PACIFIC JOURNAL OF MATHEMATICS
Vol, 69, No. 1, 1977

INVARIANT SUBSPACES OF WEAK-*DIRICHLET ALGEBRAS

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Let A be a weak-*Dirichlet algebra of $L^\infty(m)$. For $0 < p \leqq \infty$, a closed subspace M of $L^p(m)$ is called invariant if $f \in M$ and $g \in A$ imply that $fg \in M$. Let B^∞ be a weak-*closed subalgebra of $L^\infty(m)$ which contains A such that $B^\infty M \subseteqq M$ for an invariant subspace M. The main result of this paper is a characterization of the left continuous invariant subspaces for B^∞, which is a natural generalization of simply invariant subspaces. Applying this result with $B^\infty = H^\infty(m)$ (or $B^\infty = L^\infty(m)$), the simply (or doubly) invariant subspace theorem follows. Moreover this result characterizes also the invariant subspaces which are neither simply nor doubly invariant. Merrill and Lal characterized some special invariant subspaces of this kind.

1. Introduction. Recall that by definition a weak-*Dirichlet algebra, which was introduced by Srinivasan and Wang [6], is an algebra A of essentially bounded measurable functions on a probability measure space $(X, \mathscr{A}, m)$ such that (i) the constant functions lie in A; (ii) $A + \bar{A}$ is weak-*dense in $L^\infty(m)$ (the bar denotes conjugation, here and always); (iii) for all f and g in A,

$$\int_X fg\,dm = \left(\int_X f\,dm\right)\left(\int_X g\,dm\right).$$

The abstract Hardy spaces $H^p(m)$, $0 < p \leqq \infty$, associated with A are defined as follows. For $0 < p < \infty$, $H^p(m)$ is the $L^p(m)$-closure of A, while $H^\infty(m)$ is defined to the weak-*closure of A in $L^\infty(m)$. For $0 < p \leqq \infty$, $H_0^p = \left\{f \in H^p(m) \colon \int_X f\,dm = 0\right\}$.

Let B^∞ be a weak-*closed subalgebra of $L^\infty(m)$ which contains A and let $B_0^\infty = \left\{f \in B^\infty \colon \int_X f\,dm = 0\right\}$ and let I_B^∞ be a maximum weak-*closed ideal of B^∞ in B_0^∞, of which in Lemma 2 we shall show the existence. If $B^\infty = H^\infty(m)$ or $L^\infty(m)$, we know that $B_0^\infty = I_B^\infty = H_0^\infty$ or $I_B^\infty = \{0\}$ respectively. By [6, p. 226] and the following Lemma 1, it follows that $I_B^\infty \subseteqq H_0^\infty$.

Suppose $0 < p \leqq s \leqq \infty$. For any subset $M \subset L^s(m)$, denote by $[M]_p$ the $L^p(m)$-closure of M (weak-*closure for $p = \infty$). For any measurable subset E of X, the function χ_E is the characteristic function of E. If $f \in L^p(m)$, write E_f for the support set of f and write χ_f for the characteristic function of E_f.

We use the following crucial result. In the proof, the simply invariant subspace theorem for $L^p(m)$ [6, p. 227] is not used. For weak-*Dirichlet algebras it has not been published.

LEMMA 1 (Gamelin and Lumer). *Suppose* $0 < p < s \leqq \infty$. *If the set* M_s *is a closed invariant subspace of* $L^s(m)$, *then*

$$M_s = [M_s]_p \cap L^s(m) .$$

If the set M_p *is a closed invariant subspace of* $L^p(m)$, *then*

$$M_p = [M_p \cap L^s(m)]_p .$$

Proof. The proof is essentially that of Gamelin and Lumer [1, p. 131]. If v is a nonnegative function in $L^1(m)$ and

$$\int_X f v dm = \int_X f dm , \quad f \in A ,$$

then $v = 1$ a.e. By [3; Theorem 4] $H^\infty(m)$ is a logmodular algebra on the maximal ideal space of $L^\infty(m)$, i.e., that each real-valued function in $L^\infty(m)$ is the logarithm of the modulus of an invertible function in the algebra $H^\infty(m)$. There exists a Radon measure $\hat{m}$ on the maximal ideal space Y of $L^\infty(m)$ such that

$$\int_X f dm = \int_Y \hat{f} d\hat{m}$$

for all $f \in L^\infty(m)$ where $\hat{f}$ is the Gelfand transform of f. Now the measure $\hat{m}$ is a unique representing measure for the multiplicative functional m on $\widehat{H^\infty(m)}$ and $\widehat{H^\infty(m)}$ is weak-*closed in $L^\infty(\hat{m})$. By [1, p. 131] this proves lemma.

For weak-*Dirichlet algebras, the following two invariant subspace theorems are known.

(a) *If the set* M *is a closed invariant subspace of* $L^p(m)$ *which is doubly invariant*, i.e., *if* $f \in M$ *and* $g \in A$ *imply that*

$$fg \in M \quad \text{and} \quad f\bar{g} \in M ,$$

then $M = X_E L^p(m)$ *for some measurable subset* E *of* X.

(b) *If the set* M *is a closed invariant subspace of* $L^p(m)$ *which is simply invariant*, i.e., *if*

$$M \supsetneqq [A_0 M]_p$$

where $A_0 = \left\{f \in A: \int_X f dm = 0\right\}$, *then* $M = qH^p(m)$ *for* $|q| = 1$ a.e.

In general there exist many invariant subspaces which are neither doubly nor simply invariant. Consider any weak-*closed algebra B^∞ such that $H^\infty(m) \subsetneqq B^\infty \subsetneqq L^\infty(m)$ if $H^\infty(m)$ is not a maximal weak-*closed subalgebra, then $\chi_E q[B^\infty]_p$ for every χ_E in B^∞ is an invariant subspace which is not doubly or simply invarian.t We characterize such invariant subspaces under a condition which is natural as a generalization of simply invariant subspaces.

It is a consequence of the definition of a weak-*Dirichlet algebra that if f is in $H^\infty(m)$ and $\int_E f dm = 0$ for all χ_E in $H^\infty(m)$, then $\int_X f^2 dm = 0$.

DEFINITION 1. Suppose B^∞ is a weak-*closed subalgebra of $L^\infty(m)$ which contains A. We call the measure m quasi-multiplicative on B^∞ if $\int_X f^2 dm = 0$ for every f in B^∞ such that $\int_E f dm = 0$ for all χ_E in B^∞.

THEOREM. *Fix p in range $0 < p \leqq \infty$. Let the set M be a closed invariant subspace of $L^p(m)$ such that $B^\infty M \subseteqq M$ and*

$$\chi_E M \supsetneqq \chi_E[I_B^\infty M]_p$$

*for every nonzero χ_E in B^∞ so that $\chi_E M \neq \{0\}$. Let B^∞ be a weak-*closed subalgebra of $L^\infty(m)$ which contains A and on which the measure m is quasi-multiplicative. Then M has the form*

$$\chi_{E_0} q B^p$$

for some unimodular function q and some χ_{E_0} in B^∞, where $B^p = [B^\infty]_p$.

This theorem contains all known results of invariant subspaces (doubly, simply and sesqui-invariant [4]) in the context of a weak-*Dirichlet algebra.

2. Decomposition. Let A be a weak-*Dirichlet algebra of $L^\infty(m)$. H_0^∞ is a maximal weak-*closed ideal of $H^\infty(m)$ and it is clear that $H^2(m) \oplus \bar{H}_0^2 = L^2(m)$.

LEMMA 2. *Suppose B^∞ is any weak-*closed subalgebra of $L^\infty(m)$ which contains A. Then, for $1 \leqq p \leqq \infty$,*

(1) *There exists a maximum weak-*closed ideal I_B^∞ of B^∞ which is contained in B_0^∞.*

(2) *Let $I_B^\infty = [I_B^\infty]_p$. Then*

$$I_B^p = \left\{ f \in L^p(m) \colon \int_X f g dm = 0 \quad \text{for all} \quad g \in B^\infty \right\}.$$

(3) *Let* $B^p = [B^\infty]_p$. *Then*

$$B^p = \left\{f \in L^p(m): \int_X fgdm = 0 \quad \text{for all} \quad g \in I_B^\infty\right\}.$$

(4) $B^\infty + I_B^\infty$ *is weak-*dense in* $L^\infty(m)$ *and in particular* $B^2 \oplus \bar{I}_B^2 = L^2(m)$.

(5) I_B^∞ *is contained in* H_0^∞.

Proof. Suppose $I_B^\infty = \{f \in L^\infty(m): \int_X fgdm = 0 \text{ for all } g \in B^\infty\}$. Then since $H^2 \oplus \bar{H}_0^2 = L^2(m)$, it follows that $I_B^\infty \subset H_0^\infty \subset B_0^\infty$. This proves (5). It is trivial that I_B^∞ is a weak-*closed ideal of B^∞. Let V be any weak-*closed ideal of B^* which is contained in B_0^∞. Then since $B^\infty V \subseteqq V$ and $V \subset B_0^\infty$, the set $V \subseteqq I_0^\infty$ and hence the weak-*closed ideal I_B^∞ of B^∞ is maximal in B_0^∞. This implies (1). For $1 \leqq p < \infty$, it is trivial that

$$I_B^p \subseteqq M_p = \left\{f \in L^p(m): \int_X fgdm = 0 \text{ for all } g \in B^\infty\right\}.$$

Since both I_B^p and M_p are the closed invariant subspaces of $L^p(m)$, by the first half of Lemma 1, it follows that $I_B^\infty = I_B^p \cap L^\infty(m)$ and by definition, $I_B^\infty = M_p \cap L^\infty(m)$. Now by the second half of Lemma 1, it follows that $I_B^p = M_p$. This proves (2). Let

$$W^1 = \left\{f \in L^1(m): \int_X fgdm = 0 \text{ for all } g \in I_B^\infty\right\}.$$

Then since $I_B^\infty = \left\{f \in L^\infty(m): \int_X fgdm = 0 \text{ for all } g \in B^1\right\}$, by the duality relation, it follows that $W^1 = B^1$. For $1 < p \leqq \infty$, by the first half of Lemma 1, the assertion (3) is proved. If f in $L^1(m)$ annihilate $B^\infty + \bar{I}_B^\infty$, by (2) and (3), then $f \in I_B^1 \cap \bar{B}^1$. Since $\bar{f} \in B^1$, there exists a sequence $g_n \in B^\infty$ such that $g_n \to \bar{f}$ in $L^1(m)$ as $n \to \infty$. Hence, since I_B^∞ is a ideal of B^∞, it follows that $|f|^2 \in [I^\infty]_{1/2} \subset H^{1/2}(m)$. $|f|^2 = 0$ a.e. because every nonnegative $H^{1/2}(m)$ function is a constant [7]. Thus $f = 0$ a.e. This proves (4).

DEFINITION 2. Let the set M be a closed invariant subspace of $L^p(m)$ for $0 < p \leqq \infty$. (i) M is called left continuous for B^∞ if B^∞ is a weak-*closed subalgebra such that $B^\infty M \subseteqq M$ and $A \subset B^\infty$ and

$$\chi_E M \supsetneqq \chi_E[I_B^\infty M]_p$$

for every nonzero $\chi_E \in B^\infty$ so that $\chi_E M \neq \{0\}$. (ii) M is called right continuous for B^∞ if M is left continuous for B^∞ where

$$M = \left\{f \in \chi_E L^s(m): \int_X fgdm = 0 \text{ for all } g \in M\right\}$$

and E is a support set of M and $1/p + 1/s = 1$.

We shall show a decomposition theorem that any invariant subspace of $L^p(m)$ is a direct sum of a left continuous invariant subspace, a right continuous invariant subspace and a remaining invariant subspace.

THEOREM 1. *Suppose $0 < p \leqq \infty$, the set M is an invariant subspace of $L^p(m)$ and B^∞ is a weak-*closed subalgebra such that $B^\infty M \subseteqq M$ and $B^\infty \supset A$. Then*

$$M = M_0 + M_1 + M_2$$

where $M_i = \chi_{E_i} M (i = 0, 1, 2)$, $\chi_{E_i} \in B^\infty (i = 0, 1, 2)$ and $\chi_{E_i}\chi_{E_j} = 0$ as $i \neq j$. M_2 is left continuous for B^∞, M_1 is right continuous for B^∞ which contains no left continuous invariant subspace of the from $\chi_E M$ for $\chi_E \in B^\infty$, and $M_0 = [I_B^\infty M_0]_p$ and $M_0^\perp = [I_B^\infty M_0^\perp]_s$, where s is the conjugate index to p. If the algebra B^∞ is fixed, then this decomposition is unique.

Proof. If M is left continuous for B^∞, let $M_2 = M$. If M is not left continuous for B^∞, there exists at least one nonzero $\chi_E \in B^\infty$ and $\chi_E M \subseteqq [I_B^\infty M]_p$. If χ_E and χ_F in B^∞ such that $\chi_E M \subseteqq [I_B^\infty M]_p$ and $\chi_F M \subseteqq [I_B^\infty M]_p$, then it is easy to show that $\chi_{E \cup F} \in B^\infty$ and $\chi_{E\cup F} M \subseteqq [I_B^\infty M]_p$. Let

$$\alpha = \sup \{m(E) \colon \chi_E M \in B^\infty, \chi_E M \subseteqq [I_B^\infty M]_p\}$$

then we can show that there exists χ_{K_0} in B^∞ such that $m(K_0) = \alpha$ and $\chi_{K_0} M \subseteqq [I_B^\infty M]_p$. The set $\chi_{K_0^c} M$ is left continuous for B^∞ or trivial. Suppose $M_2 = \chi_{K_0^c} M$ if $\chi_{K_0^c} M \neq \{0\}$, where $E_2 = K_0^c$.

The set $\chi_{K_0} M$ coincides with $[I_B^\infty \chi_{K_0} M]_p$. Let E be the support set of M and let $K_0' = K_0 \cap E$. Suppose

$$(\chi_{K_0} M)^\perp = \left\{ f \in \chi_{K_0'} L^s(m) \colon \int_X f g dm = 0 \text{ for all } g \in \chi_{K_0} M \right\},$$

where $1/p + 1/s = 1$. Then $(\chi_{K_0} M)^\perp$ is a closed invariant subspace of $L^s(m)$ and $B^\infty (\chi_{K_0} M)^\perp \subseteqq (\chi_{F_0} M)^\perp$. Just as in the first part of the proof, we can show that there exists χ_{F_0} in B^∞ such that

$$(\chi_{K_0} M)^\perp = \chi_{F_0} (\chi_{K_0} M)^\perp + \chi_{F_0^c} (\chi_{K_0} M)^\perp,$$

where the set $\chi_{F_0^c}(\chi_{K_0} M)^\perp$ is left continuous for B^∞ and $\chi_{F_0}(\chi_{K_0} M)^\perp = [I_B^\infty \chi_{F_0} (\chi_{K_0} M)^\perp]_s$. Then

$$\chi_{K_0} M = \chi_{F_0 \cap K_0} M + \chi_{F_0^c \cap K_0} M,$$

where the set $\chi_{F_0^c \cap K_0} M$ is right continuous which contains no left continuous invariant subspace $\chi_E M$ for $\chi_E \in B^\infty$ or trivial. Let $M_1 = \chi_{E_1} M$ if $\chi_{E_1} M \neq \{0\}$ and let $M_0 = \chi_{E_0} M$ where $E_1 = F_0^c \cap K_0$ and $E_0 = F_0 \cap K_0$. It is clear that $M_0 = [I_B^\infty M_0]_p$ and $M_0^\perp = [I_B^\infty M_0^\perp]_s$. If the algebra B^∞ is fixed, then this decomposition is unique. For if $M = \chi_{E_0'} M + \chi_{E_1'} M + \chi_{E_2'} M$ is anotherde composition of M for B^∞, it is absurd that $m(E_0' \cap E_1) > 0$ or $m(E_0' \cap E_2) > 0$ since $\chi_{E_0'} M = [I_B^\infty \chi_{E_0'} M]_p$ and $(\chi_{E_0'} M)^\perp = [I_B^\infty (\chi_{E_0'} M)^\perp]_s$. Thus $E_0 = E_0'$. Both $\chi_{E_1} M$ and $\chi_{E_0'} M$ are right continuous for B^∞ and they do not contain left continuous invariant subspaces $\chi_E M$ for $\chi_E \in B^\infty$. So it is clear that $E_1' = E_1$. This proves the uniqueness.

REMARK. In this theorem, suppose $B^\infty(M) = \{g \in L^\infty(m): gM \subseteqq M\}$. The remaining invariant subspace M_0 has the properties such that $M_0 = [I_{B(M)}^\infty M_0]_p$ and $M_0^\perp = [I_{B(M)}^\infty M_0^\perp]_s$ with $1/p + 1/s = 1$. Then for every weak-*closed subalgebra B^∞ such that $B^\infty M_0 \subseteqq M_0$ and $B^\infty \supset A$, $M_0 = [I_B^\infty M_0]_p$ and $M_0^\perp = [I_B^\infty M_0^\perp]_s$. For suppose

$$D^\infty = \{f \in L^\infty(m): fM_0 \subseteqq M_0\},$$

then D^∞ is a weak-*closed subalgebra and $\chi_F D^\infty \doteq \chi_F B^\infty(M)$ where F is the support set of M_0. Let I_D^∞ be a maximal weak-*closed ideal of D^∞ in D_0^∞. By (4) of Lemma 2 and Lemma 1, it follows that $\chi_F I_D^\infty = \chi_F I_{B(M)}^\infty$ and hence $M_0 = [I_D^\infty M_0]_p$ and $M_0^\perp = [I_D^\infty M_0^\perp]_s$. If $B^\infty M_0 \subseteqq M_0$, then $B^\infty \subseteqq D^\infty$ and hence $I_D^\infty \subseteqq I_B^\infty$ by (2) of Lemma 2. Thus $M_0 = [I_B^\infty M_0]_p$ and $M_0^\perp = [I_B^\infty M_0^\perp]_s$.

Helson and Lawdenslager [2] established that there exists an invariant subspace M such that if the weak-*closed subalgebra B^∞ contains A and $B^\infty M \subseteqq M$, then $M = [I_B^\infty M]_p$ and $M^\perp = [I_B^\infty M^\perp]_s$ with $1/p + 1/s = 1$.

3. Characterization. Let A be a weak-*Dirichlet algebra of $L^\infty(m)$. In this section, we shall characterize left continuous invariant subspaces for any weak-*closed subalgebra B^∞ which contains A and on which the measure m is quasi-multiplicative. Then we can characterize right continuous invariant subspace, too.

LEMMA 3. *Suppose B^∞ is any weak-*closed subalgebra which contains A and on which the measure m is quasi-multiplicative. If v is a nonnegative function in B^1, then* (1) $\sqrt{v} \in B^1$, (2) $1/(v + \varepsilon) \in B^1$ *for any $\varepsilon > 0$ and $\chi_v \in B^1$.*

Proof is an easy consequence of Lemma 4 and Theorem 4 in §5. For $[\mathscr{L}_B^\infty]_1 = L^1(\mathscr{B})$ for some σ-algebra $\mathscr{B}$.

Now we shall show the main theorem.

THEOREM 2. *Fix p in range $0 < p \leqq \infty$. Suppose B^∞ is a weak-*closed subalgebra of $L^\infty(m)$ which contains A and on which the measure m is quasi-multiplicative.*

(1) *The set M is a left continuous invariant subspace in $L^p(m)$ for B^∞ if and only if M has the form*

$$M = \chi_E q B^p$$

where χ_E is a characteristic function in B^∞ and q is a unimodular function. If $M = \chi_E q' B^p$ with a unimodular function q', then $\chi_E q' = \chi_E F q$ where F is a unimodular function and $F, \bar{F} \in B^\infty$.

(2) *The set M is a right continuous invariant subspace in $L^p(m)$ for B^∞ if and only if M has the form*

$$M = \chi_E q I_B^p$$

where χ_E is a characteristic function in B^∞ and q is a unimodular function.

Proof. If the assertion (1) is shown, the assertion (2) follows by (2) and (3) in Lemma 2. In the assertion (1), 'if' part is easy. For if $M = \chi_E q B^p$, then

$$\chi_F[I_B^\infty M]_p = \chi_F \chi_E q I_B^p \subsetneqq \chi_F \chi_E q B^p = \chi_F M$$

for all $\chi_F \in B^\infty$ and $\chi_F M \neq \{0\}$. We shall show only 'only if' part. By Lemma 1, it suffices to consider the case $p = 2$. For when $2 < p \leqq \infty$, let M be a left continuous invariant subspace of $L^p(m)$ and let $M_2 = M \cap L^2(m)$. Then M_2 is a closed invariant subspace of $L^2(m)$ and it is left continuous by the second half of Lemma 1. Thus $M_2 = \chi_E q B^2$ and hence again by the second half of Lemma 1, $M = \chi_E q B^p$. By the first half of Lemma 1, when $0 < p < 2$, the proofs are the same one as the above.

Let M be a left continuous invariant subspace in $L^2(m)$ for B^∞ and let $R = M \ominus [I_B^\infty M]_2$. Observe that for any $f \in R$.

$$\int_X g |f|^2 dm = 0 \qquad (g \in I_B^\infty) .$$

By (3) of Lemma 2, it follows that $|f|^2$ lies in B^1 and hence by Lemma 3, it follows that $|f|$ lies in B^2 and $\chi_f \in B^\infty$. Let E be the support set of R, then there exists $f \in R$ with $E_f = E$. Now just as Merrill and Lal [4, Lemma 8], define

$$q(x) = \begin{cases} f(x)/|f(x)| & x \in E \\ 1 & x \notin E . \end{cases}$$

Define $q_\varepsilon(x) = f(x)/(|f(x)| + \varepsilon)$ for any $\varepsilon > 0$. Then q_ε lies in R. For since f is orthogonal to $I_B^\infty M$ and $1/(|f| + \varepsilon) \in B^\infty$, the function f is orthogonal to $1/(|f| + \varepsilon)I_B^\infty M$. Thus q_ε is orthogonal to $I_B^\infty M$ for any $\varepsilon > 0$. Since $q_\varepsilon \in M$, it follows that q_ε lies in R. Since $q_\varepsilon \to q\chi_E$ a.e. as $\varepsilon \to 0$ and $|q_\varepsilon| < 1$, it follows that $\chi_E q \in R$. Clearly $\chi_E q B^2 \subseteqq M$ as $B^\infty M \subseteqq M$ and $\chi_E q \in M$.

Let $g \in M \ominus \chi_E q B^2$. Then g is orthogonal to $X_E q B^\infty$. Also since $\chi_E q \in R$, we have $\chi_E q$ is orthogonal to $g I_B^\infty \subseteqq I_B^\infty M$. So $\chi_E \bar{q} g$ is orthogonal to $B^\infty + \bar{I}_B^\infty$ in $L^2(m)$, and hence is 0 a.e. by (4) of Lemma 2. But $|q| = 1$ a.e., so $\chi_E g = 0$ a.e. If $\chi_{E^c} g \neq 0$, then

$$\{0\} \neq \chi_{E^c} M \subseteqq \chi_{E^c}[I_B^\infty M]_2$$

and $\chi_{E^c} \in B^\infty$. This contradicts M being left continuous. So $\chi_{E^c} g = 0$ a.e. and hence $g = 0$ a.e. Thus $M = \chi_E q B^2$.

If $M = \chi_E q' B^p$ with a unimodular function q', then the function $\chi_E \bar{q} q'$ and $\chi_E q \bar{q}'$ lie in B^∞. Suppose $F = \chi_E \bar{q} q' + \chi_{E^c}$.

This theorem contains all known results of invariant subspaces in the context of a weak-*Dirichlet algebra as corollaries.

COROLLARY 1 (Wiener). *For $0 < p \leqq \infty$, the set M is a doubly invariant subspace in $L^p(m)$ if and only if M has the form*

$$M = \chi_E L^p(m) .$$

Proof. Since $A + \bar{A}$ is weak-*dense in $L^\infty(m)$ and M is doubly invariant, $L^\infty(m)M \subseteqq M$. Since m is clearly quasi-multiplicative on $L^\infty(m)$, apply Theorem 2 with $B^\infty = L^\infty(m)$.

COROLLARY 2 (Beurling [6, p. 244]). *For $0 < p \leqq \infty$, the set M is a simply invariant subspace in $L^p(m)$ if and only if M has the form*

$$M = qH^p(m)$$

where q is a unimodular function.

Proof. Since m is multiplicative on $H^\infty(m)$ by definition, apply Theorem 2 with $B^\infty = H^\infty(m)$.

COROLLARY 3 (Merrill and Lal [4]). *Suppose there exists at least one positive nonconstant function v in $L^1(m)$ such that the measure* vdm *is multiplicative on A. Then there exists a unimodular function Z such that $H_0^\infty = ZH^\infty(m)$. For $1 \leqq p \leqq \infty$, define*

$$I^p = \left\{f \in H^p(m) \colon \int \bar{Z}^n f dm = 0,\ n = 0, 1, 2, \cdots\right\}$$

and denote by $\mathscr{L}^p$ *the closure* (in $L^p(m)$) *of the polynomials in* Z *and* $\bar{Z}$ (for $p = \infty$, the closure is taken in the weak-*topology). *Let* M *be a closed invariant subspace of* $L^p(m)$ *such that* M *is not simply or doubly invariant. Then we call* M *sesqui-invariant.*

Fix p *in range* $1 \leqq p \leqq \infty$. *Let* M *be a closed sesqui-invariant subspace of* $L^p(m)$ *and let* E *be the support set of* M. *Let*

$$R = \left\{f \in M \cap L^s(m) \colon \int_X f\bar{g}dm = 0 \text{ for all } g \in I^\infty M\right\}$$

where s *is the conjugate index to* p. *Then* E *is the support set for* R *if and only if* M *has the form*

$$M = \chi_E q(\mathscr{L}^p + I^p)$$

where $\chi_E \in \mathscr{L}^2$ *and* q *is a unimodular function.*

Proof. Since M is sesqui-invariant, it follows that $J^\infty M \subseteqq M$ by [4, Lemma 2], where J^∞ is the weak-*closure of $\bigcup_{n=0}^\infty \bar{Z}^n H^\infty(m)$. By Theorem 5 in §5, m is quasi-multiplicative on J^∞. Hence by the remark below Theorem 4 in §5, $J^\infty = \mathscr{L}_J^\infty + I_J^\infty$, where $\mathscr{L}_J^\infty$ is a selfadjoint part of J^∞. It is clear that $I^\infty \supseteqq I_J^\infty$ and by the definition of I^∞, and by [4, Lemma 1], it follows that $H^\infty(m)I^\infty \subseteqq I^\infty$ and $\bar{Z}I^\infty \subseteqq I^\infty$. So I^∞ is a weak-*closed ideal of J^∞ in J_0^∞ and hence $I^\infty = I_J^\infty$. Since $H^2(m) = \mathscr{H}^2 + I^2$, where $\mathscr{H}^2$ is the L^2-closure of the polynomials in Z, it follows that $J^2 = [\mathscr{L}_J^\infty]_2 \oplus I^2 \supseteqq [\mathscr{L}^\infty + I^\infty]_2 \supsetneqq H^2(m)$. Since J^∞ is the minimum weak-*closed subalgebra of $L^\infty(m)$ which contains $H^\infty(m)$ properly by [5, Theorem 1], $J^\infty = \mathscr{L}_J^\infty + I^\infty = [\mathscr{L}^\infty + I^\infty]_2 \cap L^\infty(m)$. Hence by the second half of Lemma 1, it follows that $[\mathscr{L}_J^\infty]_2 + I^2 = [\mathscr{L}^\infty]_2 + I^2$ and hence $[\mathscr{L}_J^\infty]_2 = [\mathscr{L}^\infty]_2$. We can show that $[\mathscr{L}_J^\infty]_2 = [\mathscr{L}^\infty]_2 = L^2(\mathscr{B})$ for some σ-algebra β and hence $\mathscr{L}_J^\infty = \mathscr{L}^\infty$.

Let E be the support set of R. Suppose there exists some characteristic function χ_{E_0} in J^∞ such that $\chi_{E_0}M = \chi_{E_0}[I^\infty M]^p$ and $\chi_{E_0}M \neq \{0\}$. If f is any function in $L^s(m)(1/p + 1/s = 1)$ such that

$$\int_X f\bar{g}dm = 0 \quad \text{for all} \quad g \in \chi_{E_0} I^\infty M\,,$$

then

$$\int_X f\bar{g}dm = 0 \quad \text{for all} \quad g \in \chi_{E_0} M\,.$$

Therefore if $f \in R$, then $\chi_{E_0} f = 0$ a.e. This contradicts the fact the support set of M conincides with that of R. Thus M is left continuous for J^∞. Now apply Theorem 2 with $B^\infty = J^\infty$, then $M = \chi_E q J^p$ where $\chi_E \in J^\infty$ and q is a unimodular function. By Lemma 4 in §5, $J^p = \mathscr{L}^p + I^p$. It is clear that $\chi_E \in J^\infty$ if and only if $\chi_E \in \mathscr{L}^2$.

In many examples which we know, the measure m is quasi-multiplicative on every weak-*closed subalgebra which contains A. So under such a condition we would like to know the form of all invariant subspaces.

THEOREM 3. *Suppose the measure m is quasi-multiplicative on every weak-*closed subalgebra B^∞ which contains A. Suppose $0 < p \leqq \infty$, the set M is an invariant subspace and $B^\infty = \{f \in L^\infty(m): fM \subseteqq M\}$. Then*

$$M = M_0 + \chi_{E_1} q_1 I_B^p + \chi_{E_2} q_2 B^p$$

where $M_0 = (1 - \chi_{E_1} - \chi_{E_2})M$, $\chi_{E_1} q_1 I_B^p = \chi_{E_1} M$, and $\chi_{E_2} q_2 B^p = \chi_{E_2} M$, $\chi_{E_i} \in B^\infty (i = 1, 2)$ and $\chi_{E_1}\chi_{E_2} = 0$ and $q_i (i = 1, 2)$ are unimodular functions. Here $\chi_{E_1} q_1 I_B^p = \chi_{E_1} q_1 [I_B^\infty I_B^p]_p$ and $M_0 = [I_B^\infty M_0]_p$ and $M_0^\perp = [I_B^\infty M_0^\perp]_s$ with $1/p + 1/s = 1$. Moreover if I_B^∞ is left continuous for B^∞, then

$$M = \chi_{E_2} q_2 B^p .$$

Proof. By Theorem 1, we can get a decomposition of M such that $M = M_0 + M_1 + M_2$. By Theorem 2, it follows that $M_1 = \chi_{E_1} M = \chi_{E_1} q_1 I_B^p$ where $\chi_E \in B^\infty$ and q_1 is a unimodular function and, $M_2 = \chi_{E_2} M = \chi_{E_2} q_2 B^p$ where $\chi_{E_2} \in B^\infty$ and q_2 is a unimodular function.

Moreover if I_B^∞ is left continuous for B^∞, then $I_B^\infty = \chi_E q B^\infty$ by Theorem 2. So $\chi_{E_1} q_1 I_B^p = \chi_E \chi_{E_1} q_1 B^p$ and hence $\chi_{E_1} q_1 I_B^p$ is left continuous. By the above decomposition, it follows that $\chi_{E_1} q_1 I_B^p = \{0\}$. Since $M_0 = [I_B^\infty M_0]_p = q[\chi_E B^\infty M_0]_p$ and $B^\infty M_0 \subseteqq M_0$, it follows that $\bar{q} M_0 \subseteqq M_0$. As in the remark below Theorem 1,

$$\{f \in \chi_F L^\infty(m): fM_0 \subseteqq M_0\} = \chi_F B^\infty$$

where F is the support set of M_0. While since $\chi_K \chi_E \bar{q} \notin B^\infty$ for every $\chi_K \in B^\infty$ and $\chi_K \chi_E \neq 0$, if $\chi_F \neq 0$,

$$\{f \in \chi_F L^\infty(m): fM_0 \subseteqq M_0\} \neq \chi_F B^\infty .$$

Thus $M_0 = \{0\}$ and hence $M = \chi_{E_2} q_2 B^p$.

4. Remarks. Our definition of left continuous invariant subspaces is natural as a generalization of simply invariant subspaces. Because it is immedeate that if M is a simply invariant subspace,

then M is a left continuous invariant subspace. Suppose M is a closed invariant subspace of $L^p(m)$. We call M a sesqui-invariant subspace for B^∞ under the following condition: Let B^∞ be a weak-*closed subalgebra which contains A, let E be the support set of M and let

$$R = \left\{f \in M \cap L^s(m) \colon \int_X f\bar{g}dm = 0 \quad \text{for all} \quad g \in I_B^\infty M\right\}$$

where $1/p + 1/s = 1$, then E is the support set for R. This definition is a natural generalization of sesqui-invariant subspaces by Merrill and Lal [4]. However it is somewhat unnatural. If M is a sesqui-invariant subspace in $L^1(m)$, even if the measure m is not quasi-multiplicative on B^∞, we can characterize it. For we can easily show that if v is a nonnegative function in any weak-*closed subalgebra which contains A, then (1) $\sqrt{v} \in B^\infty$, (2) $1/(v + \varepsilon) \in B^\infty$ for any $\varepsilon > 0$ and (3) $\chi_v \in B^\infty$. Then we can show that $M = \chi_E q B^1$ just as the proof of Theorem 2. But we can not characterize any sesqui-invariant subspace for $p \neq 1$. If M is a sesqui-invariant subspace, then it is clear that M is a left continuous invariant subspace.

5. Quasi-multiplicative.

To our regrect, we have been unable to prove the conjucture; Every left continuous invariant subspace can be characterized. However we characterized left continuous invariant subspaces for the weak-*closed subalgebra B^∞ on which the measure m is quasi-multiplicative. In this section, we investigate when the measure m is quasi-multiplicative.

Let B^∞ be any weak-*closed subalgebra of $L^\infty(m)$ which contains A and let $\mathscr{L}_B^\infty$ be a self-adjoint part of B^∞. Suppose

$$\mathscr{I}_B^\infty = \left\{f \in B^\infty \colon \int_E f dm = 0 \quad \text{for all} \quad \chi_E \in B^\infty\right\},$$

then $B_0^\infty \supseteqq \mathscr{I}_B^\infty \supseteqq I_B^\infty$. If $B^\infty = H^\infty(m)$ or $B^\infty = L^*(m)$, then $\mathscr{I}_B^\infty = I_B^\infty$.

LEMMA 4.

$$B^\infty = \mathscr{L}_B^\infty \oplus \mathscr{I}_B^\infty$$

where $\oplus$ denotes algebraic direct sum. Moreover for $1 \leqq p < \infty$

$$B^p = [\mathscr{L}_B^\infty]_p \oplus [\mathscr{I}_B^\infty]_p .$$

Proof. The set $\mathscr{L}_B^\infty$ is a weak-*closd subalgebra of B^∞ and hence it is a commutative von Neumann algebra as an algebra of operators on $L^2(m)$. Let $\mathscr{B}$ be the σ-algebra of Borel subsets E of X for which the characteristic functions χ_E lie in B^∞. Then $\mathscr{L}_B^\infty$ coincides the set of essentially bounded measurable functions $L^\infty(\mathscr{B})$ on a probability measure space $(X, \mathscr{B}, m)$ and $[\mathscr{L}_B^\infty]_p = L^p(\mathscr{B})$ for $1 \leqq p < \infty$.

If $f \in B^\infty$, then f defines a bounded linear functional on $L^1(m)$ which induces a bounded linear functional on $L^1(\mathscr{B})$. Let

$$\phi_f(v) = \int_X vf dm$$

for any v in $L^1(\mathscr{B})$. Since $L^\infty(\mathscr{B})$ is a dual space of $L^1(\mathscr{B})$, there exists a function F in $L^\infty(\mathscr{B})$ such that

$$\int_X vfdm = \int_X vFdm$$

for all v in $L^1(\mathscr{B})$. By definition of $\mathscr{I}_B^\infty$, $f - F$ lies in $\mathscr{I}_B^\infty$. Hence $B^\infty = \mathscr{L}_B^\infty \oplus \mathscr{I}_B^\infty$. To show the second assertion, as [1, Lemma 5], it suffices to show that whenever $f = u + F$ for $u \in \mathscr{L}_B^\infty$, and $F \in \mathscr{I}_B^\infty$ then for $1 < p < \infty$, $\left(\int_X |u|^p dm\right)^{1/p} \leqq \left(\int_X |f|^p dm\right)^{1/p}$.

$$\begin{aligned}\left(\int_X |u|^p dm\right)^{1/p} &\\ &= \sup \left|\int_X sudm\right|, \ s \in L^q(\mathscr{B}) \int_X |s| qdm < 1 \\ &= \sup \left|\int_X s(u + F)dm\right| \leqq \left(\int_X |u + F|^p dm\right)^{1/p}.\end{aligned}$$

Thus $B^p = [\mathscr{L}_B^\infty]_p \oplus [\mathscr{I}_B^\infty]_p$.

Let the set M be a closed invariant subspace of $L^p(m)$, let $B^\infty M \subseteqq M$ and suppose $\chi_E M \supsetneqq \chi_E[\mathscr{I}_B^\infty M]_p$ for every nonzero $\chi_E \in B^\infty$ and $\chi_E M \neq \{0\}$. Then we can show that $M = \chi_F q B^p$ as in the proof of Theorem 2. However we do not know whether $\chi_E B^p \supsetneqq \chi_E[\mathscr{I}_B^\infty B^p]_p$ for every nonzero $\chi_E \in B^\infty$. We shall show that this is equivalent to the measure m being quasi-multiplicative.

THEOREM 4. *Let B^∞ be a weak-*closed subalgebra which contains A. Then the following are evuivalent.*

(1) *The measure m is quasi-multiplicative on B^∞.*

(2) *For every real-valued function u in B^2, there exist real-valued functions u_n in B^∞ such that $\int_X |u - u_n|^2 dm \to 0$ as $n \to \infty$.*

(3) $\mathscr{I}_B^\infty = I_B^\infty$

(4) $B^\infty \mathscr{I}_B \subseteqq \mathscr{I}_B^\infty$

(5) $\chi_E B^\infty \supsetneqq \chi_E[\mathscr{I}_B^\infty B^\infty]_\infty$ *for every nonzero χ_E in B^∞.*

Proof. Suppose $S = B^2 \ominus I_B^2$, then S is the self-adjoint part of

B^2 by (4) of Lemma 2. By Lemma 4

$$[\mathscr{L}_B^\infty]_2 \oplus [\mathscr{I}_B^\infty]_2 = S \oplus I_B^2 .$$

This shows that $(2) \Leftrightarrow (3)$.

$(1) \Rightarrow (3)$. The assertion (1) implies that $fg \in B_0^\infty$ for every f and g in $\mathscr{I}_B^\infty$ and hence $\mathscr{I}_B^\infty$ is orthogonal to $\bar{\mathscr{I}}_B^\infty$. This is that B^∞ is orthogonal to $\bar{\mathscr{I}}_B^\infty$. By (4) of Lemma 2, it follows that $\mathscr{I}_B^\infty = I_B^\infty$.

$(3) \Rightarrow (5)$. Since I_B^∞ is a weak-*closed ideal of B^∞ and $\mathscr{I}_B^\infty = I_B^\infty$, for every nonzero χ_E in B^∞, $\chi_E B^\infty \geqq \chi_E I_B^\infty = \chi_E[\mathscr{I}_B^\infty B^\infty]$.

$(5) \Rightarrow (4)$. By Lemma 1, we may assume $\chi_E B^2 \supsetneqq \chi_E[\mathscr{I}_B^\infty B^2]_2$ for every nonzero χ_E in B^∞. Let $R = B^2 \ominus [\mathscr{I}_B^\infty B^2]_2$, then for any $f \in R$

$$\int_X g|f|^2 dm = 0 \quad (g \in \mathscr{I}_B^\infty) .$$

By (3) of Lemma 2, it follows that $|f|^2$ lies in B^1. Since $|f|^2 \in B^1$ annihilate $\mathscr{I}_B^\infty$, by Lemma 4, it follows that $|f|^2$ lies in $[\mathscr{L}_B^\infty]_1 = L^1(\mathscr{B})$ for some σ-algebra $\mathscr{B}$. So $|f| \in B^2$, $1/(|f| + \varepsilon) \in B^\infty$ for any $\varepsilon > 0$ and $\chi_f \in B^\infty$. As the proof of Theorem 2, we can show that $B^2 = qB^2$ for some unimodular function q in $R \cap \mathscr{L}_B^\infty$. Since $\mathscr{L}_B^\infty R \subseteqq R$, it follows that the constant function 1 lies in R and hence $B^2 = [\mathscr{L}_B^\infty]_2 \oplus [\mathscr{I}_B^\infty B^2]_2$, and hence $B^\infty \mathscr{I}_B^\infty \subseteqq \mathscr{I}_B^\infty$ by Lemma 4.

$(4) \Rightarrow (1)$ is trivial.

Now by the above theorem, if the measure m is quasi-multiplicative on the weak-*closed subalgebra B^∞ which contains A, then B^∞ or $H^\infty(m)$ has the form

$$B^\infty = \mathscr{L}_B^\infty \oplus I_B^\infty$$

or

$$H^\infty(m) = \mathscr{H}_B^\infty \oplus I_B^\infty$$

where $\mathscr{H}_B^\infty = H^\infty(m) \cap \mathscr{L}_B^\infty$.

We shall search for the weak-*closed subalgebra which contains A and on which the measure m is quasi-multiplicative. $H^\infty(m)$ and $L^\infty(m)$ are typical such subalgebras.

THEOREM 5. *Let B^∞ be a weak-*closed subalgebra which contains A and let $I_B^\infty = \chi_E q B^\infty$ for some χ_E in B^∞ and some unimodular function q. Suppose D^∞ is the weak-*closure of $\bigcup_{n=0}^\infty (\chi_E \bar{q})^n B^\infty$. If the* *measure m is quasi-multiplication on B^∞, then it is quasi-multiplicative on $\chi_F D^\infty + \chi_{F^c} L^\infty(m)$ for some χ_F in D^∞.*

Proof. Let S be a weak-*closed linear span of $\chi_E q^n \mathscr{L}_B^\infty$ for all positive integers n. Then

$$B^2 = [S]_2 \oplus I_D^2 .$$

For suppose $K = B^2 \ominus [S]_2$, then since m is quasi-multiplicative on B^∞, by (2) of Theorem 4, the set $K \subset I_B^2$. Since $I_B^2 = \chi_E q B^2$ and $\chi_E \bar{q} K$ is orthogonal to S, the set $\chi_E \bar{q} K \subset K$ and hence $\bar{S} K \subset K$. If $f \in K$ and $g \in B^\infty$, then $fg \in B^2$. If $k \in S$, then $\bar{k} f \in K \subset I_B^2$ and hence by (4) of Lemma 2 $\int_X \bar{k} f g dm = 0$. Thus fg lies in K, i.e., $B^\infty K \subseteqq K$ and hence $D^\infty K \subseteqq K$. By the definition of K, the subspace K contains I_D^2. Again by (2) of Lemma 2, $K \cap L^\infty(m)$ coincides I_D^∞ and hence $K = I_D^2$ by Lemma 1.

Now we can show that m is quasi-multiplicative on D^∞. For by the above assertion,

$$\begin{aligned} L^2(m) &= B^2 \oplus \bar{I}_B^2 \\ &= [S]_2 \oplus I_D^2 \oplus \overline{(I_B^2 \ominus I_D^2)} \oplus \bar{I}_D^2 \end{aligned}$$

and $I_B^2 \ominus I_D^2$ is contained in $[S]_2$. Thus $D^2 = [S]_2 \oplus \overline{(I_B^2 \ominus I_D^2)} \oplus I_D^2$ and hence m is quasi-multiplicative D^∞ by (2) of Theorem 4. For some $\chi_F D^\infty$, suppose $D_F^\infty = \chi_F D^\infty + \chi_{F^c} L^\infty(m)$. Then

$$D_F^\infty = \chi_F \mathscr{L}_D^\infty + \chi_{F^c} L^\infty(m) + \chi_F I_D^\infty ,$$

by the remark below Theorem 4, since m is quasi-multiplicative on D^∞. By Lemma 4 and (3) of Theorem 4, it follows that m is quasi-multiplicative on D_F^∞.

6. Applications. Let A be the algebra of continuous, complex-valued functions on the torus $T^2 = \{(z, w) \in C^2 : |z| = |w| = 1\}$ which are uniform limits of polynomials in $z^n w^m$ where

$$(n, m) \in \Gamma = \{(n, m) : m > 0\} \cup \{(n, 0) : n \geqq 0\} .$$

Denote by m the normalized Haar measure on T^2, then A is a weak-*Dirichlet algebra of $L^\infty(m)$. Merrill and Lal [4] characterized completely the invariant subspaces of $L^p(m)(1 \leqq p \leqq \infty)$, together with known results.

If M is an invariant subspace of $L^p(m)(1 \leqq p \leqq \infty)$, then M has the next forms;

(1) $M = \chi_E L^p(m)$ for some measurable set $E \subseteqq T^2$.

(2) $\mathscr{L}^p$ is the $L^p(m)$-closure of the polynomials in z and $\bar{z}$ and

I^p is the L^p-closure of the polynomials in $z^n w^m$ for $m \geqq 1$. Then,

$$M = \chi_{E_1} L^p(m) + \chi_{E_2} q(\mathscr{L}^p + I^p)$$

where q is a unimodular function, E_1 is some measurable set of T_2, $\chi_{E_2} \in \mathscr{L}^2$ and $\chi_{E_1} \cdot \chi_{E_2} = 0$.

(3) $M = qH^p(m)$ for some unimodular function q.

Our Theorem 3 implies that if M is an invariant subspace of $L^p(m)(0 < p \leqq \infty)$, then M has the form

$$M = \chi_E q B^p$$

where $B^\infty = \{f \in L^p(m): fM \subseteqq M\}$, q is a unimodular function and $\chi_E \in B^\infty$.

There exist many examples to which the theorem of Merrill and Lal [4] is not applied. However our theorem is applied. We shall give those examples.

First example: Let A be a weak-*Dirichlet algebra. Suppose there exists at least one positive nonconstant function v in $L^1(m)$ such that the measure vdm is multiplicative on A. Then let J^∞ be the minimum weak-*closed subalgebra of $L^\infty(m)$ which contains $H^\infty(m)$ properly and suppose $\chi_f \in J^\infty$ for every $f \in H^\infty(m)$.

By [5, Theorem 1], it follows that J^∞ is the weak-*closure of $\bigcup_{n=0}^\infty \bar{Z}^n H^\infty$, where $H_0^\infty = ZH^\infty(m)$. Since m is multiplicative on $H^\infty(m)$, by Theorem 5, m is quasi-multiplicative on $\chi_E J^\infty + \chi_{E^c} L^\infty(m)$ for $\chi_E \in J^\infty$. Since $\chi_f \in J^\infty$ for every $f \in J^\infty$, by [5, Theorem 4], we know that each weak-*closed subalgebras which contains $H^\infty(m)$ has the form; $\chi_E J^\infty + \chi_{E^c} L^\infty(m)$ for $\chi_E \in J^\infty$. Hence by Theorem 3, it follows that if M is an invariant subspace of $L^p(m)(0 < p \leqq \infty)$, then M has the form

$$M = M_0 + \chi_{E_1} q_1 I_B^p + \chi_{E_2} q_2 B^p$$

where $B^\infty = \{f \in L^\infty(m): fM \subseteqq M\}$ and $\chi_{E_i} \in B^\infty$ and q_i is a unimodular function.

If I_J^∞ is left continuous for J^∞, then $I_{J_E}^\infty$ is left continuous for $J_E^\infty = \chi_E J^\infty + \chi_{E^c} L^\infty(m)(\chi_E \in J^\infty)$. For $I_{J_E}^\infty = \chi_E I_J^\infty$. Thus by Theorem 3 every invariant subspace M has the form

$$M = \chi_E q B^p$$

where $B^\infty = \{f \in L^\infty(m): fM \subseteqq M\}$, q is a unimodular function and $\chi_E \in B^\infty$.

Second example: Let A be the algebra of continuous, complex-valued functions on the polydisc $T^n = \{(z_1, \cdots, z_n) \in C^n : |z_1| = \cdots = |z_n| = 1\}$ which are uniform limits of polynomials in $z_1^{\ell_1}, \cdots, z_n^{\ell_n}$ where

$$(\ell_1, \cdots, \ell_n) \in \Gamma = \{\ell_1, \cdots, (\ell_n) : \ell_n > 0\} \cup \{\ell_1, \cdots, \ell_{n-1}, 0) : \ell_{n-1} > 0\} \cup \cdots \cup \{(\ell_1, 0, \cdots, 0) : \ell_1 > 0\} .$$

Denote by m the normalized measure on T^n, then A is a weak-* Dirichlet algera of $L^\infty(m)$. For $n = 1$, we know forms of all invariant subspaces of $L^p(m)$. For $n = 2$, Merrill and Lal [4] characterized all invariant subspaces of $L^p(m)$. However their result is not applied to $n \geqq 3$. We shall show that for $n = 3$, if M is an invariant subspace of $L^p(m)$ $(0 < p \leqq \infty)$, then M has the form $M = \chi_E q B^p$ where $B^p = \{f \in L^\infty(m) : fM \subseteq M\}$. For $n > 3$, we can show it similarly. By Theorem 3, it suffices to show that m is quasi-multiplicative on every weak-*closed subalgebra B^∞ which contains A and every I_B^∞ is left continuous.

Suppose J_1^∞ is the weak-*closure of $\bigcup_{n=0}^\infty \bar{z}_1^n H^\infty(m)$ and suppose J_2^∞ is the weak-*closure of $\bigcup_{n=0}^\infty \bar{z}_2^n J_1^\infty$. By Theorem 5, m is quasi-multiplicative on every weak-*closed subalgebra B^∞ which has form $B^\infty = \chi_{E_1} J_1^\infty + \chi_{E_2} J_2^\infty + \chi_{E_3} L^\infty(m)$ for $\chi_{E_1} \in J_1^\infty$ and $\chi_{E_i} \in J_2^\infty (i = 2, 3)$. I_B^∞ for such a subalgebra is clear left continuous. Thus it suffices to show that every weak-*subalgebra B^∞ which contains A has the form $B^\infty = \chi_{E_1} J_1^\infty + \chi_{E_2} J_2^\infty + \chi_{E_3} L^\infty(m)$ or $B^\infty = H^\infty(m)$.

Let B^∞ be any weak-*closed subalgebra which contains A. By [5, Theorem 1], it follows that if $B^\infty \supsetneqq H^\infty(m)$, then $B^\infty \supseteqq J_1^\infty$. Then B^∞ is an invariant subspace such that $J_1^\infty B^\infty \subseteqq B^\infty$. Since m is quasi-multiplicative on J_1^∞ and $I_{J_1}^\infty$ is left continuous, by Theorem 1 and Theorem 2, B^∞ has the form $\chi_{E_1^c} B^\infty + \chi_{E_1} q J_1^\infty$ for $\chi_{E_1} \in J_1^\infty$, where $\chi_{E_1^c} B^\infty = \chi_{E_1^c}[I_{J_1}^\infty B^\infty]_\infty$. Since χ_{E_1} lies in $\chi_{E_1} q J_1^\infty$, it follows that $B^\infty = \chi_{E_1^c} B^\infty + \chi_{E_1} J^\infty$. Since $I_{J_1}^\infty = z_2 J_1^\infty$, $\bar{z}_2 \chi_{E_1^c} B^\infty \subseteqq \chi_{E_1^c} B^\infty$ and hence $J_2^\infty \chi_{E_1^c} B^\infty \subseteqq \chi_{E_1^c} B^\infty$. Similarly as the above $\chi_{E_1^c} B^\infty = \chi_{F^c} \chi_{E_1^c} B^\infty + \chi_F \chi_{E_1^c} J_2^\infty$ and $\bar{z}_3 \chi_{F^c} \chi_{E_1^c} B^\infty \subseteqq \chi_{F^c} \chi_{E_1^c} B^\infty$. Since $L^\infty(m)$ is the weak-*closure of $\bigcup_{n=0}^\infty \bar{z}_3^n J_2^\infty$, $\chi_{F^c} \chi_{E_1^c} B^\infty = \chi_{F^c} \chi_{E_1^c} L^\infty(m)$. Let E_2 be $F \cap E_1^c$ and let E_3 be $F^c \cap E_1^c$. Then $B^\infty = \chi_{E_1} J_1^\infty + \chi_{E_2} J_2^\infty + \chi_{E_3} L^\infty(m)$.

Third example: Let K be the Bohr compactification of the real line. Let A be the algebra of continuous, complex-valued functions on $K \times K$ which are uniform limits of polynomials in $\chi_{\tau_1} \chi_{\tau_2}$ where

$$(\tau_1, \tau_2) \in \Gamma = \{(\tau_1, \tau_2) : \tau_2 > 0\} \cup \{(\tau_1, 0) : \tau_1 \geqq 0\}$$

and denote by χ_{τ_i} the characters on K, where τ_i in the real line. Denote by m the normalized measure on $K \times K$, then A is a weak-* Dirichlet algebra of $L^\infty(m)$. Then there exist no positive nonconstant functions in $L^1(m)$ which are multiplicative on A. If M is a simply invariant subspace of $L^p(m)$ or a doubly invariant subspace of $L^p(m)$, then the characterization of M is known.

Suppose M is neither simply nor doubly invariant. Suppose there exists $\tau_1 > 0$ such that $\bar{\chi}_{\tau_1} M \subseteq M$. Let V^∞ be the weak-*closure of $\bigcup_{\tau_1 \geqq 0} \bar{\chi}_{\tau_1} H^\infty(m)$, then $H^\infty(m) \subsetneqq V^\infty \subsetneqq L^\infty(m)$ and V^∞ is a weak-*closed subalgebra. Then $\chi_f \in V^\infty$ for every $B \in H^\infty(m)$ [5, Example 3]. By (2) of Theorem 4, we can easily show that m is quasi-multiplicative on V^∞ and hence on every weak-*closed subalgebra which contains V^∞ by [5, Theorem 3]. From the hypothesis, it follows that

$$V^\infty \subseteq B^\infty = \{g \in L^\infty(m) \colon gM \subseteq M\} .$$

Thus if M is left continuous, we can characterize the form of M.

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Received September 23, 1975.

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