e^{-\alpha q^B}) = 1 - 0 = 1 .\end{aligned}$$

Observe that the functional equations (8) and (4.10) are identical. It follows that the relation

$$S_n(qx) - F(qx, \theta) = S_n(x) - F(x, \theta) \tag{4.11}$$

holds for all positive $x$ and all real $\theta$. Of special interest for us is the following

COROLLARY 2. *In studying the behavior of the difference* $S_n(x) - F(x, \theta_n)$, *we loose no generality in assuming that* $x$ *is restricted to the interval* $1 \leqq x \leqq q$.

LEMMA 4. *Let*

(4.12) $2 \leqq n_1 < n_2 < \cdots$ *be an increasing sequence of integers,*

(4.13) $\eta_1, \eta_2, \cdots$ *be a sequence of reals,* $0 \leqq \eta_\nu \leqq 1$ ,

*and let* $x$ *be restricted to the interval*

$$1 \leqq x \leqq q . \tag{4.14}$$

*Then the series*

$$\sum_{s=-\infty}^{\infty} q^{\eta_\nu+s}\left(1 - x\frac{q^{\eta_\nu+s}}{n_\nu}\right)_+^{n_\nu-1} \tag{4.15}$$

*converges uniformly with respect to all data* $n_\nu$, $\eta_\nu$ *and* $x$ *satisfying the conditions* (4.12), (4.13) *and* (4.14).

*Proof.* By Lemma 2 the series (4.15) is termwise dominated by the series

$$\sum_s q^{\eta_\nu+s}2e^{-xq^{\eta_\nu+s}} ,$$

and this last series is termwise dominated by the series

$$\sum_{s=-\infty}^{\infty} q^{1+s}2e^{-q^s} \tag{4.16}$$

whose terms are independent of all the data. Since (4.16) converges, the lemma is established.

LEMMA 5. *If we add to* (4.13) *the assumption*

$$\lim_{\nu\to\infty} \eta_\nu = \eta \tag{4.17}$$

*then*

$$\lim_{\nu\to\infty} \sum_s q^{\eta_\nu+s}\left(1 - x\frac{q^{\eta_\nu+s}}{n_\nu}\right)_+^{n_\nu-1} = \sum_s q^{\eta+s}e^{-xq^{\eta+s}} \tag{4.18}$$

*uniformly in $x$ satisfying* (4.14).

*Proof.* By Lemma 1

$$\lim_{\nu\to\infty} q^{\eta_\nu+s}\left(1 - x\frac{q^{\eta_\nu+s}}{n_\nu}\right)_+^{n_\nu+1} = q^{\eta+s}e^{-xq^{\eta+s}} ,$$

and the uniform convergence of the series on the left side of (4.18) (Lemma 4), implies the relation (4.18).

In view of the uniform convergence in $x$ we can integrate termwise the limit relation (4.18) with respect to $x$ between the limits of integration 1 and $x$, where we assume (4.14). These integrations are immediately performed because the expressions to be integrated were obtained by (4.6) and (4.7) by differentiation. We state the result as

LEMMA 6. *If we assume* (4.12), (4.13), (4.14), *and* (4.17) *to hold, then*

$$\begin{aligned} &\lim_{\nu\to\infty} \sum_s \left\{\left(1 - \frac{1}{n_\nu}q^{\eta_\nu+s}\right)_+^{n_\nu} - \left(1 - \frac{x}{n_\nu}q^{\eta_\nu+s}\right)_+^{n_\nu}\right\} \\ &\qquad = \sum_s \left(e^{-q^{\eta+s}} - e^{-xq^{\eta+s}}\right) , \end{aligned} \tag{4.19}$$

*uniformly in $\eta_\nu$, $\eta$, and in $x$.*

An approximation of the $q$-spline $S_n(x)$ is now obtained as follows. We apply Lemma 6 for the special case that $n_\nu = \nu$, and that $\eta_\nu$ does not depend on $\nu$, $\eta_\nu = \eta$ for all $\nu$. We obtain from (4.19) that

$$\lim_{n\to\infty} \sum_s \left\{\left(1 - \frac{1}{n}\, q^{\eta+s}\right)_+^n - \left(1 - \frac{x}{n}\, q^{\eta+s}\right)_+^n\right\} = \sum_s \left(e^{-q^{\eta+s}} - e^{-xq^{\eta+s}}\right) , \tag{4.20}$$

uniformly in $\eta (0 \leqq \eta \leqq 1)$ and in $x$ $(1 \leqq x \leqq q)$.

This means the following: To every positive $\varepsilon$, there corresponds an $N_\varepsilon$ such that the error term in (4.20) is in absolute value $<\varepsilon$ if $n > N_\varepsilon$, and this for all $\eta$ and $x$ in their respective intervals. *But then it is clear that we may let $\eta$ also depend on $n$.* Choosing in particular

$$\eta = \theta_n = \log n/\log q - [\log n/\log q]$$

we obtain by (4.3) and (4.4) the following

THEOREM 4. *We have the relation*

$$S_n(x) = F\left(x, \frac{\log n}{\log q}\right) + o(1) \quad as \quad n \to \infty \ , \tag{4.21}$$

*the error term $o(1)$ being uniform for all positive values of $x$.*

Notice that indeed $F(x, \log n/\log q) = F(x, \theta_n)$, due to the periodicity of $F(x, \theta)$ as a function of $\theta$. In extending the uniformity of $o(1)$ from $[1, q]$ to $(0, \infty)$ we have used Corollary 2.

The numbers $\theta_n$ being $\equiv \log n/\log q$, mod 1, it is clear that the sequence $(\theta_n)$ is everywhere dense in $[0, 1]$. Given $\theta$ in $[0, 1]$ we can therefore choose an increasing sequence $(n_\nu)$ such that $\theta_{n_\nu} \to \theta$ as $\nu \to \infty$. It now follows from (4.21) and the continuity of $F(x, \theta)$ that

$$\lim_{\nu\to\infty} S_{n_\nu}(x) = F(x, \theta) \ . \tag{4.22}$$

In view of these remarks it is clear that the relation (4.21) implies the truth of Theorem 3.

**5. The Fourier series of $F(x, \theta)$.** Surely Theorem 4 shows the role played by the periodic function

$$F(x, \theta) = \sum_{s=-\infty}^{\infty} (e^{-q^{\theta+s}} - e^{-xq^{\theta+s}}) \tag{5.1}$$

in the behavior of $S_n(x)$ for large values of $n$. Let us derive its Fourier expansion

$$F(x, \theta) = \sum_{\nu=-\infty}^{\infty} a_\nu(x) e^{2\pi i\nu\theta} \ . \tag{5.2}$$

We find that

$$a_0(x) = \int_0^1 \sum_s (e^{-q^{\theta+s}} - e^{-xq^{\theta+s}}) d\theta = \int_{-\infty}^{\infty} (e^{-q^t} - e^{-xq^t}) dt$$

and setting $q^t = u$, hence $q^t \log q \, dt = du$, we find that

$$\begin{aligned} a_0(x) &= \frac{1}{\log q}\int_0^\infty (e^{-u} - e^{-xu})u^{-1}du = \frac{1}{\log q}\int_0^\infty\Big(\int_1^x e^{-uv}dv\Big)du \\ &= \frac{1}{\log q}\int_1^x\Big(\int_0^\infty e^{-uv}du\Big)dv = \frac{1}{\log q}\int_1^x \frac{dv}{v} \end{aligned}$$

and finally

$$a_0(x) = \frac{\log x}{\log q} \cdot \tag{5.3}$$

Notice that the average of $F(x, \theta)$ over a period is equal to the presumed limit of $S_n(x)$. At this point we observe that $F(x, \theta)$ being continuous in $\theta$, there certainly exists $a\ \theta_x$ such that

$$F(x, \theta_x) = \log x/\log q \ .$$

The last paragraph of § 4 now implies the

COROLLARY 2. *To every positive $x$ corresponds a sequence $(n_\nu)$ such that*

$$\lim_{\nu\to\infty} S_{n_\nu}(x) = \frac{\log x}{\log q} \ . \tag{5.4}$$

If $\nu \neq 0$, we find that

$$\begin{aligned} a_\nu(x) &= \int_0^1 e^{-2\pi i\nu\theta}F(x, \theta)d\theta = \int_0^1 e^{-2\pi i\nu\theta}\sum_s (e^{-q^{\theta+s}} - e^{-xq^{\theta+s}})d\theta \\ &= \sum_s \int_0^1 e^{-2\pi i\nu\theta}(e^{-q^{\theta+s}} - e^{-xq^{\theta+s}})d\theta \\ &= \int_{-\infty}^\infty e^{-2\pi i\nu t}(e^{-q^t} - e^{-xq^t})dt \ . \end{aligned}$$

Setting again $q^t = u$ we obtain

$$a_\nu(x) = \frac{1}{\log q}\int_0^\infty u^{-(2\pi i\nu/\log q)-1}(e^{-u} - e^{-xu})du \ . \tag{5.5}$$

This is easily evaluated in terms of the $\Gamma$-function. If we add $\varepsilon(\varepsilon > 0)$ to the exponent of $u$ in the integrand, we may then write the integral as a difference of two $\Gamma$-integrals which amounts to

$$\frac{1}{\log q}\Gamma\Big(\varepsilon - \frac{2\pi i\nu}{\log q}\Big)(1 - x^{(2\pi i\nu/\log q)-\varepsilon}) \ .$$

On letting $\varepsilon \to 0$ we obtain that

$$a_\nu(x) = \frac{1}{\log q}\Gamma\Big(-\frac{2\pi i\nu}{\log q}\Big)(1 - x^{2\pi i\nu/\log q}) \,, \quad (\nu \neq 0) \ . \tag{5.6}$$

From (4.4) we see that $F(1, \theta) = 0$ for all $\theta$. Moreover (4.10) shows that $F(q^r, \theta) = r$ for all $\theta$. We therefore see that the interval of variability $I(x)$, defined by (4.5), reduces of the point $r$ if $x = q^r$ ($r$ integer). The relations (5.6) allow us to prove the converse, to the effect that $I(x)$ reduces to a single point, namely $\log x/\log q$, only if $x$ is a power of $q$. For if $I(x)$ is a single point, then the unicity theorem of Fourier series shows that all coefficients (5.6) vanish. The $\Gamma$-function being $\neq 0$, we conclude that

$$1 = x^{2\pi i\nu/\log q} = e^{2\pi i\nu(\log x/\log q)} \text{ if } \nu \neq 0$$

and this indeed implies that $\log x/\log q = r$ is an integer, hence $x = q^r$.

In view of Theorem 3 we have just established

COROLLARY 3. *The relation*

$$\lim_{n\to\infty} S_n(x) = \frac{\log x}{\log q} \tag{5.7}$$

*is valid if and only if $x = q^r$ for an integer $r$.*

The Fourier series (5.2), (5.3), (5.6) allows us to write the approximation formula (4.21) in explicit terms. Observing that

$$e^{2\pi i\nu(\log n/\log q)} = n^{2\pi i\nu/\log q}$$

and using the Fourier series we obtain the following

COROLLARY 4. *For every fixed positive $x$ we have*

$$\begin{aligned} S_n(x) = \frac{\log x}{\log q} + \frac{1}{\log q} \sum_{\nu\neq 0} \Gamma\Big(-\frac{2\pi i\nu}{\log q}\Big)(1 - x^{2\pi i\nu/\log q})n^{2\pi i\nu/\log q} \\ + o(1)\ , \quad \text{as } n \to \infty\ , \end{aligned} \tag{5.8}$$

*the error term $o(1)$ being uniform in $x > 0$.*

This will be used in § 7.

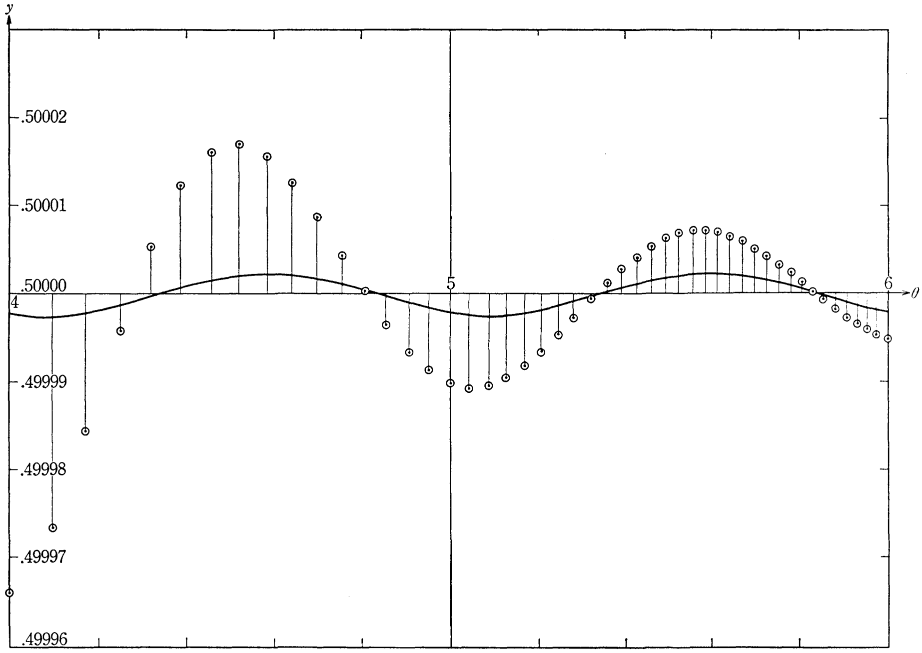
**6. A graph for the case $q = 2$ and $x = \sqrt{2}$.** We select

$$q = 2\ , \quad x = \sqrt{2}\ , \quad \text{hence } \frac{\log x}{\log q} = \frac{1}{2}\ . \tag{6.1}$$

Figure 1 shows the graph of the function

$$y = F(\sqrt{2}, \theta) \text{ in the interval } 4 \leqq \theta \leqq 6 \tag{6.2}$$

and also the 49 points




This sheet represents in the $(\theta, y)$-plane the rectangle defined by

$$4 \leqq \theta \leqq 6\,, \qquad .49996 \leqq y \leqq .50003\,.$$

The curve is $y = F(\sqrt{2}, \theta)$, $4 \leqq \theta \leqq 6$.

The 49 dots represent the points $(\log n/\log 2, S_n(\sqrt{2}))$, $n = 16, 17, \cdots, 64$.

Figure 1

$$\left(\frac{\log n}{\log 2}, S_n(\sqrt{2})\right) \text{ for } n = 16, 17, \cdots, 64\,. \tag{6.3}$$

Notice the tremendously magnified scale in the $y$-direction, and that the abscissae of the points (6.3) also vary from 4 to 6. Thus Figure 1 exhibits graphically all terms of the relation (4.21) or

$$S_n(\sqrt{2}) = F\left(\sqrt{2}, \frac{\log n}{\log 2}\right) + o(1) \text{ for } n = 16, 17, \cdots, 64\,. \tag{6.4}$$

If we were to extend the graph of Figure 1 from $4 \leqq \theta \leqq 6$ to $4 \leqq \theta \leqq +\infty$, then the abscissae of consecutive dots would get closer together indefinitely, while the oscillations of their ordinates would decrease so that the dots would tend to the curve $y = F(\sqrt{2}, \theta)$, according to the relation (6.4).

Figure 1 reveals the following apparent situation: The "curve" through the points (6.3) seems to intersect the horizontal line $y = 1/2$ in practically the same points as the curve $y = F(\sqrt{2}, \theta)$. *It therefore seems that* $S_n(\sqrt{2})$ *is very close to its "correct" limit* $1/2$ *whenever* $n$ *is such that* $\log n/\log 2$ *is close to one of the points* $\theta$

*where* $F(\sqrt{2}, \theta)$ *assumes its average value* 1/2. We have no explanation for this phenomenon, if true.

A word on the computations underlying Figure 1. In the first place we needed the ordinates of the points (6.3). Since $x = \sqrt{2}$ is in the interval $[1, q]$ we have $S_n(x) = P_n(x)$, by Theorem 1. Moreover the relation (1.2) shows that

$$\frac{\log x}{\log q} - P_n(x) = (-1)^n \Delta^n \frac{x^t - 1}{q^t - 1}\bigg|_{t=0} . \tag{6.5}$$

Here $\Delta^n$ denotes the ordinary $n$th order forward difference (with step 1) with respect to $t$. Our colleague C. de Boor kindly computed these differences for the data (6.1) and for $n = 1, 2, 3, \cdots, 70$. Beyond this point round-off difficulties became apparent. This gave us the points (6.3).

The function (6.2) was obtained from its Fourier series (5.2). It so happens that for $x = \sqrt{2}$ the coefficients (5.6) vanish for all even values of $\nu \neq 0$. For this reason by taking only the fundamental ($\nu = \pm 1$) in the expansion (5.2) we already obtain better than 15-place accuracy. The coefficients $a_1(\sqrt{2})$ and $a_{-1}(\sqrt{2})$ were computed by means of the fine tables in [1], (p. 277), and we found with 11-place accuracy that

$$F(\sqrt{2}, \theta) = \frac{1}{2} + A \sin 2\pi(\theta - \alpha) , \tag{6.6}$$

where

$$A = .238061 \times 10^{-5} , \quad \alpha = .340\,775 . \tag{6.7}$$

7. **The summability of the sequence $S_n(x)$.** We rewrite (5.8) as

$$S_n(x) = \frac{\log x}{\log q} + \sum_{\nu \neq 0} a_\nu(x) n^{2\pi i \nu / \log q} + o(1) \tag{7.1}$$

and are looking for a summability method that will produce the limit $\log x / \log q$. For this it is necessary that the method should assign for the sequence

$$(n^{i\lambda})_{n=1,2,\cdots} \quad \text{(for constant } \lambda \neq 0) \tag{7.2}$$

the limit zero. Neither the Cesàro nor the Abel method will do that. However, the *logarithmic means* (see [2], §§ 3.8 and 4.16) will be found to work yielding the following

THEOREM 5. *We have that*

$$\lim_{n\to\infty} \frac{1}{\log n}\left\{S_1(x) + \frac{1}{2}S_2(x) + \cdots + \frac{1}{n}S_n(x)\right\} = \frac{\log x}{\log q} \tag{7.3}$$

*uniformly in* $x$, $x > 0$.

*Proof.* Referring to (7.1) and using the fact that

$$\sum_{\nu\neq 0} |a_\nu(x)| < \frac{2}{\log q}\sum_{\nu\neq 0}\left|\Gamma\left(-\frac{2\pi i\nu}{\log q}\right)\right| < \infty ,$$

it is clear that (7.1) will imply (7.3) as soon as we use the following

LEMMA 6. *For real* $\lambda$ *we have*

$$|1^{i\lambda-1} + 2^{i\lambda-1} + \cdots + n^{i\lambda-1}| \leqq 1 + \log n \text{ for all } n , \tag{7.4}$$

*and*

$$\lim_{n\to\infty}\frac{1}{\log n}(1^{i\lambda-1} + 2^{i\lambda-1} + \cdots + n^{i\lambda-1}) = 0 \text{ if } \lambda \neq 0 . \tag{7.5}$$

*Proof.* 1. The left side of (7.4) is less than

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \leqq 1 + \int_1^n \frac{dx}{x} = 1 + \log n .$$

2. Writing $f(x) = x^{i\lambda-1}$ we use the Euler-MacLaurin summation formula

$$\sum_1^n f(k) = \int_1^n f(x)dx + \frac{1}{2}(f(1) + f(n)) + \int_1^n\left(x - [x] - \frac{1}{2}\right)f'(x)dx$$

which shows that

$$\begin{aligned}\left|\sum_1^n f(k)\right| &< \left|\int_1^n f(x)dx\right| + \frac{1}{2}(1 + n^{-1}) + \int_1^n\left|f'(x)\right|dx \\ &= |n^{i\lambda} - 1|\lambda^{-1} + \frac{1}{2}(1 + n^{-1}) + |i\lambda - 1|\int_1^n x^{-2}dx .\end{aligned}$$

The last expression is $O(1)$ and therefore $o(\log n)$.

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