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LOCAL AND GLOBAL BIFURCATION FROM NORMAL EIGENVALUES

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This paper studies the bifurcation of solutions of nonlinear eigenvalue problems of the form $Lu = \lambda u + H(\lambda, u)$, where L is linear and H is $o(\| u \|)$ on bounded λ intervals. It is shown that isolated normal eigenvalues of L having odd algebraic multiplicity are bifurcation points, and moreover possess branches of solutions which satisfy an alternative theorem. A related situation is studied, and an application explored.

Introduction. In this paper we study the bifurcation of solutions of nonlinear eigenvalue problems. The equations to be studied are of the form

$$Lu = \lambda u + H(\lambda, u) \tag{0.1}$$

where all operators are defined in a real Banach space $\mathscr{B}$. L is assumed to be linear, bounded or unbounded; I, the identity map; and H, compact and $o(\| u \|)$ near $u = 0$. Clearly, $(\lambda, 0)$ is a solution for each real λ, and these are called the trivial solutions of (0.1). Of more interest are the nontrivial solutions, pairs (λ, u) satisfying (0.1) with $u \neq 0$. In particular, one is interested in how solutions of (0.1) are related to solutions of the linear equation

$$Lu = \lambda u\ . \tag{0.2}$$

The study of this led to the following definition.

DEFINITION. A point $(\lambda_0, 0)$ is a bifurcation point for (0.1) if every neighborhood of $(\lambda_0, 0)$ in $\boldsymbol{R} \times \mathscr{B}$ contains a nontrivial solution of (0.1).

Under quite general conditions, it is easy to show that in order for $(\lambda_0, 0)$ to be a bifurcation point of (0.1), it is necessary that λ_0 be in the spectrum of L.

The first general existence theorem for bifurcation points was obtained by Krasnoseljskii [2]. He considered equations of the type

$$u = \lambda Lu + H(\lambda, u) \tag{0.3}$$

where L is linear and compact, I and H being as before. He proved that if λ_0 is a characteristic value of L having odd algebraic multiplicity, then $(\lambda_0, 0)$ is a bifurcation point.

More recently, Rabinowitz [5] studied the same problem as

Krasnoseljskii and proved a much stronger result. The bifurcation from such points is a global property, with a continuous branch of solutions joining $(\lambda_0, 0)$ to infinity in $\boldsymbol{R} \times \mathscr{B}$, or if the branch is bounded, containing $(\lambda_1, 0)$ with $\lambda_1 \neq \lambda_0$.

Turner [8] discovered a global result for (0.3) somewhat different from that of Rabinowitz. Let $[a, b]$ be an interval containing an odd number of characteristic values of L counting multiplicities with $1/a$ and $1/b$ in the resolvent set of L. Select C, a simple curve in $\boldsymbol{R} \times \boldsymbol{R}_+$ joining $(a, 0)$ to $(b, 0)$. Then (0.3) has at least two nontrivial solutions $(\lambda^{(1)}, u^{(1)})$ and $(\lambda^{(2)}, u^{(2)})$ such that $(\lambda^{(i)}, \|u^{(i)}\|)$ lie on C. A similar result holds when the assumptions on H are weakened: $H(\lambda, u) = J(\lambda, u)u$ where $J(\lambda, u)$ is a compact linear operator taking $\mathscr{B}$ into $\mathscr{B}$ and $J(\lambda, u)u$ denotes $J(\lambda, u)$ operating on u.

The main result of this paper is that the compactness assumption on L is not needed. The proofs of the theorems mentioned involve the use of degree theory. In order to apply degree theory in this new situation, it is shown that (0.1) is equivalent to a compact perturbation of the identity for certain values of λ. In looking for bifurcation points we will consider the isolated normal eigenvalues of L.

DEFINITION. An eigenvalue λ of L is defined to be normal if

(i) the multiplicity of λ is finite

(ii) $\mathscr{B}$ is the direct sum of subspaces, $\mathscr{L}_\lambda \oplus \mathscr{N}_\lambda$, where $\mathscr{L}_\lambda = \bigcup_{j=1}^{\infty} \ker (L - \lambda)^j$, $\mathscr{N}_\lambda$ is invariant under L, and $(L - \lambda)$ is invertible on $\mathscr{N}_\lambda$.

An eigenvalue λ of L is isolated if there exists $\varepsilon > 0$ such that $(\lambda - \varepsilon, \lambda + \varepsilon)$ contains no other members of $\operatorname{sp} L$.

It should be noted that all nonzero eigenvalues of a linear compact operator are normal and isolated.

Section 1 contains a generalization of Krasnoseljskii's result. If λ_0 is an isolated normal eigenvalue of L having odd multiplicity, then $(\lambda_0, 0)$ is a bifurcation point for (0.1). Since the concept of normal eigenvalue is crucial to the proof, § 1 concludes with a set to sufficient conditions under which an eigenvalue of L is a normal eigenvalue.

Section 2 generalizes Rabinowitz's result. Since L is no longer compact, it is necessary to modify his second alternative and introduce a third one. Examples are given demonstrating that these three alternatives are nonvacuous. Section 3 generalizes Turner's result to noncompact operators L in a way similar to the two preceding theorems. Section 4 concludes the paper by applying these theorems to a class of ordinary differential equations of Sturm-

Liouville type on a semi-infinite interval.

1. A local bifurcation theorem. Let $\mathscr{B}$ be a real Banach space and let $\mathscr{E}$ denote $\boldsymbol{R} \times \mathscr{B}$ with the product topology. By a nonlinear eigenvalue problem we mean an equation of the form

$$Lu = \lambda u + H(\lambda, u) \tag{1.1}$$

where $L: \mathscr{B} \to \mathscr{B}$ is linear and $H: \mathscr{E} \to \mathscr{B}$ is a nonlinear operator satisfying hypothesis H-1:

(H-1) (i) H is compact, and
(ii) H is $o(\|u\|)$ for u near 0 uniformly on each bounded λ interval.

A nontrivial solution of (1.1) is a pair (λ, u) with $u \neq 0$ which satisfies (1.1), and the trivial solutions are the pairs $(\lambda, 0)$.

In what follows, $L: \mathscr{B} \to \mathscr{B}$ will be a densely defined linear operator (bounded or unbounded) with domain $\operatorname{dom}(L)$. The resolvent set of L, $\rho(L)$, will be all real values of λ for which there exists a bounded linear operator $C: \mathscr{B} \to \mathscr{B}$ such that

$$\begin{aligned} &C(L - \lambda)x = x,\ x \in \operatorname{dom}(L) \\ &(L - \lambda)Cx = x,\ x \in \operatorname{range}(L - \lambda)\,. \end{aligned} \tag{1.2}$$

C will be denoted by $(L - \lambda)^{-1}$.

Definition 1.1. The (algebraic) multiplicity of an eigenvalue λ of L is defined to be the dimension of the subspace $\bigcup_{j=1}^{\infty} \ker (L - \lambda)^j$ where $\ker (L - \lambda)^j$ denotes the nullspace of $(L - \lambda)^j$. $\bigcup_{j=1}^{\infty} \ker (L - \lambda)^j$ will be referred to as the principal manifold of L associated with λ.

Definition 1.2. An eigenvalue λ of L is defined to be normal if

(i) the multiplicity of L is finite

(ii) $\mathscr{B}$ is the direct sum of subspaces $\mathscr{L}_\lambda \oplus \mathscr{N}_\lambda$ where $\mathscr{L}_\lambda = \bigcup_{j=1}^{\infty} \ker (L - \lambda)^j$, $\mathscr{N}_\lambda$ is invariant under L, and $(L - \lambda)$ is invertible on $\mathscr{N}_\lambda$.

The projection of $\mathscr{B}$ onto $\mathscr{L}_\lambda$ along $\mathscr{N}_\lambda$ is denoted by P_λ. Hence $P_\lambda \mathscr{B} = \mathscr{L}_\lambda$ and $(I - P_\lambda)\mathscr{B} = \mathscr{N}_\lambda$.

An eigenvalue λ of L is isolated if there exists $\varepsilon > 0$ such that $(\lambda - \varepsilon, \lambda + \varepsilon)$ contains no other members of $\operatorname{sp} L$. The set of isolated normal eigenvalues of L is called the discrete spectrum of L which we denote by $\operatorname{sp}_d(L)$. The remaining part of the spectrum will be called nondiscrete and is denoted by $\operatorname{sp}_{nd}(L)$.

Remark. If L is self-adjoint, the nondiscrete spectrum is the essential spectrum of L.

DEFINITION 1.3. $(\lambda, 0)$ is a bifurcation point for (1.1) if every neighborhood in $\mathscr{E}$ of $(\lambda, 0)$ contains a nontrivial solution of (1.1).

DEFINITION 1.4. If $\mathscr{V}$ is a subset of $\mathscr{E}$, $\mathscr{V}^{\lambda}$ and $\mathscr{V}_R$ are defined to be $\mathscr{V}^{\lambda} = \{u \mid (\lambda, u) \in \mathscr{V}\}$ and $\mathscr{V}_R = \{\lambda \mid (\lambda, u) \in \mathscr{V}$ for some $u\}$. For $W \subset \boldsymbol{R}$, $\mathscr{B}$, or $\mathscr{E}$, $\bar{W}$ denotes the closure of W in the respective space.

The first theorem shows that bifurcation from an isolated eigenvalue λ_0 of L having odd multiplicity is not dependent upon L being compact, but rather on the behavior of $L-\lambda$ near the eigenvalue λ_0.

THEOREM 1.1. *Let L be as above and let H satisfy* H-1. *If λ_0 is an isolated normal eigenvalue of L having odd multiplicity, then* $(\lambda_0, 0)$ *is a bifurcation point for* (1.1).

Proof. In order to prove this theorem, (1.1) will be rewritten in the form $u - C(\lambda, u) = 0$ where C is compact. Let $Q_{\lambda_0} = I - P_{\lambda_0}$ and split (1.1) by

$$\begin{aligned} LP_{\lambda_0}u &= \lambda P_{\lambda_0}u + P_{\lambda_0}H(\lambda, u) \\ LQ_{\lambda_0}u &= \lambda Q_{\lambda_0}u + Q_{\lambda_0}H(\lambda, u) . \end{aligned} \tag{1.3}$$

A solution of (1.1) is equivalent to a simultaneous solution of the two equations in (1.3). Select $\mu_0 \in \rho(L)$. Instead of (1.3) we may write

$$\begin{aligned} P_{\lambda_0}u &= \frac{(L-\mu_0)P_{\lambda_0}u}{\lambda-\mu_0} - \frac{P_{\lambda_0}H(\lambda, u)}{\lambda-\mu_0} \\ Q_{\lambda_0}u &= (L-\lambda)^{-1}Q_{\lambda_0}H(\lambda, u) \end{aligned} \tag{1.4}$$

where $(L-\lambda)^{-1}$ is to be interpreted as $(L-\lambda)^{-1} \mid \mathscr{N}_{\lambda_0}$. Thus, (1.4) is valid for $\lambda \in \{\lambda_0\} \cup \{\rho(L)\backslash\{\mu_0\}\}$. Adding these equations we get

$$\begin{aligned} u &= C_1(\lambda, u) + C_2(\lambda, u) \\ C_1(\lambda, u) &= \frac{(L-\mu_0)P_{\lambda_0}u}{\lambda-\mu_0} \\ C_2(\lambda, u) &= \left((L-\lambda)^{-1}Q_{\lambda_0} - \frac{P_{\lambda_0}}{\lambda-\mu_0}\right)H(\lambda, u) . \end{aligned} \tag{1.5}$$

Note that $C_1\colon \mathscr{E} \rightarrow \mathscr{B}$ is compact and linear in u for each fixed λ. $C_2\colon \mathscr{E} \rightarrow \mathscr{B}$ satisfies H-1. Define

$$\Phi(\lambda, \cdot) = I - C_1(\lambda, \cdot) - C_2(\lambda, \cdot) . \tag{1.6}$$

Clearly, (1.5) or $\Phi(\lambda, u) = 0$ is equivalent to (1.1) for the specified values of λ when L is bounded. If L is unbounded, the question arises as to whether u is in dom(L) if (λ, u) is a zero of Φ. Noting (1.4), which is obtained from (1.5) by projecting onto $\mathscr{L}_{\lambda_0}$, $\mathscr{N}_{\lambda_0}$ respectively, we see that $Q_{\lambda_0}u$ is in dom(L). Since $P_{\lambda_0}u$ is in an eigenspace of L, $u = P_{\lambda_0}u + Q_{\lambda_0}u$ is in dom(L).

If the assertion of the theorem is not true we can find a neighborhood $\mathscr{O}$ of $(\lambda_0, 0)$ such that the only solutions of (1.1) in $\bar{\mathscr{O}}$ are trivial solutions and $\rho(L)\backslash\bar{\mathscr{O}}_R \neq \emptyset$. Select $\mu_0 \in \rho(L)\backslash\bar{\mathscr{O}}_R$ such that (1.1) is equivalent to (1.5) for all $\lambda \in \bar{\mathscr{O}}_R$. Select $\varepsilon > 0$ such that $[-\varepsilon + \lambda_0, \lambda_0 + \varepsilon] \times \{0\} \subset \mathscr{O}$. Applying the homotopy property of degree theory we obtain

$$\deg(\Phi(\lambda, \cdot), \mathscr{O}^\lambda, 0) = \text{constant} \qquad |\lambda - \lambda_0| < \varepsilon . \tag{1.7}$$

Select $\underline{\lambda}$ and $\bar{\lambda}$ such that $\lambda_0 - \varepsilon < \underline{\lambda} < \lambda_0 < \bar{\lambda} < \lambda_0 + \varepsilon$. Then

$$\begin{aligned} &\deg(\Phi(\underline{\lambda}, \cdot), \mathscr{O}^{\underline{\lambda}}, 0) = \text{index}(I - C_1(\underline{\lambda}, \cdot), (\underline{\lambda}, 0)) \\ &\deg(\Phi(\bar{\lambda}, \cdot), \mathscr{O}^{\bar{\lambda}}, 0) = \text{index}(I - C_1(\bar{\lambda}, \cdot), (\bar{\lambda}, 0)) . \end{aligned} \tag{1.8}$$

Thus, using (1.7) and (1.8),

$$\begin{aligned} &\text{index}(I - C_1(\underline{\lambda}, \cdot), \underline{\lambda}, 0)) \\ &\quad = \text{index}(I - C_1(\bar{\lambda}, \cdot), (\bar{\lambda}, 0)) . \end{aligned} \tag{1.9}$$

However, since the multiplicity of λ_0 is odd,

$$\begin{aligned} &\text{index}(I - C_1(\underline{\lambda}, \cdot), (\underline{\lambda}, 0)) \\ &\quad = -\text{index}(I - C_1(\bar{\lambda}, \cdot), (\bar{\lambda}, 0)) . \end{aligned} \tag{1.10}$$

Since the indices in (1.9) and (1.10) are either $+1$ or -1, we have a contradiction. Thus, such a neighborhood $\mathscr{O}$ can never be found. This proves that $(\lambda_0, 0)$ is a bifurcation point.

REMARK. If $\lambda_0 \neq 0$ is an eigenvalue of L having odd multiplicity, then the hypotheses of Theorem 1 are satisfied if L is compact or if L is self-adjoint with λ_0 isolated in sp L.

It is possible to give conditions under which an eigenvalue of a linear operator L is normal. In the following, $\hat{\mathscr{B}}$ will denote the complexification of $\mathscr{B}$, and $\hat{L}$ will be the unique linear extension of L to $\hat{\mathscr{B}}$. $\hat{\mathscr{B}}$ will be thought of as $\mathscr{B} \times \mathscr{B}$ and for a pair $(x, y) \in \hat{\mathscr{B}}$, we define the norm $\|(x, y)\|_{\hat{\mathscr{B}}} = \sqrt{\|x\|^2 + \|y\|^2}$ where $\|\cdot\|$ is the norm in $\mathscr{B}$.

LEMMA 1.1. $\operatorname{sp} L = \boldsymbol{R} \cap \operatorname{sp} \hat{L}$. *If λ is in the point, continuous, or residual spectrum of L, then it is also in the point, continuous, or residual spectrum of L respectively. If λ is a real eigenvalue for $\hat{L}$, then its multiplicity* (finite or infinite) *is the same for L as for $\hat{L}$.*

Proof. It is easily seen that $\operatorname{sp} L \subset \boldsymbol{R} \cap \operatorname{sp} \hat{L}$, with parts corresponding. To consider the reverse inclusion, select a real λ in the point spectrum of $\hat{L}$. Then there exists $(x, y) \neq (0, 0)$ such that $(\hat{L} - \lambda)(x, y) = (0, 0)$. Thus, at least one of x and y is a nonzero eigenvector of L associated with λ.

Now select a real λ which is an approximate eigenvalue of $\hat{L}$, but not an eigenvalue of $\hat{L}$. Then $(\hat{L} - \lambda)$ is injective. Since $(\hat{L} - \lambda)$ is not invertible, there exists $\{(x_n, y_n)\}_{n=1,2..}$, each term of unit $\widehat{\mathscr{B}}$-norm, such that

$$\| (L - \lambda)x_n \|^2 + \| (L - \lambda)y_n \|^2 = \| (\hat{L} - \lambda)(x_n, y_n) \|^2_{\hat{\mathscr{B}}} < \frac{1}{n^2} .$$

For each n we may select z_n as one of the pair x_n, y_n such that $\| (L - \lambda)z_n \| < 1/n$ and $\| z_n \| \geqq 1/2$. Since $(L - \lambda)$ is injective, λ is an approximate eigenvalue of L, but not an eigenvalue.

Finally, let a real λ be in the residual spectrum of $\hat{L}$. $(\hat{L} - \lambda)$ is injective, thus showing $(L - \lambda)$ is also injective. There exists $(z_1, z_2) \in \widehat{\mathscr{B}}$ and $\varepsilon > 0$ such that $\| (z_1, z_2) - (\hat{L} - \lambda)(x, y) \|^2_{\hat{\mathscr{B}}} > \varepsilon$ for all $(x, y) \in \widehat{\mathscr{B}}$. In particular.

$$\| z_1 - (L - \lambda)x \|^2 + \| z_2 - (L - \lambda)y \|^2 > \varepsilon . \tag{1.11}$$

It follows that $\| z_1 - (L - \lambda)x \| > \varepsilon/2$ or $\| z_2 - (L - \lambda)x \| > \varepsilon/2$ for all x. Hence λ is in the residual spectrum of L. We now also know that the real part of the continuous spectrum of $\hat{L}$ is the continuous spectrum of L.

Suppose λ is an eigenvalue of L and $\hat{L}$, and let $\mathscr{L}_\lambda$ and $\widehat{\mathscr{L}_\lambda}$ denote the principal manifolds associated with λ. Let $(x, y) \in \widehat{\mathscr{L}_\lambda}$. Then it is known that $\| (\hat{L} - \lambda)^n(x, y) \|_{\hat{\mathscr{B}}} = 0$ for some n. Since $\| (L - \lambda)^n x \| \leqq \| (\hat{L} - \lambda)^n(x, y) \|_{\hat{\mathscr{B}}}$, we know

$$\| (L - \lambda)^n x \| = \| (L - \lambda)^n y \| = 0$$

thus proving that $\widehat{\mathscr{L}_\lambda} \subset \mathscr{L}_\lambda \times \mathscr{L}_\lambda$. Now select $(x, y) \in \mathscr{L}_\lambda \times \mathscr{L}_\lambda$. Then there exists n such that $\| (L - \lambda)^n x \| = \| (L - \lambda)^n y \| = 0$. Thus $\| (\hat{L} - \lambda)^n(x, y) \|_{\hat{\mathscr{B}}} = 0$ which shows that $\mathscr{L}_\lambda \times \mathscr{L}_\lambda \subset \widehat{\mathscr{L}_\lambda}$.

THEOREM 1.2. *Let λ_0 be an eigenvalue of a bounded linear* *operator L having finite multiplicity and isolated in* $\operatorname{sp} \hat{L}$. *Then* λ_0 *is a normal eigenvalue of* L. *Moreover,* $\operatorname{sp}(L \mid \mathscr{N}_{\lambda_0}) \subset \operatorname{sp} L \backslash \{\lambda_0\}$.

Proof. Since λ_0 is isolated in $\operatorname{sp} \hat{L}$ and is of finite multiplicity, λ_0 is a normal eigenvalue for $\hat{L}$.

The projection $\hat{P}_{\lambda_0}$ of $\widehat{\mathscr{B}}$ onto $\widehat{\mathscr{L}}_{\lambda_0}$ is given explicitly by

$$\hat{P}_{\lambda_0} = -\frac{1}{2\pi i} \int_{\partial D} (\hat{L} - z\hat{I})^{-1} dz \tag{1.12}$$

where D is a bounded domain in the complex plane with λ_0 in its interior and $\operatorname{sp} \hat{L} \backslash \{\lambda_0\}$ in its exterior. From (1.12) it is clear that $\hat{P}_{\lambda_0}$ is bounded. Moreover, if $\widehat{\mathscr{N}}_{\lambda_0} = (\hat{I} - \hat{P}_{\lambda_0})\widehat{\mathscr{B}}$, then

$$\hat{L} \mid \widehat{\mathscr{N}}_{\lambda_0} : \widehat{\mathscr{N}}_{\lambda_0} \longrightarrow \widehat{\mathscr{N}}_{\lambda_0} \quad \text{and} \quad \operatorname{sp} \hat{L} \mid \widehat{\mathscr{N}}_{\lambda_0} = \operatorname{sp} \hat{L} \backslash \{\lambda_0\} .$$

Any $x \in \mathscr{B}$ can be written uniquely as

$$(x, 0) = (x_1, y) + (x_2, -y) \tag{1.13}$$

where $(x_1, y) \in \widehat{\mathscr{L}}_{\lambda_0}$ and $(x_2, -y) \in \widehat{\mathscr{N}}_{\lambda_0}$. Let us define $P_{\lambda_0}: \mathscr{B} \to \mathscr{B}$ by $P_{\lambda_0} x = x_1$. Since

$$(P_{\lambda_0} x, 0) = (x_1, 0) + (0, 0) , \tag{1.14}$$

we have $P_{\lambda_0}(P_{\lambda_0} x) = P_{\lambda_0} x$, making P_{λ_0} a projection with range in the principal manifold of L associated with λ_0. Moreover, let x be in that manifold. Then $(x, 0) = (x, 0) + (0, 0)$ uniquely, showing $P_{\lambda_0} x = x$. Thus the range of P_{λ_0} is the principal manifold of L associated with λ_0. Denote the range of P_{λ_0} by $\mathscr{L}_{\lambda_0}$ and the range of $I - P_{\lambda_0}$ by $\mathscr{N}_{\lambda_0}$. It is clear that $L: \mathscr{N}_{\lambda_0} \to \mathscr{N}_{\lambda_0}$ since $\hat{L}: \widehat{\mathscr{N}}_{\lambda_0} \to \widehat{\mathscr{N}}_{\lambda_0}$. It remains to show that $\operatorname{sp}(L \mid \mathscr{N}_{\lambda_0}) \subset \operatorname{sp} L \backslash \{\lambda_0\}$. Select a real $\lambda \notin \operatorname{sp} L \backslash \{\lambda_0\}$. According to Riesz-Nagy [6] and Lemma 1.1, $\hat{L} - \lambda$ is invertible on $\widehat{\mathscr{N}}_{\lambda_0}$. For $x \in \mathscr{N}_{\lambda_0}$ (1.13) shows there exists $y \in \mathscr{L}_{\lambda_0}$ such that $(x, y) \in \widehat{\mathscr{N}}_{\lambda_0}$. If (x, y) and (x, z) are in $\widehat{\mathscr{N}}_{\lambda_0}$ with y and z in $\mathscr{L}_{\lambda_0}$, then $(0, y - z) \in \mathscr{N}_{\lambda_0} \cap \mathscr{L}_{\lambda_0}$, showing $y = z$. $(\hat{L} - \lambda)^{-1}(x, y)$ must be of the form (x', y') with $y' \in \mathscr{L}_{\lambda_0}$ and $(x', y') \in \widehat{\mathscr{N}}_{\lambda_0}$. Thus, since $(x', 0) = (0, -y') + (x', y') \in \widehat{\mathscr{L}}_{\lambda_0} + \widehat{\mathscr{N}}_{\lambda_0}$, we see that $x' \in \mathscr{N}_{\lambda_0}$. Therefore $L - \lambda$ is injective and surjective on $\mathscr{N}_{\lambda_0}$.

If $T: \widehat{\mathscr{B}} \to \mathscr{B}$ is defined by $T(x, y) = x$, we see that $P_{\lambda_0} = T \circ \hat{P}_{\lambda_0}$. Since T and $\hat{P}_{\lambda_0}$ are bounded, P_{λ_0} is continuous and $\mathscr{N}_{\lambda_0} = \{u \mid P_{\lambda_0} u = 0\}$ is a closed subspace. We now know

(i) $(L - \lambda)\mathscr{N}_{\lambda_0} = \mathscr{N}_{\lambda_0}$

(ii) $(L - \lambda)$ is a closed map

(iii) $(L - \lambda)\mathscr{N}_{\lambda_0}$ is of second category

(iv) $(L - \lambda)^{-1}$ is well defined on $\mathscr{N}_{\lambda_0}$.

The bounded inverse theorem states that $\|(L-\lambda)^{-1}\| < \infty$. This shows that $\mathrm{sp}\,(L \mid \mathscr{N}_{\lambda_0}) \subset \mathrm{sp}\, L\backslash\{\lambda_0\}$.

COROLLARY 1.1. *Let $\lambda_0, \lambda_1, \cdots, \lambda_n$ be eigenvalues of a bounded linear operator L having finite multiplicity and isolated from $\mathrm{sp}\,\hat{L}\backslash\{\lambda_0, \lambda_1, \cdots, \lambda_n\}$. Then each of $\lambda_0, \lambda_1, \cdots, \lambda_n$ is a normal eigenvalue of L and $P = P_{\lambda_0} + P_{\lambda_1} + \cdots + P_{\lambda_n}$ is a bounded projection onto $\bigcup_{j=0}^{n} \mathscr{L}_{\lambda_j} = \mathscr{L}$. Moreover, if $\mathscr{N} = (I-P)\mathscr{B}$, $\mathrm{sp}\,(L \mid \mathscr{N}) \subset \mathrm{sp}\, L\backslash\{\lambda_0, \lambda_1, \cdots, \lambda_n\}$.*

Proof. The result follows using a proof similar to the preceding one, observing that $P_{\lambda_j} \circ P_{\lambda_k} = 0$ whenever $j \neq k$.

2. A global alternative theorem. In this section we will show that the local bifurcation exhibited in Theorem 1.1 is a global property with an alternative-type result.

For $\mathscr{V} \subset \mathscr{E}$, a subcontinuum of $\mathscr{V}$ is a subset of $\mathscr{V}$ which is closed and connected in $\mathscr{E}$. $\mathscr{S}$ will denote the closure of the set of nontrivial solutions of (1.1) in $\mathscr{E}$. Let $\mathscr{C}_{\lambda_0}$ denote the maximal subcontinuum of $\mathscr{S} \cup (\lambda_0, 0)$ containing $(\lambda_0, 0)$. B_ρ will denote the open ball in $\mathscr{B}$ centered at 0 and having radius ρ. L and H will be as in § 1.

LEMMA 2.1. *Let K be a compact metric space and A and B disjoint closed subsets of K. Then either there exists a subcontinuum of K meeting both A and B, or $K = K_A \cup K_B$ where K_A and K_B are disjoint compact subsets of K containing A and B respectively.*

Proof. See [5].

The following lemma is due in part to Rabinowitz [5].

LEMMA 2.2. *Suppose λ_0 is an isolated normal eigenvalue of L having finite multiplicity. Assume $\mathscr{C}_{\lambda_0}$ is bounded, $(\overline{\mathscr{C}_{\lambda_0}})_R \cap \mathrm{sp}_{nd}(L) = \emptyset$, and $\mathscr{C}_{\lambda_0} \cap \{\boldsymbol{R} \times \{0\}\} = (\lambda_0, 0)$. Then $\mathscr{C}_{\lambda_0}$ is compact and there exists a bounded open set $\mathscr{O} \subset \mathscr{E}$ such that $\mathscr{C}_{\lambda_0} \subset \mathscr{O}$, $\partial\mathscr{O} \cap \mathscr{S} = \emptyset$, $(\overline{\mathscr{O}_R}) \cap \mathrm{sp}_{nd}(L) = \emptyset$, the trivial solutions contained in $\mathscr{O}$ are the points $(\lambda, 0)$ where $|\lambda - \lambda_0| < \varepsilon$ for some $\varepsilon < \varepsilon_0 = \mathrm{dist}\,(\lambda_0, \mathrm{sp}\, L\backslash\{\lambda_0\})$, and $\|(\lambda, u) - (\mu, 0)\| \geqq 2\varepsilon_1$ for some positive ε_1 whenever $(\lambda, u) \in \partial\mathscr{O}$ and $\mu \in \mathrm{sp}\, L$.*

Proof. $\mathscr{C}_{\lambda_0}$ is a compact set. Indeed, let $\{(\lambda_n, u_n)\}$ be any sequence in $\mathscr{C}_{\lambda_0}$. By hypothesis the sequence $\{\lambda_n\}$ is bounded away from $\mathrm{sp}_{nd}(L)$. By passing to a subsequence $\mathscr{N}_1 \subset \mathscr{N} = \{1, 2, \cdots\}$

we can obtain $\lim_{n\to\infty, n\in\mathscr{N}_1} \lambda_n = \lambda$, and $\lim_{n\to\infty, n\in\mathscr{N}_1} H(\lambda_n, u_n) = w$ for some $\lambda \in \boldsymbol{R}$, $w \in \mathscr{H}$. Since $\mathscr{C}_{\lambda_0}$ is bounded, we then know that $\lim_{n\to\infty, n\in\mathscr{N}_1} (L - \lambda)u_n = w$. Since $\lambda \notin \mathrm{sp}_{nd}(L)$, λ is either in the resolvent of L or is a normal eigenvalue. In the first case $(L-\lambda)^{-1}$ is well defined, yielding $\lim_{n\to\infty, n\in\mathscr{N}_1} u_n = (L-\lambda)^{-1}w$. In the second case, let P be the projector onto the eigenspace corresponding to λ. Then $\lim_{n\to\infty, n\in\mathscr{N}_1} (I-P)u_n = (L-\lambda)^{-1}(I-P)w$. By passing to another subsequence $\mathscr{N}_2 \subset \mathscr{N}_1$ we can find a $v \in \mathscr{H}$ such that $\lim_{n\to\infty, n\in\mathscr{N}_2} u_n = v + (L-\lambda)^{-1}(I-P)w$. In either case, continuity shows that the limit point is in $\mathscr{C}_{\lambda_0}$.

Since $\mathscr{C}_{\lambda_0}$ is compact, we may find a δ-neighborhood U_δ of $\mathscr{C}_{\lambda_0}$ such that $\overline{(U_\delta)_{\boldsymbol{R}}} \cap \mathrm{sp}_{nd}(L) = \varnothing$ and $\overline{U_\delta}$ contains no trivial solutions other than points $(\lambda, 0)$ where $|\lambda - \lambda_0| < \varepsilon < \varepsilon_0$ for some $\varepsilon > 0$.

$K = \overline{U_\delta} \cap \mathscr{S}$ is a compact metric space (with the induced metric). The proof of this fact is similar to the proof of the compactness of $\mathscr{C}_{\lambda_0} \cdot \mathscr{C}_{\lambda_0}$ and $\partial\overline{U_\delta} \cap \mathscr{S}$ are disjoint closed subsets of K, and K does not contain a subcontinuum which meets both $\mathscr{C}_{\lambda_0}$ and $\partial\overline{U_\delta} \cap \mathscr{S}$. Thus, using Lemma 2.1, there exist disjoint compact sets K_A and K_B such that $K = K_A \cup K_B$, $\mathscr{C}_{\lambda_0} \subset K_A$, and $\partial\overline{U_\delta} \cap \mathscr{S} \subset K_B$. Select an $\varepsilon' > 0$ such that $\varepsilon' < \mathrm{dist}(K_A, K_B)$ and define $\mathscr{O}_1$ to be the ε'-neighborhood of K_A. Finally, let $\mathscr{O} = U_\delta \cap \mathscr{O}_1$. In case $\mathscr{O} \cap \{\boldsymbol{R} \times \{0\}\} \neq (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \times \{0\}$, we may add $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \times B_r$ to $\mathscr{O}$, for r sufficiently small.

$\Gamma = \mathrm{sp}\, L \cap \overline{\mathscr{O}_{\boldsymbol{R}}}$ has finitely many elements. Since

$$(\overline{\mathscr{O}_{\boldsymbol{R}}}) \cap \mathrm{sp}_{nd}(L) = \varnothing \quad \text{and} \quad \partial\mathscr{O} \cap \{\Gamma \times \{0\}\} = \varnothing ,$$

it is clear that $\mathrm{dist}(\partial\mathscr{O}, \{\mathrm{sp}\, L \times \{0\}\}) > 0$. Select a positive ε_1 such that $2\varepsilon_1 < \mathrm{dist}(\partial\mathscr{O}, \mathrm{sp}\, L \times \{0\})$.

LEMMA 2.3. *Suppose λ_0 and λ_1 are distinct normal eigenvalues of L. Then $\mathscr{B} = \mathscr{L}_{\lambda_0} \oplus \mathscr{L}_{\lambda_1} \oplus \mathscr{N}$, a direct sum of subspaces, where $\mathscr{N} = \mathscr{N}_{\lambda_0} \cap \mathscr{N}_{\lambda_1}$, and $P = P_{\lambda_0} + P_{\lambda_1}$ projects onto $\mathscr{L}_{\lambda_0} \oplus \mathscr{L}_{\lambda_1}$ along $\mathscr{N}$.*

Proof. Since λ_0 is normal, we may write $\mathscr{B} = \mathscr{L}_{\lambda_0} + \mathscr{N}_{\lambda_0}$ as described in Definition 1.3. For $x_1 \in \mathscr{L}_{\lambda_1}$, let us write

$$x_1 = x_0 + x_2 (\in \mathscr{L}_{\lambda_0} \oplus \mathscr{N}_{\lambda_0}) .$$

Then

$$\lambda_1 x_1 = \lambda_0 x_0 + L x_2$$

with $Lx_2 \in \mathscr{N}_{\lambda_0}$. However, $\lambda_1 x_1 = \lambda_1 x_0 + \lambda_1 x_2$. Thus $x_0 = 0$ and $\mathscr{L}_{\lambda_1} \subset \mathscr{N}_{\lambda_0}$.

Select a $y \in \mathscr{N}_{\lambda_0}^*$. It can be written uniquely as $y_1 + y_2$ with $y_1 \in \mathscr{L}_{\lambda_1}^*$ and $y_2 \in \mathscr{L}_{\lambda_1}^*$. Since $y_2 = y - y_1$, we see that $y_2 \in \mathscr{N}_{\lambda_0}^* \cap \mathscr{N}_{\lambda_1}^*$.

Since $\mathscr{L}_{\lambda_1}^* \subset \mathscr{N}_{\lambda_0}^*$, it is clear that $P_{\lambda_1} \circ P_{\lambda_0} = P_{\lambda_0} \circ P_{\lambda_1} = 0$. Moreover, $\mathscr{N}$ is the nullspace of P. Thus $P = P_{\lambda_0} + P_{\lambda_1}$ is indeed the projector onto $\mathscr{L}_{\lambda_0} \oplus \mathscr{L}_{\lambda_1}$ along $\mathscr{N}$.

The following theorem is modeled after an alternative theorem which Rabinowitz proved for (0.3) when L is compact.

Theorem 2.1. *Suppose λ_0 is an isolated normal eigenvalue of L of odd multiplicity. L is as before and H satisfies* H-1. *Then $(\lambda_0, 0)$ is a bifurcation point of (1.1) possessing a continuous branch $\mathscr{C}_{\lambda_0}$ such that one and only one of the following alternative occurs.*

(i) *$\mathscr{C}_{\lambda_0}$ is unbounded*

(ii) *$\mathscr{C}_{\lambda_0}$ is bounded and $\overline{(\mathscr{C}_{\lambda_0})_R} \cap \mathrm{sp}_{nd}(L) \neq \varnothing$*

(iii) *$\mathscr{C}_{\lambda_0}$ is compact, $(\mathscr{C}_{\lambda_0})_R \cap \mathrm{sp}_{nd}(L) = \varnothing$ and $\mathscr{C}_{\lambda_0}$ contains $(\lambda_1, 0)$ where λ_1 is a normal eigenvalue of L different from λ_0.*

Proof. Assume the theorem is false. Then we may find a set $\mathscr{O}$ and a positive constant ε as specified in Lemma 2.2. Let σ_0 denote a closed interval with $\overline{\mathscr{O}_R}$ in its interior and contained in $R\backslash \mathrm{sp}_{nd}(L)$. If $\sigma_0 \cap \mathrm{sp}_d(L) = \{\lambda_0, \lambda_1, \cdots, \lambda_n\}$, let $P = P_{\lambda_0} + P_{\lambda_1} + \cdots + P_{\lambda_n}$ (each λ_j, $0 \leqq j \leqq n$, is a normal eigenvalue of L). Then, using the same derivation as in Theorem 1.1, we may show that (1.1) is equivalent to

$$
\begin{aligned}
&u = C_1(\lambda, u) + C_2(\lambda, u)\\
(2.1)\qquad &C_1(\lambda, u) = \frac{(L - \mu_0)Pu}{\lambda - \mu_0}\\
&C_2(\lambda, u) = \Big((L - \lambda)^{-1}(I - P) - \frac{P}{\lambda - \mu_0}\Big)H(\lambda, u)
\end{aligned}
$$

for $\lambda \in \sigma_0$, $\mu_0 \notin \sigma_0$.

Define $\Phi(\lambda, u) = u - C_1(\lambda, u) - C_2(\lambda, u)$ as before. For $0 < |\lambda_0 - \lambda| \leqq \varepsilon$, $(\lambda, 0)$ is an isolated solution of (1.1) in $\{\lambda\} \times \mathscr{B}$. Thus, there exists $\rho(\lambda) > 0$ such that $(\lambda, 0)$ is the only solution of (1.1) in $\{\lambda\} \times \overline{B_{\rho(\lambda)}}$. Let $\rho_0(\lambda) = \mathrm{dist}\,((\lambda, 0), \mathscr{S})$ and choose $\rho(\lambda) = 1/2(\rho_0(\lambda))$. Define $\rho(\lambda) = \rho(\lambda_0 + \varepsilon)$ for $\lambda \geqq \lambda_0 + \varepsilon$ and $\rho(\lambda) = \rho(\lambda_0 - \varepsilon)$ for $\lambda \leqq \lambda_0 - \varepsilon$. We may select $\rho(\lambda_0 - \varepsilon)$ and $\rho(\lambda_0 + \varepsilon)$ sufficiently small so that $\overline{B_{\rho(\lambda)}} \cap (\partial\mathscr{O})^\lambda = \varnothing$ for $|\lambda - \lambda_0| \geqq \varepsilon$. Since (1.1) has no solutions on $\partial(\mathscr{O}^\lambda - \overline{B_{\rho(\lambda)}})$ for $\lambda \neq \lambda_0$, $\deg\,(\Phi(\lambda, \cdot), \mathscr{O}^\lambda - \overline{B_{\rho(\lambda)}}, 0)$ is well defined for such λ. We will prove that

$$(2.2)\qquad \deg\,(\Phi(\lambda, \cdot), \mathscr{O}^\lambda - \overline{B_{\rho(\lambda)}}, 0) = 0$$

for those λ.

Let $\lambda > \lambda_0$ and $\lambda_* > \lambda$ such that $\lambda_* \in \sigma_0 \backslash \overline{\mathcal{O}_R}$. Define

$$\rho = \inf \{\rho(\mu) \mid \mu \in [\lambda, \lambda_*]\} ,$$

which is positive due to the definition of $\rho(\lambda)$. Let $\mathcal{U} = \mathcal{O} - [\lambda, \lambda_*] \times \overline{B_\rho}$. $\mathcal{U}$ is a bounded open set in $[\lambda, \lambda_*] \times \mathcal{B}$ and $\Phi(\gamma, u) \neq 0$ for $(\gamma, u) \in \partial\mathcal{U}$ (the boundary of $\mathcal{U}$ in $[\lambda, \lambda_*] \times \mathcal{B}$). By the homotopy of degree, for $\gamma \in [\lambda, \lambda_*]$,

$$\deg (\Phi(\gamma, \cdot), \mathcal{O}^\gamma - \overline{B_\rho}, 0) = \text{constant} . \tag{2.3}$$

Since $\mathcal{O}^{\lambda *} = \emptyset$,

$$\deg (\Phi(\lambda_*, \cdot), \mathcal{O}^{\lambda *} - \overline{B_\rho}, 0) = 0 . \tag{2.4}$$

$\Phi(\lambda, \cdot)$ has no solution in $\{\lambda\} \times (\overline{B_{\rho(\lambda)}} - B_\rho)$. Thus

$$\deg (\Phi(\lambda, \cdot), \overline{B_{\rho(\lambda)}} - B_\rho, 0) = 0 . \tag{2.5}$$

Combining (2.3), (2.4), and (2.5) and using the additivity of degree we get

$$\deg (\Phi(\lambda, \cdot), \mathcal{O}^\lambda - \overline{B_{\rho(\lambda)}}, 0) = 0 . \tag{2.6}$$

Similarly, (2.6) holds for $\lambda < \lambda_0$.

Once again applying the homotopy of degree,

$$\deg (\phi(\lambda, \cdot), \mathcal{O}^\lambda, 0) = \text{constant} \tag{2.7}$$

for $|\lambda - \lambda_0| < \varepsilon$.

Select $\underline{\lambda}, \bar{\lambda}$ such that $\lambda_0 - \varepsilon < \underline{\lambda} < \lambda_0 < \bar{\lambda} < \lambda_0 + \varepsilon$. Using the additivity of degree, we see that

$$\begin{aligned} \deg (\Phi(\underline{\lambda}, \cdot), \mathcal{O}^{\underline{\lambda}}, 0) &= \operatorname{index} (\Phi(\underline{\lambda}, \cdot), (\underline{\lambda}, 0)) \\ &\quad + \deg (\Phi(\underline{\lambda}, \cdot), \mathcal{O}^{\underline{\lambda}} - \overline{B_{\rho(\underline{\lambda})}}, 0) \\ \deg (\Phi(\bar{\lambda}, \cdot), \mathcal{O}^{\bar{\lambda}}, 0) &= \operatorname{index} (\Phi(\bar{\lambda}, \cdot), (\bar{\lambda}, 0)) \\ &\quad + \deg (\Phi(\bar{\lambda}, \cdot), \mathcal{O}^{\bar{\lambda}} - \overline{B_{\rho(\bar{\lambda})}}, 0) . \end{aligned} \tag{2.8}$$

Applying (2.6) and (2.7) to (2.8) yields

$$\operatorname{index} (\Phi(\underline{\lambda}, \cdot), (\underline{\lambda}, 0)) = \operatorname{index} (\Phi(\bar{\lambda}, \cdot), (\bar{\lambda}, 0)). \tag{2.9}$$

These numbers are either $+1$ or -1 and since λ_0 has odd multiplicity, they differ by a factor of -1. This is incompatible with (2.9), proving that the hypotheses of Lemma 2.2 do not occur in this situation. Thus (i), (ii) or (iii) must occur.

LEMMA 2.4. *Suppose λ_0 is an isolated eigenvalue of L having finite multiplicity. Assume $\mathcal{C}_{\lambda_0}$ is bounded, $(\overline{\mathcal{C}_{\lambda_0}})_R \cap \mathrm{sp}_{nd}(L) = \emptyset$,* *and* $\mathscr{C}_{\lambda_0} \cap \{\boldsymbol{R} \times \{0\}\} = A \times \{0\}$ *where* $A = \{\lambda_1, \cdots, \lambda_n\}$ *and* $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. *Then* $\mathscr{C}_{\lambda_0}$ *is compact and there exists a bounded open set* $\mathscr{O} \subset \mathscr{E}$ *such that* $\mathscr{C}_{\lambda_0} \subset \mathscr{O}$, $\partial\mathscr{O} \cap \mathscr{S} = \varnothing$, $\overline{(\mathscr{O}_R)} \cap \mathrm{sp}_{nd}(L) = \varnothing$, *the trivial solutions contained in* $\mathscr{O}$ *are points* $(\lambda, 0)$ *where* $\lambda_1 - \varepsilon < \lambda < \lambda_n + \varepsilon$ *for some* $\varepsilon < \varepsilon_0 = \mathrm{dist}\,(A, \mathrm{sp}\, L\backslash A)$, *and* $\|(\lambda, u) - (\mu, 0)\| \geqq 2\varepsilon_1$ *for some positive* ε_1 *whenever* $(\lambda, u) \in \partial\mathscr{O}$ *and* $\mu \in \mathrm{sp}\, L$.

Proof. The method employed in the proof of Lemma 2.2 applies here.

THEOREM 2.2. *Suppose* λ_0 *is an isolated normal eigenvalue of* L *of odd multiplicity.* L *is as before and* H *satisfies* H-1. *Then* $(\lambda_0, 0)$ *is a bifurcation point of* (1.1) *possessing a continuous branch* $\mathscr{C}_{\lambda_0}$ *such that one and only one of the following alternatives occur.*

(i) $\mathscr{C}_{\lambda_0}$ *is unbounded*

(ii) $\mathscr{C}_{\lambda_0}$ *is bounded and* $\overline{(\mathscr{C}_{\lambda_0})_R} \cap \mathrm{sp}_{nd}(L) \neq \varnothing$

(iii)′ $\mathscr{C}_{\lambda_0}$ *is compact,* $(\mathscr{C}_{\lambda_0})_R \cap \mathrm{sp}_{nd}(L) = \varnothing$, *and* $\mathscr{C}_{\lambda_0} \cap \{\boldsymbol{R} \times \{0\}\} = \{\lambda_0, \lambda_1, \cdots, \lambda_n\} \times \{0\}$ *where* $\lambda_1, \cdots, \lambda_n$ *are normal eigenvalues of* L *distinct from* λ_0, *and the sum of the multiplicities of* $\lambda_0, \lambda_1, \cdots, \lambda_n$ *is even.*

Proof. Suppose (i), (i), and (iii)′ do not occur. Then $\mathscr{C}_{\lambda_0}$ is compact, $(\mathscr{C}_{\lambda_0})_R \cap \mathrm{sp}_{nd}(L) = \varnothing$, $\mathscr{C}_{\lambda_0} \cap \{\boldsymbol{R} \times \{0\}\} = \{\lambda_0, \lambda_1, \cdots, \lambda_n\}$, and the sum of the multiplicities of these eigenvalues is odd. We may suppose $\lambda_0 < \lambda_1 < \cdots < \lambda_n$.

Construct an open set $\mathscr{O}$ and select $\varepsilon > 0$ as specified in Lemma 2.4. Also, define σ_0, P, and $\Phi(\lambda, u)$ as in Theorem 2.1. Then $\deg\,(\Phi(\lambda, \cdot), \mathscr{O}_\lambda, 0)$ is well defined for $\lambda_0 - \varepsilon < \lambda < \lambda_n + \varepsilon$, and moreover,

$$\deg\,(\Phi(\lambda, \cdot), \mathscr{O}^\lambda, 0) = \text{constant} \tag{2.10}$$

for $\lambda_0 - \varepsilon < \lambda < \lambda_n + \varepsilon$. Select $\underline{\lambda}$ and $\bar{\lambda}$ such that $\lambda_0 - \varepsilon < \underline{\lambda} < \lambda_0$ and $\lambda_n < \bar{\lambda} < \lambda_n + \varepsilon$. Then, using degree arguments from Theorem 2.1, we see that

$$\begin{aligned} &\mathrm{index}\,(I - C_1(\underline{\lambda}, \cdot), (\underline{\lambda}, 0)) \\ &\qquad = \mathrm{index}\,(I - C_1\cdot(\bar{\lambda}, \cdot), (\bar{\lambda}, 0)) . \end{aligned} \tag{2.11}$$

These numbers are either $+1$ or -1. However, the assumption that the sum of the multiplicities is odd implies that

$$\begin{aligned} &\mathrm{index}\,(I - C_1(\underline{\lambda}, \cdot), (\underline{\lambda}, 0)) \\ &\qquad = -\mathrm{index}\,(I - C_1(\bar{\lambda}, \cdot), (\bar{\lambda}, 0)) . \end{aligned} \tag{2.12}$$

This contradiction proves that one of the alternatives (i), (ii), (iii)′ occurs.

REMARK. If λ_0 is an isolated normal eigenvalue of L having even multiplicity and if $(\lambda_0, 0)$ is a bifurcation point, then one of the alternatives in Theorem 2.2 must occur. Note that (iii)′ occurs even if $\mathscr{C}_{\lambda_0}$ loops back to $(\lambda_0, 0)$.

Examples of the three alternatives. Examples of (i) are common. In particular, this situation occurs whenever (1.1) is linear (i.e., $H \equiv 0$). Examples of (ii) and (iii)′ are more difficult to construct.

Let $\mathscr{B}_1 = \boldsymbol{R}^2$ with general element $u = (u_1, u_2)$, the normal inner product $(\cdot, \cdot)_1$, and norm $\|\cdot\|_1$. Define $L\colon \mathscr{B}_1 \to \mathscr{B}_1$ and $B(u)\colon \boldsymbol{R} \times \mathscr{B}_1 \to \mathscr{B}_1$ by means of the matrices

$$L = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$B(u) = \begin{pmatrix} \|u\|_1^2 & \|u\|_1^2/4 \\ -\|u\|_1^2/4 & \|u\|_1^2 \end{pmatrix} \tag{2.13}$$

and consider

$$Lu = \lambda[u - B(u)u] \,. \tag{2.14}$$

First let us show that $\|u\|_1 \leqq 1$ whenever (λ, u) is a solution of (2.14) in $\mathscr{C}_{1/2}$ or $\mathscr{C}_1$. Direct computation yields

$$(B(u)u, u)_1 = \|u\|_1^4 \,.$$

Since the only solution of (2.14) of the form $(0, u)$ is $(0, 0)$, it is clear that $\mathscr{C}_{1/2}$ and $\mathscr{C}_1$ consist of solutions (λ, u) with $\lambda \geqq 0$. Assuming $\lambda \geqq 0$, take the inner product of both sides of (2.14) with u yielding

$$\|u\|_1^2 - (B(u)u, u)_1 \geqq 0 \,.$$

In other words,

$$\|u\|_1^4 \leqq \|u\|_1^2 \,. \tag{2.15}$$

Now assume $\{(\lambda_n, u_n)\}_{n=1,2,\dots}$ are nontrivial solution of (2.14) with $\lambda_n \geqq n$. Dividing (2.14) by λ and inserting these solutions yields

$$\frac{Lu_n}{\lambda_n} = u_n - B(u_n)u_n \,. \tag{2.16}$$

Since $\|u_n\|_1 \leqq 1$ for all n, a subsequence of $\{u_n\}_{n=1,2,\dots}$ must converge to some w, a solution of

$$B(v)v = v \,.$$

The only such w is $(0, 0)$ since

$$B(u)u = \| u \|_1^2(u_1, u_2) + \frac{\| u \|_1^2}{4}(+u_2, -u_1) .$$

The second term is nonzero whenever $u \neq (0, 0)$ and is always orthogonal to u. Assume $\lim_{n\to\infty} \| u_n \|_1 = 0$ and divide (2.16) by $\| u_n \|_1$ yielding

$$\frac{1}{\lambda_n} L\left(\frac{u_n}{\| u_n \|_1}\right) = \frac{u_n}{\| u_n \|_1} - B(u_n)\left(\frac{u_n}{\| u_n \|_1}\right) . \tag{2.17}$$

We may find N such that $n > N$ implies

$$\left\| \frac{1}{\lambda_n} L \frac{u_n}{\| u_n \|_1} \right\|_1 < \frac{1}{4}$$

and

$$\left\| B(u_n)\left(\frac{u_n}{\| u_n \|_1}\right) \right\|_1 < \frac{1}{4} .$$

This contradiction along with the result that $\| u \| \leqq 1$ implies that $\mathscr{C}_{1/2}$ and $\mathscr{C}_1$ are bounded in $\boldsymbol{R} \times \mathscr{B}_1$. Thus, (2.14) is a finite-dimensional example of (iii)′.

Let $\mathscr{B}_2$ be a real Hilbert space with an orthonormal basis $\{\varphi_k\}_{k=1,2,\ldots}$, inner product $(\cdot, \cdot)_2$, and norm $\| \cdot \|_2$. Define $\mathscr{B} = \mathscr{B}_1 \oplus \mathscr{B}_2$ with general element (x, y). If $(x_j, y_j) \in \mathscr{B}$ for $j = 1, 2$, define $((x_1, y_1), (x_2, y_2)) = (x_1, x_2)_1 + (y_1, y_2)_2$ and let $\| \cdot \|$ be the corresponding norm. Using this framework, the preceding example can be modified to exhibit (ii) and (iii)′ in the infinite-dimensional case.

Let $M = \sup \{\lambda \mid (\lambda, u) \in \mathscr{C}_1$ (of the 2-dimensional problem)$\}$. Define a linear operator $L_A: \mathscr{B} \to \mathscr{B}$ by $L_A(x, y) = (Lx, (M + 1)y)$ and $H_A: \boldsymbol{R} \times \mathscr{B} \to \mathscr{B}$ by $H_A(\lambda, (x, y)) = (+\lambda B(x)x, 0)$. Then $\mathscr{C}_1$ for

$$L_A u = \lambda u - H_A(\lambda, u) \tag{2.18}$$

is an example of (iii)′.

If instead of L_A we defined a linear operator $L_B: \mathscr{B} \to \mathscr{B}$ by $L_B(x, 0) = (Lx, 0)$ and $L_B(0, \varphi_k) = (0, M + 1/k)\varphi_k)$, then $\mathscr{C}_1$ for

$$L_B u = \lambda u - H_A(\lambda, u) \tag{2.19}$$

is an example of (ii).

3. Another global result.

In this section we give another global result for (1.1). This result was initially proven by Turner [8] in the case where L is compact. While being related to the work in § 2, this result gives additional information concerning $\mathscr{S}$. The restrictions on $H(\lambda, u)$ can now be relaxed. In addition to those H's satisfying H-1, we can now admit $H(\lambda, u)$ satisfying hypothesis H-2:

(H-2) $H(\lambda, u) = J(\lambda, u)u$ where for each (λ, u) in $\boldsymbol{R} \times \mathscr{B}$, $J(\lambda, u)$ is a compact linear map and $J(\lambda, u)u$ is the result of applying $J(\lambda, u)$ to u.

L and $\mathscr{B}$ are defined as before.

For $\mu \in \boldsymbol{R}$, define

$$n(\mu) = \limsup_{\substack{r \to 0 \, \|u\| \leq r \\ |\lambda - \mu| \leq r}} \frac{\|(L - \lambda)^{-1} H(\lambda, u)\|}{\|u\|} \tag{3.1}$$

where $n(\mu) = \infty$ if $(L - \mu)^{-1}$ does not exist. We let

$$\tilde{\rho} = \{\mu \mid n(\mu) < 1\} . \tag{3.2}$$

$\tilde{\rho}$ is clearly a subset of $\rho(L)$, and whenever H satisfies H-1 they are the same set since $n(\mu) = 0$ for $\mu \in \rho(L)$.

THEOREM 3.1. *Let H satisfy* H-1 *or* H-2 *and let $[a, b]$ be an interval in $\boldsymbol{R}/\mathrm{sp}_{nd}(L)$ containing an odd number of eigenvalues of L counting multiplicities with $n(a) < 1$ and $n(b) < 1$. Given a simple curve joining $(a, 0)$ to $(b, 0)$ in $\boldsymbol{R} \times \boldsymbol{R}_+$ missing $(\boldsymbol{R} - \{a\} - \{b\}, 0)$ and $(\mathrm{sp}_{nd}(L) \times \boldsymbol{R}_+)$, there are at least two nontrivial solutions $(\lambda^{(1)}, u^{(1)})$ and $(\lambda^{(2)}, u^{(2)})$ of* (1.1) *such that $(\lambda^{(i)}, \|u^{(i)}\|)$ lie on the curve.*

Proof. We begin by showing that there is a neighborhood of $(a, 0)$ in $\mathscr{E}$ such that none of the problems

$$Lu = \lambda u + tH(\lambda, u) \quad (0 \leqq t \leqq 1) \tag{3.3}$$

has a nontrivial solution (λ, u) in that neighborhood. If there were a sequence $0 \leqq t_n \leqq 1$ and nontrivial solutions (λ_n, u_n) of (3.3) such that $\lambda_n \to a$ and $\|u_n\| \to 0$ as $n \to \infty$, then it would follow that

$$\frac{u_n}{\|u_n\|} = \frac{(L - \lambda_n)^{-1} t_n H(\lambda_n, u_n)}{\|u_n\|} \tag{3.4}$$

for all n, implying that $n(a) \geqq 1$. The same result holds for $(b, 0)$.

Let C be any simple curve in $\boldsymbol{R} \times \boldsymbol{R}_+$ which connects $(a, 0)$ to $(b, 0)$ and misses $(\boldsymbol{R} - \{a\} - \{b\}, 0)$ and $\mathrm{sp}_{nd}(L) \times \boldsymbol{R}_+$. Because there are neighborhoods of $(a, 0)$ and $(b, 0)$ in $\mathscr{E}$ which do not contain nontrivial solutions (λ, u) of (1.1), showing there are a pair of solutions $(\lambda, \|u\|)$ on a simple curve $\mathscr{C}_1$ joining (a, α) to (b, α) for a suitably small $\alpha > 0$ ($\mathscr{C}_1 \subset \{\boldsymbol{R} \backslash \mathrm{sp}_{nd}(L) \times \boldsymbol{R}_+\}$) is equivalent to proving the theorem. Choose such a $\mathscr{C}_1$ and let it be given by $\mathscr{C}_1 = \{(\lambda(\eta),$

$r(\eta)) \mid 1 \leqq \eta \leqq 2\}$. Let $\mathscr{C}_t$ $(0 \leqq t \leqq 1)$ be the curve $\{(\lambda_t(\eta), r_t(\eta)) \mid 1 \leqq \eta \leqq 2\}$ where

$$\begin{aligned} \lambda_t(\eta) &= t\lambda(\eta) + (1-t)(a + (\eta - 1)(b-a)) \\ r_t(\eta) &= tr(\eta) + (1-t)\alpha \, . \end{aligned} \tag{3.5}$$

$\{\mathscr{C}_t\}_{0 \leqq t \leqq 1}$ is a continuous family of curves which deforms $\mathscr{C}_1$ into $\mathscr{C}_0$, the horizontal segment joining (a, α) to (b, α).

Let $\operatorname{sp} L \cap (\mathscr{C}_1)_R = \{\lambda_1, \cdots, \lambda_n\}$, a subset of $\operatorname{sp}_d(L)$, and define $P = P_{\lambda_1} + \cdots + P_{\lambda_n}$. Rewrite (1.1) as

$$\begin{aligned} u &= C_1(\lambda, u) + C_2(\lambda, u) \\ C_1(\lambda, u) &= \frac{(L - \mu_0)Pu}{\lambda - \mu_0} \\ C_2(\lambda, u) &= \left((L - \lambda)^{-1}(I - P) - \frac{P}{\lambda - \mu_0}\right)H(\lambda, u) \end{aligned} \tag{3.6}$$

for λ in a neighborhood of $(\mathscr{C}_1)_R$ and $\mu_0 \in \boldsymbol{R}\backslash(\mathscr{C}_1)_R$ chosen such that $\operatorname{sp}((L - \mu_0)P) \subset \boldsymbol{R}_+$. Note that if H satisfies H-1 or H-2, then C_2 does also.

We let

$$\Omega = \{u \in \mathscr{B} \mid 1 < \| u \| < 2\} \tag{3.7}$$

and for each $t \in [0, 1]$ define

$$\begin{aligned} \Phi_t(u) = u &- C_1(\lambda_t(\| u \|), u) \\ &- \frac{t \| u \| C_2}{r_t(\| u \|)}\left(\lambda_t(\| u \|), \frac{r_t(\| u \|)u}{\| u \|}\right) \end{aligned} \tag{3.8}$$

taking Ω into $\mathscr{B}$. Φ_t is well defined, for $(\mathscr{C}_t)_R \subset (\mathscr{C}_1)_R$ for $0 \leqq t \leqq 1$.

If $\Phi_1(u) = 0$ for some $u \in \Omega$, then multiplying through (3.8) by $(r_1(\| u \|))/\| u \|$ shows that $\lambda_0 = \lambda_1(\| u \|)$ and $u_0 = (r_1(\| u \|))u/\| u \|$ is a solution of (1.1) with $(\lambda_0, \| u_0 \|)$ on $\mathscr{C}_1$. We will show that $\Phi_1(u) = 0$ has at least two solutions in Ω by showing

$$|\deg(\Phi_1, \Omega)| \equiv |\deg(\Phi_1, \Omega, 0)| = 2 \, .$$

To do this we will prove $\deg(\Phi_0, \Omega) = \deg(\Phi_1, \Omega)$ and then solve the simpler problem involving Φ_0.

It must be shown that $\deg(\Phi_t, \Omega)$ is well defined for each $t \in [0, 1]$. Let us assume $\Phi_t(u) = 0$ with $\| u \| = 1$. Using (3.5), $\lambda_t(\| u \|) = a$ and $r_t(\| u \|) = \alpha$. Looking at $[r_t(\| u \|)/\| u \|](\Phi_t(u)) = 0$ we see a member of the family of equations in (3.3) has a solution (λ, v) with $\| v \| = \alpha$ and $\lambda = a$. This is impossible, showing $\Phi_t(u) = 0$ implies $\| u \| \neq 1$. Similarly, $\Phi_t(u) = 0$ implies $\| u \| \neq 2$. Thus, $\deg(\Phi_t, \Omega)$ is well defined and the homotopy invariance of degree shows that

$$\deg(\Phi_0, \Omega) = \deg(\Phi_1, \Omega) . \tag{3.9}$$

It remains to show that

$$|\deg(\Phi_0, \Omega)| = 2 \tag{3.10}$$

where

$$\Phi_0(u) = u - \frac{(L - \mu_0)Pu}{(a + (\|u\| - 1)(b - a)) - \mu_0} . \tag{3.11}$$

Thus, the zeros of Φ_0 are found from the solutions of the linear eigenvalue problem $(L - \mu_0)Pu = \lambda u$. For the remainder of the proof assume the eigenvalues of $(L - \mu_0)P$ are simple. If not, and $\{\mu_1, \cdots, \mu_n\}$ are the repeated nonzero eigenvalues of $(L - \mu_0)P$ and $E_1, \cdots, E_n$ are the corresponding rank-one eigenprojectors, then for $\varepsilon > 0$ sufficiently small $(L - \mu_0)P + \varepsilon \sum_{j=1}^{n} jE_j$ has simple eigenvalues and yields a corresponding Φ_0^ε with $\deg(\Phi_0, \Omega) = \deg(\Phi_0^\varepsilon, \Omega)$.

The solutions of $\Phi_0(u) = 0$ must satisfy $E_k u = u$ for some k. If the eigenvalues of L in $[a, b]$ are $\lambda_1, \cdots, \lambda_n$ $(\lambda_j = \mu_j + \mu_0)$, then

$$1 < \|u\| = 1 + \frac{\lambda_k - a}{b - a} < 2 . \tag{3.12}$$

There are two such u's in Ω. Let us select one and call it u_k.

Since $(L - \mu_0)P$ has finite-dimensional range $B = P\mathscr{B}$,

$$\deg(\Phi_0, \Omega) = \deg(\Phi_0, \Omega \cap B) .$$

Thus,

$$\deg(\Phi_0, \Omega_k) = \sum_{k=1}^{n} [\operatorname{index}(\Phi_0, u_k) + \operatorname{index}(\Phi_0, -u_k)] \tag{3.13}$$

where the indices are calculated in B. Let us calculate index (Φ_0, u_k). We may assume B has a norm coming from an inner product $(\cdot, \cdot)$ which at u_k agrees with the original norm. Moreover, we may assume $(u_j, u_k) = 0$ when $j \neq k$. Using the same notation $\|\cdot\|$ for the new norm, we may differentiate (3.11) and get a map taking w in B to

$$w - \frac{(L - \mu_0)Pw}{\lambda_k - \mu_0} + \frac{(u_k, w)(b - a)(L - \mu_0)Pu_k}{\|u_k\| [a + (\|u_k\| - 1)(b - a) - \mu_0]^2} \tag{3.14}$$

which simplifies to

$$w - \frac{(L - \mu_0)w}{\lambda_k - \mu_0} + \frac{(u_k, w)(b - a)u_k}{\|u_k\| (\lambda_k - \mu_0)} . \tag{3.15}$$

Assume $\lambda_1 < \lambda_2 < \cdots$. The map in (3.15) has no zeros near u_k and has n eigenvalues, those in $(-\infty, 0]$ being

$$1 - \frac{\lambda_j - \mu_0}{\lambda_k - \mu_0}, \quad k < j \leqq n .$$

Thus, the Leray-Schauder index theorem shows $\operatorname{index}(\Phi_0, u_k) = (-1)^{n-k}$. The map in (3.14) results from $-u_k$ also, yielding

$$\operatorname{index}(\Phi_0, u_k) + \operatorname{index}(\Phi_0, -u_k) = 2(-1)^{n-k} . \tag{3.16}$$

The sign in (3.16) changes as we go from u_k and $-u_k$ to u_{k+1} and $-u_{k+1}$ showing $|\deg(\Phi_0, \Omega)| = 2$.

COROLLARY 3.1. *Under the hypotheses of Theorem* 3.1, *there is a continuum of pairs* $(\lambda, \|u\|)$, *where* (λ, u) *is a solution of* (1.1), *joining* $([a, b], 0)$ *to*

(i) *infinity in* $\boldsymbol{R} \times \boldsymbol{R}^+$ *or*
(ii) $\operatorname{sp}_{nd}(L) \times \boldsymbol{R}^+$ *or*
(iii) $(\operatorname{Sp}(L)/[a, b], 0)$.

4. **Applications.** In this section we will demonstrate the application of Theorem 2.2 to a class of differential equations. We will consider equations of the form

$$Du(x) = \lambda u(x) + H(\lambda, u)(x),\ x \in \Omega \subseteqq \boldsymbol{R}^n,\ \lambda \in \boldsymbol{R}^1 \tag{4.1}$$

where D is a real differential operator. In the case that Ω is bounded, D usually defines an operator L in a real Banach space which has an inverse A. In this case, the equation

$$u(x) = \lambda Au(x) + AH(\lambda, u)(x)$$

can be studied. In the situation where Ω is bounded, A is frequently compact and the equation can be studied using existing theory. Equations of this type are treated in [5] and [8].

In the case that Ω is unbounded, this approach fails since A is usually not compact. I wish to treat such a class of equations:

$$-(p(x)u'(x))' + q(x)u(x) = \lambda u(x) + H(\lambda, u)(x) \quad x \in (0, \infty) , \qquad u(0) = 0 \tag{4.2}$$

where prime denotes differentiation with respect to x. This equation was studied by Stuart [7] when H was a k-set contraction. In the case where H is compact, further information can be gained about the solutions, and all normal eigenvalues can be treated in contrast to only a special subset of them. Conditions on H, p, and q will be given below.

Our first step is to select a space of functions on which to define our operators. Let L^2 denote the Banach space of all real measurable functions u on $[0, \infty)$ such that

$$\| u \|_2 = \left(\int_0^\infty u^2(x)dx\right)^{1/2} .$$

Let $\mathscr{C}_0^\infty$ denote the space of all infinitely differentiable functions with compact support in $(0, \infty)$. Let H_0^1 denote the closure of $\mathscr{C}_0^\infty$ in the Sobolev space $W_2^1(0, \infty)$ with norm

$$||| u ||| = (\| u \|^2 + \| u' \|^2)^{1/2} .$$

We make the following assumptions about p and q:

(H-3) $p\colon [0, \infty) \rightarrow \boldsymbol{R}$ is continuous and continuously differentiable in $(0, \infty)$ with p' bounded and $0 < P_1 \leqq p(x) \leqq P_2 < \infty$ for all $x \in [0, \infty)$.

(H-4) $q\colon [0, \infty) \rightarrow \boldsymbol{R}$ is continuous with $0 < Q \leqq q(x) \leqq Q_2 < \infty$ for all $x \in [0, \infty)$.

Let $\tilde{L}$ denote the operator defined by

$$D(\tilde{L}) = \mathscr{C}_0^\infty$$

$$\tilde{L}u(x) = (-p(x)u'(x))' + q(x)u(x) \quad (x \in (0, \infty),\ u \in D(\tilde{L}))$$

where $D(\tilde{L})$ denotes the domain of $\tilde{L}$.

LEMMA 4.1. *Under hypotheses* (H-3) *and* (H-4), $\tilde{L}$ *has a unique self adjoint extension* L *in* L^2 *with*

$$D(L) = H_0^1 \cap W_2^2(0, \infty)$$

and $\mathrm{sp}_{nd}(L) \subseteqq [Q, \infty)$ *where* $Q = \lim_{x\to\infty} \inf q(x)$.

Proof. [1].

LEMMA 4.2. *Suppose* (H-3) *and* (H-4) *are satisfied.*

(a) *If* λ_0 *is a normal eigenvalue of* L, *then the multiplicity of* λ_0 *is one.*

(b) L^{-1} *exists and is a bounded operator from all of* L^2 *into itself.*

(c) L *is a positive self-adjoint operator in* L^2. *Moreover,* $L^{1/2}$ *is a linear homeomorphism of* H_0^1 *onto* L^2 *where* $L^{1/2}$ *denotes the positive square root of* L.

Proof. (a) This follows Theorems 6.10 and 6.14 of Chapter 13 of [1].

(b) For $\phi \in \mathscr{C}_0^\infty$,

$$(L\phi, \phi) = \int_0^\infty \{p(x)[\phi'(x)]^2 + q(x)[\phi(x)]^2\}dx .$$

Thus, $\phi = 0$ implies $\phi = 0$ almost everywhere since p and q are bounded from zero, and L^{-1} exists. Clearly $(L\phi, \phi) \geqq P_1 \|\phi\|^2$, so L^{-1} is bounded.

(c) In (b) it was shown that L is positive. Thus, L has a unique positive self-adjoint square root, $L^{1/2}$. Since $L^{1/2}$ is closed, $D(L^{1/2}) = H_0^1$; since $L^{1/2}(L^{1/2}(H_0^1)) = L^2$, range $(L^{1/2}) = L^2$.

REMARK. (H-3) and (H-4) can be relaxed as long as the results of Lemmas 4.1 and 4.2 hold.

A point $(\lambda, u) \in H_0^1 \times \boldsymbol{R}$ is called a weak solution of (4.2) if

$$\int_0^\infty \{p(x)u'(x)\phi'(x) + q(x)u(x)\phi(x)\}dx$$
$$= \lambda \int_0^\infty \{u(x) + H(\lambda, u)(x)\}\phi(x)dx$$

for all $\phi \in \mathscr{C}_0^\infty$.

LEMMA 4.3. *Let H satisfy* (H-1) *and let* (λ, u) *be a weak solution of* (4.2). *Then* $u \in W_2^2 \cap H_0^1 = D(L)$ *and*

$$Lu = \lambda u + H(\lambda, u) .$$

Hence, u satisfies 4.2.

Proof. [7].

A point $(\lambda, 0)$ is a trivial solution of (4.2). Let

$$S = S \cup \{(\lambda, 0) \mid \lambda \text{ is a normal eigenvalue of } L\}$$

where S denotes the set of all nontrivial solutions of (4.2).

THEOREM 4.1. *Let* H-1, H-3, *and* H-4 *be satisfied, and let* λ_0 *denote a normal eigenvalue of* L (all operators are defined in H_0^1). *Then* $\mathscr{C}_{\lambda_0} \subseteqq \boldsymbol{R} \times (H_0^1 \cap W_2^2)$ *and* $\mathscr{C}_{\lambda_0}$ *satisfies only one of*

(i) $\mathscr{C}_{\lambda_0}$ *is unbounded*

(ii) $\mathscr{C}_{\lambda_0}$ *is bounded and* $\overline{(\mathscr{C}_{\lambda_0})_{\boldsymbol{R}}} \cap \mathrm{sp}_{nd}(L) \neq \emptyset$

(iii) $\mathscr{C}_{\lambda_0}$ *is compact,* $(\mathscr{C}_{\lambda_0})_{\boldsymbol{R}} \cap \mathrm{sp}_{nd}(L) = \emptyset$, *and* $\mathscr{C}_{\lambda_0} \cap \{\boldsymbol{R} \times \{0\}\} = \{\lambda_0, \lambda_1, \cdots, \lambda_n\} \times \{0\}$ *where* $\lambda_1, \cdots, \lambda_n$ *are normal eigenvalues of* L *distinct from* λ_0, *and the sum of the multiplicities of* $\lambda_0, \lambda_1, \cdots, \lambda_n$ *is even.*

Proof. The alternatives follow from Theorem 2.2. The result of the nature of the elements of $\mathscr{C}_{\lambda_0}$ follows from the fact that

$\mathscr{C}_{\lambda_0} \subseteqq \boldsymbol{R} \times D(L)$ (see § 1).

Much more knowledge of the nature of $\mathscr{C}_{\lambda_0}$ can be gained if the choice λ_0 and H are more restrictive. In particular, let us consider those eigenvalues of L which lie below the essential $\mathrm{sp}_{nd}(L)$, namely those characterized by

$$\lambda_n = \sup_{V \in \mathscr{F}_n} \inf \{\| L^{1/2}u \|^2 : u \in H_0^1, \| u \| = 1, u \in V^\perp\}$$

where $\mathscr{F}_n$ is the class of all $(n-1)$-dimensional subspaces of L^2. Clearly $\lambda_n < \lambda_{n+1}$ as long as $\lambda_n \notin \mathrm{sp}_{nd}(L)$. The eigenfunction associated with λ_n in the corresponding linear problem,

$$(-p(x)u'(x))' + q(x)u(x) = \lambda u(x) \quad (x \in (0, \infty),\ u(0) = 0)\ ,$$

possesses exactly $(i-1)$ simple zeroes in $(0, \infty)$. (see [1], pages 1480 and 1547). Since these eigenvalues are simple, it follows from Theorem 2.4 of [1] that near $(\lambda_i, 0)$, $\mathscr{C}_{\lambda_i}$ is a simple curve. Thus $\mathscr{C}_{\lambda_i}/(\lambda_i, 0)$ consists of at most two components $\mathscr{C}_{\lambda_i}^+$ and $\mathscr{C}_{\lambda_i}^-$. (This applies to all normal eigenvalues λ_0 of L.)

These components can be studied in greater detail if the nonlinear term H satisfies more stringent conditions. For instance:

(H-5) $H(\lambda, u)(x) = u(x)[G(\lambda, u)(x)]$ for all $x \geqq 0$ where

$$G(\lambda, u): [0, \infty) \longrightarrow [0, \infty),\ | G(\lambda, u)(x) | \leqq M | u(x) |$$

for $x \geqq 0$, and $| G(\lambda, u)(x) | \leqq N$.

THEOREM 4.2. *Suppose all the conditions* H-1, H-3, H-4, *and* H-5 *are satisfied. Then for* $\lambda_i \notin \mathrm{sp}_{nd}(L)$, $\mathscr{C}_{\lambda_i}$ *has the following properties*:

(1) *There is a neighborhood* $\mathscr{O}$ *of* $(\lambda_i, 0)$ *in* $\boldsymbol{R} \times H!$ *such that* $\mathscr{C}_{\lambda_i} \cap \mathscr{O}$ *is a simple curve and if* $(\lambda, u) \in \mathscr{S} \cap \mathscr{O}$, u *has exactly* $(i-1)$ *simple zeroes in* $(0, \infty)$.

(2) $\mathscr{C}_{\lambda_i}$ *consists of at most two components*, $\mathscr{C}_{\lambda_i}^+$ *and* $\mathscr{C}_{\lambda_i}^-$.

(3) *If* $(\lambda, u) \in \mathscr{C}_{\lambda_i}$, *then* u *has exactly* $(i-1)$ *simple zeroes in* $(0, \infty)$.

(4) *If* $(\lambda, u) \in \mathscr{C}_{\lambda_i}$, *then* $0 < \lambda \leqq \lambda_i$.

(5) $\{\| u \| \mid (\lambda, u) \in \mathscr{C}_{\lambda_i}\}$ *is unbounded.*

Proof. The first part of (1) and (2), are proven in [5] and (5) follows from (4) through the application of Theorem 2.2. Thus, only (3), (4), and the last part of (1) remain to be proven.

In a similar setting these have been shown by Stuart [7], and his techniques apply in the present situation.

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