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ON FINITE REGULAR RINGS

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Several new properties are derived for von Neumann finite rings. A comparison is made of the properties of von Neumann finite *regular* rings and unit regular rings, and necessary and sufficient conditions are given for a matrix ring over a regular ring to be respectively von Neumann finite or unit regular. The converse of a theorem of Henriksen is proven, namely that if $R_{n\times n}$, the $n \times n$ matrix ring over ring R, is unit regular, then so is the ring R. It is shown that if $R_{2\times 2}$ is finite regular then $a \in R$ is unit regular if and only if there is $x \in R$ such that $R = aR + x(a^0)$, where a^0 denotes the right annihilator of a in R.

1. **Introduction.** In [13], Henriksen posed the question whether a finite regular ring is unit regular. This was subsequently proven in part by Ehrlich [4] for a particular class of regular rings. An example of finite regular rings which are not unit regular was recently given by Bergman (1974) (see Handelman [8]). In his paper [8], Handelman showed that a regular ring R is unit regular if and only if, for any finitely generated projective right R-modules A, B, and $C, A\oplus B \cong A\oplus C$ implies $B\cong C$. He also characterized unit regular rings by perspectivity on the lattices of their principal right ideals. The purpose of this paper, however, is to characterize finite regular rings and to compare their properties with unit regular rings. Some of the results of the theory of generalized inverses [1] are used to show that in a regular ring, the properties of finiteness and unit-regularity each correspond to a suitable cancellation law for principal ideals. These cancellation laws are closely related to the substitution property of Fuchs [5], and the cancellation law of Ehrlich [4]. We shall use a result by Vidav [15] to show that if the matrix ring $R_{n\times n}$ is unit regular then so is ring R. Let us begin by defining our concepts and by stating some useful general results. A ring R is called *regular* if for all $a \in R$, there is a solution $a^- \in R$ to the equation $axa = a$. The element a^- is called as inner or 1-inverse of a [1]. Similarly, any solution to $axa = a, xax = x$ is called a reflexive or $1-2$ inverse of a, and will be denoted by a^+. For example, a^-aa^- is always such a solution. An element $a \in R$ is said to have a group inverse $a^\# \in R$ if it is a group member, i.e., a belongs to some multiplicative group of R. Necessary and sufficient conditions for $a^\#$ to exist are that $a^2R = aR$, $Ra^2 = Ra$, or $axa = a$, $xax = x$, $ax = xa$, for some $x \in R$ [10]. A regular ring R with unity is called *unit regular*

[3] if every $a \in R$ has a unit inner inverse $a^- \in R$. R is called von Neumann or Dedekind *finite*, finite for short, if it contains a unity 1 and $ab = 1 \Rightarrow ba = 1$. It is easily seen that unit-regular rings are finite. An idempotent $e^2 = e \in R$ is said to be finite if eRe is finite. The ring of $n \times n$ matrices over R will be denoted by $R_{n\times n}$, while R^n denotes the free module of $n \times 1$ columns over R. Isomorphisms will be denoted by $\cong$, similarity by $\approx$, and *internal* and *external* direct sums by $\dotplus$ and $\oplus$ respectively. As usual, the right and left annihilator of a will be denoted by $a^0 = \{x \in R;\ ax = 0\}$ and ${}^0a = \{x \in R;\ xa = 0\}$ respectively, while the Jacobson radical is given by $\mathscr{J}(R)$. Throughout this paper all rings are assumed to have unity 1.

For two idempotents $e, f \in R$, $e \sim f$ means that $e = p\hat{p}$, $f = \hat{p}p$ for some $p \in eRf$ and $\hat{p} \in fRe$, while $e \leqq f$ denotes the well-known [14] ordering for compatible idempotents defined by $e = ef = fe$, or equivalently by $eRe \subseteq fRf$.

We shall make continued use of the following facts which hold for idempotents e and f in any ring R with unity [14].

$$
(1.1)\qquad
\begin{aligned}
&\text{(i)} && e \sim f \Longleftrightarrow eR \cong fR \text{ as right } R\text{-modules}\\
& && \qquad\ \ \Longleftrightarrow Re \cong Rf \text{ as left } R\text{-modules.}\\
&\text{(ii)} && e \approx f \Longleftrightarrow e \sim f \text{ and } 1 - e \sim 1 - f.\\
&\text{(iii)} && eR = fR \Longrightarrow e \approx f.\\
&\text{(iv)} && e = pfq,\ p,\ q \text{ invertible} \Longrightarrow e \approx f.
\end{aligned}
$$

In addition we shall use the result that

LEMMA 1. *If R is a ring, and A, B, C, D are matrices in $R_{n\times n}$, then the following are equivalent in pairs.*

(α) (i) *$AR_{n\times n} \oplus BR_{n\times n} \cong CR_{n\times n} \oplus DR_{n\times n}$ as $R_{n\times n}$-modules.*

(ii) *$AR^n \oplus BR^n \cong CR^n \oplus DR^n$ as R-modules.*

(β) (i) *$AR_{n\times n} \dotplus BR_{n\times n} \cong CR_{n\times n} \dotplus DR_{n\times n}$ as $R_{n\times n}$-modules.*

(ii) *$AR^n \dotplus BR^n \cong CR^n \dotplus DR^n$ as R-modules.*

Indeed, if $\phi\begin{bmatrix} AX \\ BY \end{bmatrix} = \phi\begin{bmatrix} A \\ 0 \end{bmatrix}X + \phi\begin{bmatrix} 0 \\ B \end{bmatrix}Y$ is the given isomorphism in (αi) then the mapping χ, given by

$$\chi\begin{bmatrix} A\boldsymbol{x} \\ B\boldsymbol{y} \end{bmatrix} = \phi\begin{bmatrix} A \\ 0 \end{bmatrix}\boldsymbol{x} + \phi\begin{bmatrix} 0 \\ B \end{bmatrix}\boldsymbol{y}$$

will be a desired isomorphism for (αii), while conversely if $\chi\begin{bmatrix} A\boldsymbol{x} \\ B\boldsymbol{y} \end{bmatrix} = \chi\begin{bmatrix} A \\ 0 \end{bmatrix}\boldsymbol{x} + \chi\begin{bmatrix} 0 \\ B \end{bmatrix}\boldsymbol{y}$ is the isomorphism given by (αii) then the map ϕ defined by

$$\phi\begin{bmatrix} A[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n] \\ B[\boldsymbol{y}_1, \cdots, \boldsymbol{y}_n] \end{bmatrix} = \left[\chi\begin{bmatrix} A\boldsymbol{x}_1 \\ B\boldsymbol{y}_1 \end{bmatrix}, \cdots, \chi\begin{bmatrix} A\boldsymbol{x}_n \\ B\boldsymbol{y}_n \end{bmatrix} \right]$$

is one of the desired $R_{n\times n}$ isomorphisms for (αi). Similar maps can be defined for case (β). In particular, if E and F are idempotents in $R_{n\times n}$, then

$$E \sim F \Longleftrightarrow ER_{n\times n} \cong FR_{n\times n} \Longleftrightarrow ER^n \cong FR^n \ . \tag{1.2}$$

In addition, if for example, $E = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}$ and $F = \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}$ then

$$E \sim F \Longleftrightarrow \begin{bmatrix} e_1R \\ e_2R \end{bmatrix} \cong \begin{bmatrix} f_1R \\ f_2R \end{bmatrix} \Longleftrightarrow e_1R \oplus e_2R \cong f_1R \oplus f_2R \ . \tag{1.3}$$

The dual result for left modules is obvious. Lastly, we note that in any ring R,

$$aR = bR \Longrightarrow Ra \cong Rb \ . \tag{1.4}$$

Indeed, if $ax = b$ and $a = by$, then the mapping $ra \mapsto rax$ is a desired isomorphism. Let us now turn to some useful results concerning finite rings.

2. Finite rings.

THEOREM 1. *Let R be a ring with unity 1 and suppose that e and f are arbitrary idempotents in R. The following are equivalent.*

(2.1)
- (i) *R is finite*
- (ii) $eR \subseteqq fR,\ e \sim f \Longrightarrow eR = fR$.
- (iii) $Re \subseteqq Rf,\ e \sim f \Longrightarrow Re = Rf$.
- (iv) $e \sim 1 \Longrightarrow e = 1$.

Proof. (i) $\Rightarrow$ (ii). Let $eR \subseteqq fR$ and $e \sim f$. Then $e = fe$ and $e = p\hat{p}$, $f = \hat{p}p$ for some $p \in eRf$, $\hat{p} \in fRe$.

Consider now

$$x = \hat{p} + (1-f)(1-e) = \hat{p} + 1 - f,\ y = p + (1-e)(1-f) = p + 1 - e - f + ef \ .$$

Then $xy = 1$ and hence, since R is finite, $y = x^{-1}$.

But now $fy = fp + f - fe - f + fef = fp - e + ef \in feRf + eR = eRf + eR \subseteqq eR$. Thus $f \in eRy^{-1} = eR$ and so $fR = eR$. We may again replace $e \sim f$ by either $eR \cong fR$ or $Re \cong Rf$.

(ii) $\Rightarrow$ (iv). Let $e \sim 1$ and $eR \subseteqq R$. By (ii), $eR = R$, which implies that $1 = ex$ for some x, or $e = ex = 1$.

(iv) $\Rightarrow$ (i). Let $ab = 1$ and set $e = ba = e^2$. Then $e \sim 1$, because $e = p\hat{p}$, $1 = \hat{p}p$ where $p = b$ and $\hat{p} = a$. In fact, $p \in eR1 = baR$ as $p = bab = b$ and $\hat{p} \in 1Re = Rba$ as $\hat{p} = aba = a$. Hence by (iii) $e = 1$ and so $ba = 1$ as desired. Part (iii) follows by left-right symmetry.

REMARKS 1. The last part should be compared with the result of Vidav [15], and Fuchs [5], which states that a regular ring is unit regular if and only if $e \sim f \Rightarrow e \approx f$.

2. The second part is best possible in that neither e nor f may fail to be idempotent. Indeed, if R is finite then $aR \cong R \not\Rightarrow aR = R$, as seen from the example of the ring of integers with $a = 2$. On the other hand, a ring R with the property that $aR \cong R \Rightarrow aR = R$ must be finite, yet need not be regular as seen from the following counterexample.

EXAMPLE 1. Let $R_1 = \begin{bmatrix} R & R \\ 0 & R \end{bmatrix}$, where $R = \boldsymbol{R}$ is the real field. Then clearly R_1 is not regular since $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Suppose now that $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}R_1 \cong R_1$. We claim that $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}R_1 = R_1$. Let $\phi\colon \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}R_1 \to R_1$ denote the given isomorphism such that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \phi\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}X\right) = \phi\begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix}$. Now $a' \neq 0$ since $a' = 0$ would imply that

$$\phi\left(\begin{bmatrix} 0 & b' \\ 0 & c' \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \phi\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 ,$$

while $\phi\left(\begin{bmatrix} 0 & b' \\ 0 & c' \end{bmatrix}\right)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Next we claim that $c' \neq 0$. For if $c' = 0$, then $\phi\left(\begin{bmatrix} a' & b' \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & b' \\ 0 & -a' \end{bmatrix}\right) = \phi\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$, while $\phi\left(\begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix}\right)\begin{bmatrix} 0 & b' \\ 0 & -a' \end{bmatrix} = \begin{bmatrix} 0 & b' \\ 0 & -a' \end{bmatrix}$, which implies that $a' = 0$, a contradiction. Thus $a' \neq 0 \neq c'$ and $\begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix}$ is invertible. Hence

$$R_1 = \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix}R_1 \subseteq \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}R_1 \subseteq R_1$$

as desired.

COROLLARY 1. *If R is finite ring and if $e^2 = e \in R$, then the corner ring eRe is also finite.*

Proof. Suppose that $(eae)(ebe) = e$. Then $eR = eaeR$ and $Re = Rebe$. Hence $Reae = Rebe(eae)$ and thus by (1.4) $eR = eaeR \cong ebe(eae)R \subseteqq eR$, ensuring by Theorem 1, that $eR = (ebe)eaeR$. This implies that $e = ebe(eae)$, as desired. We may consequently order the compatible idempotents with the usual partial ordering relation "$\leqq$". The following corollary is *equivalent* to the one above.

COROLLARY 2. *If R is a finite ring and B is a subring with unity, then B is finite.*

Proof. Let R be a finite ring and let B be a subring of R with unity e. Suppose that $x, y \in B$ and $xy = e$. If $f = yx = f^2$, then $(fxf)(fyf) = f$ and thus since fRf is finite by Corollary 1, it follows that $y^2x^2 = fyf(fxf) = yx$. On premultiplication by x and post-multiplication by y this yields $yx = eyxe = e$. It is obvious that Corollary 2 implies Corollary 1. It should be noted here, that the latter result can also be proven directly, as pointed out to us by the referee.

COROLLARY 3. *If R is a finite ring and $e^2 = e \in R$, then the centralizer $C_e = \{x \in R \mid xe = ex\}$ is finite.*

COROLLARY 4. *If R is a finite ring and $e^2 = e \in R$, then eae is a unit in eRe if and only if $eaeR = eR$.*

Proof. If $eaeebe = e$ then clearly $eR = eaeR$. Conversely, if $eR = eaeR$, then for some $x \in R$, $eaex = e = (eae)exe$. Since eRe is finite, it follows that $e = exe(eae)$ implying that eae is a unit in eRe.

We remark that this also follows from the fact that the maximal subgroup H_e containing e is the group of units in eRe and that $H_e = \{x \in R;\ xR = eR,\ Rx = Re\}$. Finiteness shows that $eaeR = eR \Leftrightarrow Re = Reae$.

We next obtain as a corollary the result by Kaplansky [14] p. 11, which says that an idempotent e *finite* if it is not equivalent to a smaller idempotent.

COROLLARY 5. *Let R be a ring and and let e, f be idempotents in R such that fRf is finite. Then*

$$eRe \subseteqq fRf,\ e \sim f \Longrightarrow e = f\ .$$

Proof. Since $e = ef = fe$, $eRf = f(eRf) \subseteqq fRf$. Now eRf and fRf are isomorphic as right fRf-modules, since the map $erf \mapsto \hat{p}erf$, with $e = p\hat{p}$, $f = \hat{p}p$, $p \in eRf = e(fRf)f$, $\hat{p} \in fRe = f(fRf)e$, is an example of such an isomorphism. Hence by Theorem 1, $eRf = fRf$, which ensures that $f = exf$ for some x or $e = ef = f$ as desired.

We remark that if R is a unit regular ring then $eRe \cong R \not\Rightarrow eRe = R$, as seen from the example where $R = \prod_{i=1}^{\infty} R_i$, $R_i = D$(a division ring) and $e = (0, 1, 1, \cdots)$. The following is a partial converse to Corollary 1.

PROPOSITION 0 (Savage). *If R is a ring with unity* 1 *such that fRf is finite for every idempotent element $f \neq 1$, then R is finite.*

Proof. Replace B by R and e by 1 in the proof of Corollary 2.

PROPOSITION 1. *Suppose R is a ring with unity* 1 *and let $\mathscr{J}(R)$ denote its Jacobson radical. Then R is finite exactly when $R/\mathscr{J}(R)$ is finite.*

Proof. $\Rightarrow$: Suppose we denote $R/\mathscr{J}(R)$ by $\bar{R}$ and the elements from $\bar{R}$ by $\bar{a}$, $\bar{b}$ etc. Let $\bar{a}\bar{b} = \bar{1}$. Then $1 - ab \in \mathscr{J}(R)$, implying that $1 - (1 - ab)1 = ab$ is a unit. That is $abc = 1 = cab$ for some c. Since R is finite it follows that $bca = 1$ and so $(\bar{b}\bar{c})\bar{a} = 1$. Hence $\bar{a}$ has left and right inverses and thus is a unit with $(\bar{a})^{-1} = \bar{b}$.

$\Leftarrow$: Conversely, suppose $\bar{R}$ is finite and that $ab = 1$. Then $\bar{a}\bar{b} = \bar{1} \Rightarrow \bar{b}\bar{a} = \bar{1} \Rightarrow 1 - ba \in \mathscr{J}(R)$. And so $1 - (1 - ba)1 = ba$ is a unit in R implying that $bac = 1 = cba$ for some $c \in R$. Hence a has left and right inverses and thus is a unit with inverse $b = a^{-1}$. This result should be contrasted with the fact that $R/\mathscr{J}(R)$ may be unit regular without R being regular (cf. [5], Lemma). Indeed, if R is any nonsemisimple Artinian ring, then R cannot be regular while $R/\mathscr{J}(R)$ is semisimple Artinian and hence unit regular.

Finiteness is closely related to the existence of group and Drazin inverses of the elements in R [2], [10]. We recall that the left (right) index $l(a)(r(a))$ of $a \in R$ is the *smallest* value of $p(q)$ for which $a^{p+1}R = a^pR(Ra^{q+1} = Ra^q)$, and that if both are finite, they have to be equal, [2]. This common value is called the *index* $i(a)$ of a. A ring is called strongly π-regular if every a in R possesses a Drazin inverse. Now a has a group inverse $a^\#$ in R exactly when $i(a) = 0$ or 1 that is exactly when $a^2R = aR$ and $Ra^2 = Ra$ or when $axa = a$, $xax = x$ and $ax = xa$ for some $x \in R$. The following is a generalization of the concept of finiteness.

We say that a ring R satisfies property (k, l) where k and l are nonnegative integers if, for $a \in R$, $a^{k+1}R = a^kR \Rightarrow Ra^{l+1} = Ra^l$. Clearly all strongly π-regular rings satisfy this property [2].

PROPOSITION 2. *Let R be a ring with unity* 1. *Suppose R satisfies property* (k, l) *for some nonnegative integers k and l. Then R is finite.*

Proof. This follows immediately from the fact that if $ab = 1$ then $i(a) = r(a) = l(a) = 0$.

It is unknown if the converse is true in general, but we shall see shortly that this is the case for subrings of finite regular rings. Presently we do have the following result dealing with the ring $\begin{bmatrix} R & R \\ 0 & R \end{bmatrix}$. This ring will be used repeatedly as a source of examples and counter examples.

PROPOSITION 3. *Let R be a ring with unity and let* $R_1 = \begin{bmatrix} R & R \\ 0 & R \end{bmatrix}$. *Then*

(i) *R is finite if and only if R_1 is finite*

(ii) *if R has property* (1, 1) *then so does R_1, in which case both R and R_1 are finite.*

Proof. Since part (i) is easily established we turn to (ii). Suppose that $M = \begin{bmatrix} u & v \\ 0 & w \end{bmatrix}$, $X = \begin{bmatrix} a & c \\ 0 & d \end{bmatrix}$ and $M^2X = M$. Equating entries shows that

$$\text{(2.3)} \qquad \begin{aligned} &\text{(i)} \quad u^2a = u \\ &\text{(ii)} \quad w^2d = w \\ &\text{(iii)} \quad u^2c + (uv + vw)d = v\ . \end{aligned}$$

Since R has property (1, 1), $u^2R = uR \Rightarrow Ru^2 = Ru$ and thus $u^\#$ exists. Similarly $w^\# \in R$. Hence

$$uu^\#u^2 = u^2 \quad \text{and} \quad wd = w^\#w^2d = w^\#w\ .$$

Then multiplying (2.3)-iii through on the left by $uu^\#$ yields

$$\text{(2.4)} \qquad u^2c + uvd + uu^\#vw^\#w = uu^\#v\ .$$

Substracting this from (2.3)-iii gives

$$\text{(2.5)} \qquad (1 - uu^\#)v(1 - ww^\#) = 0\ .$$

This, however, is exactly the consistency condition needed for M to have a group inverse: $M^\# = \begin{bmatrix} u^\# & c \\ 0 & w^\# \end{bmatrix}$, where $c = -u^\#vw^\# + (1 - uu^\#)vw^{\#2} + u^{\#2}v(1 - ww^\#)$, satisfying $MM^\#M = M$, $M^\#MM^\# = M^\#$ and $MM^\# = M^\#M$.

It is clear that $M^{\#} \in R_1$. For semi-simple rings the first part of this proposition may be proven with aid of Proposition 1. It is not known if the (k, l) property is inherited by $R_{2\times 2}$.

3. **Finite regular rings.** Let us first give a couple of preliminary observations for regular rings.

PROPOSITION 4. *Let R be a regular ring and let $a, b \in R$.*

$$(3.1) \qquad \text{(i)} \quad Ra \cong Rb \Longrightarrow aR \cong bR$$

(ii) *If a^- and $a^=$ are two inner inverses of a, then*

$$aa^- \sim aa^=,\ a^-a \sim a^=a,\ 1 - aa^- \sim 1 - aa^=,\ \textit{and}\ 1 - a^-a \sim 1 - a^=a\ .$$

Proof. (i) Let $Ra \cong Rb$. Then $Ra^+a \cong Rb^+b$ and so $a^+a \sim b^+b$. Hence $a^+a = p\hat{p}$, $b^+b = \hat{p}p$ for some $p \in a^+aRb^+b = a^+Rb$, and $\hat{p} \in b^+bRa^+a = b^+Ra$. If we set $q = apb^+$ and $\hat{q} = b\hat{p}a^+$, then clearly $q\hat{q} = aa^+$, $\hat{q}q = bb^+$, while $q \in aRb^+$ and $\hat{q} \in bRa^+$. Thus $aR = aa^+R \cong bb^+R = bR$. The converse follows by symmetry.

(ii) Clearly $aa^-R = aa^=R$, $Ra^-a = Ra^=a$ and $a^0 = (1 - a^-a)R = (1 - a^=a)R$, ${}^0a = R(1 - aa^-) = R(1 - aa^=)$.

If $RS(A)$ and $CS(A)$ denote the row-space and column space of a matrix A respectively [1], then we have:

COROLLARY 6. *For rectangular matrixes over a regular ring R,*

$$RS(A) \cong RS(B) \Longleftrightarrow CS(A) \cong CS(B)\ .$$

For future reference we add the following.

PROPOSITION 5. *In a regular ring R,*

$$(3.2) \qquad \begin{array}{ll} \text{(i)} & abR = aR \Longleftrightarrow {}^0b \cap Ra = (0) \Longleftrightarrow bR + (1 - a^-a)R = R \\ \text{(ii)} & Rab = Rb \Longleftrightarrow a^0 \cap bR = (0) \Longleftrightarrow Ra + R(1 - bb^-) = R\ . \end{array}$$

Proof. First we recall that $abR = aR \Rightarrow {}^0(ab) = {}^0a$. Now let $x \in {}^0b \cap Ra$, that is $xb = 0$ and $x = ya$ for some $y \in R$. Thus if $abR = aR$, then $0 = xb = yab \Rightarrow 0 = ya = x$. If we now write ${}^0b \cap Ra = {}^0b \cap {}^0(1 - a^-a) = {}^0(bR + (1 - a^-a)R)$ then ${}^0b \cap Ra = (0) \Rightarrow bR + (1 - a^-a)R = R$. It is obvious that the latter implies $abR = aR$. Symmetry now yields the second result.

THEOREM 2. *Let R be a regular ring with unity 1 and let a, b, c be elements of R. Suppose that I and J are right ideals in R. The following are equivalent.*

$$
(3.3)\quad
\begin{array}{rll}
\text{(i)} & R \textit{ is finite} & \\
\text{(ii)} & aR \subseteqq bR \textit{ together with } aR \cong bR \textit{ or } Ra \cong Rb & \Longrightarrow aR = bR \\
\text{(iii)} & I \subseteqq bR,\ I \cong bR & \Longrightarrow I = bR \\
\text{(iv)} & I \subseteqq J,\ I \cong J,\ J \textit{ complemented} & \Longrightarrow I = J \\
\text{(v)} & I \cong R & \Longrightarrow I = R \\
\text{(vi)} & aR \cong R & \Longrightarrow aR = R \\
\text{(vii)} & aR \oplus I \cong cR,\ cR \subseteqq aR & \Longrightarrow I = (0) \\
\text{(viii)} & aR \oplus bR \cong aR & \Longrightarrow b = 0 \\
\text{(ix)} & \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} \sim \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} & \Longrightarrow f = 0 \\
\text{(x)} & aR \dotplus bR \cong aR & \Longrightarrow b = 0 \\
\text{(xi)} & Ra^k = Ra^{k+1} \Longrightarrow a^kR = a^{k+1}R \textit{ for some } k \geqq 0. &
\end{array}
$$

An analogous and equivalent result holds for left ideals.

Proof. (i) $\Rightarrow$ (ii): Let $e = aa^+$ and $f = bb^+$. Then $e \sim f$ and $eR \subseteqq fR$. By Theorem 1 $eR = fR$ and so $aR = bR$.

(ii) $\Rightarrow$ (iii): Since $I \cong bR \Rightarrow I = aR$ for some $a \in R$, the result is clear.

(iii) $\Rightarrow$ (iv): Any complemented one-sided ideal is principal in a ring with unity.

(iv) $\Rightarrow$ (v): Take $J = R$ in (iv).

(v) $\Rightarrow$ (vi): Clear with $aR = I$.

(vi) $\Rightarrow$ (i): If $ab=1$ then $Rb=R \Rightarrow bR \cong R$. By (vi) $bR=R$, implying that b is a unit with a as inverse.

(ii) $\Rightarrow$ (vii): Let $\phi\colon aR \oplus I \longrightarrow cR$ be the isomorphism. Then $\phi(a, 0) = cx$ for some $x \in R$, $\phi(0, I) = I_1 \subseteqq cR$ and $\phi(aR, 0) = cxR$. Hence $aR \cong cxR \subseteqq aR$ and so by (iii) $aR = cxR \subseteqq cR \subseteqq aR$. Now $I_1 = aR \cap I_1 = cxR \cap I_1 = \phi(aR, 0) \cap \phi(0, I) = \phi[(aR, 0) \cap (0, I)] = \phi(0, 0) = (0)$. Hence $I \cong I_1 = (0)$ and so $I = (0)$ as desired.

(vii) $\Rightarrow$ (viii): Trivial.

(viii) $\Leftrightarrow$ (ix): By (1-3) $\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} \sim \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \Leftrightarrow eR \oplus fR \cong eR$ and hence setting $e = aa^+$ and $f = bb^+$ yields the desired equivalence.

(viii) $\Rightarrow$ (x): External cancellation laws always imply the corresponding internal cancellation laws since the directness of the sum $aR \dotplus bR$ implies that $aR \dotplus bR \cong aR \oplus bR$.

(x) $\Rightarrow$ (i): Let $e^2 = e$, then $eR \cong R = eR \dotplus (1 - e)R$ implies by (x) that $e = 1$, ensuring by Theorem 1 that R is finite.

(ii) $\Rightarrow$ (xi): If $Ra^k = Ra^{k+1}$ then $a^kR \cong a^{k+1}R \subseteqq a^kR$ and hence $a^kR = a^{k+1}R$ guaranteeing that $(a^k)^\#$ exists.

(xi) $\Rightarrow$ (i): This was shown in Proposition 2.

The left analogue follow by left-right symmetry.

COROLLARY 7. *Over a finite regular ring* $aR \not\cong R \oplus R$, *and* $aR \cong aR \oplus aR$ *implies* $a = 0$. *In particular* $R \not\cong R \oplus R$.

REMARK. In Example 2 we shall see that the isomorphic inclusion law for right ideals:

$$I \subseteqq J, I \cong J \Longrightarrow I = J \, , \tag{3.4}$$

will not be true in general in a finite regular ring, if J is not a principal ideal.

PROPOSITION 6. *Let* R *be a finite regular ring. Then*

(i) $aR \subseteqq baR \Rightarrow aR = baR$ *and* $Ra = Rba$

(iii) $Ra \subseteqq Rab \Rightarrow Ra = Rab$ *and* $aR = abR$.

Proof. (i). $aR \subseteqq baR \Rightarrow a = ba(ba)^+a \Rightarrow Ra = Rba(ba)^+a \subseteqq Ra(ba)^+a \subseteqq Ra$. Hence by Proposition 4, $aR \cong a(ba)^+aR \subseteqq aR$ and thus again by (3.3) $aR = a(ba)^+aR \Rightarrow baR = ba(ba)^+aR = aR$. Lastly, $Rba \cong Ra \supseteqq Rba$ and so by (3.3) $Rba = Ra$. Part (ii) follows by symmetry. We note in passing that Proposition 5 applies and that, for example, in (i) $aR = b^n aR$, $Ra = Rb^n a$ for all $n = 1, 2, \cdots$.

COROLLARY 8. *In a finite regular ring* R

(i) $aR \subseteqq bacR \Rightarrow aR = bacR$ *and* $Rac = Rbac$.

(ii) $Ra \subseteqq Rcab \Rightarrow Ra = Rcab$ *and* $caR = cabR$.

PROPOSITION 7. *Let* R *be a subring of a finite regular ring* R^* *and let* $a \in R$. *Then*

$$aR = a^2R \Longrightarrow Ra = Ra^2 \, .$$

Proof. Since $aR = a^2R$, there exists $x \in R$, such that $a = a^2x$ and so $aR^* = a^2R^*$. Now R^* is finite regular and thus by (3.3) $R^*a = R^*a^2$, which implies that $a = ya^2$ for some $y \in R^*$. Hence $ya = y(a^2x) = (ya^2)x = ax \in R$. Now let $b = ya = ax$, then $yb = yax = bx \in R$ and also $ba = ya^2 = a$. Thus $(yb)a^2 = ya^2 = a$ or $Ra^2 = Ra$ as desired.

COROLLARY 9. *If* R *is a subring of a finite regular ring and* $a \in R$, *then*

$$a^{k+1}R = a^kR \Longrightarrow Ra^{k+1} = Ra^k \, .$$

Proof. Since $a^{k+1}R = a^kR \Leftrightarrow a^{2k}R = a^kR$, and $R = a^{k+1} = Ra^k \Leftrightarrow Ra^{2k} = Ra^k$, we may apply Proposition 7 to a^k.

It is clear from Corollaries 1 and 3 and the identity $eae[e(eae)^{-}e]eae = eae$, that if R is finite *regular*, then so is the corner ring eRe, where $e^2 = e \in R$. This should be compared with the following result by Kaplansky (unpublished), which states that eRe inherits the unit regularity from a ring R. We add the proof for completeness. [W. Desch, private communication].

PROPOSITION 8 (Kaplansky). *Let R be a unit regular ring and e be an idempotent element in R. Then eRe is unit regular.*

Proof. Let ere be an arbitrary element in eRe and $u = (ere + 1 - e)^{-}$ be a unit. Since $(1-e)u(1-e) = 1-e$, $ereu(1-e) = 0$, $(1-e)uere = 0$, $eu(1-e) = u(1-e) - (1-e)$ and $(1-e)ue = (1-e)(u-1)$, we have $ere(e(u - u(1-e)u)e)ere = ere$ and $e(u - u(1-e)u)e \cdot eu^{-1}e = e = eu^{-1}e \cdot e(u - u(1-e)u)e$.

Related to the corner ring is the following well-known result which generalizes a result of [15].

PROPOSITION 9. *Let M be a right unital A-modules where A is a ring with unity 1, and let $R = \mathrm{End}_A(M)$. If $e^2 = e \in R$ and $E = \mathrm{End}_A(eM)$ then $eRe \cong E$.*

It should be observed here with aid of Corollary 4, that if u is a unit in a *regular* ring R and $e = e^2 \in R$, then eue is a unit in eRe exactly when $(1-e)u^{-1}(1-e)$ is a unit in $(1-e)R(1-e)$. Indeed, from (3.2) we see that

$$\begin{aligned} eueR = eR &\Longleftrightarrow ueR + (1-e)R = R \Longleftrightarrow eR + u^{-1}(1-e)R = R \\ &\Longleftrightarrow (1-e)u^{-1}(1-e)R = (1-e)R\,. \end{aligned} \tag{3.5}$$

4. Cancellation laws. We begin by defining four strong cancellation laws for internal direct sums of principal right ideals. Let R be a ring with unity and let $a, b, c, d \in R$. We define,

$$\begin{aligned}
&C_0^{\text{in}}\colon && aR \dotplus bR = cR \dotplus dR,\ aR \cong cR \Longrightarrow bR \cong dR \\
&C_1^{\text{in}}\colon && aR \dotplus bR \cong cR \dotplus dR,\ aR \cong cR \Longrightarrow bR \cong dR \\
&C_2^{\text{in}}\colon && aR \dotplus bR \cong aR \dotplus dR \qquad\qquad\quad \Longrightarrow bR \cong dR \\
&C_3^{\text{in}}\colon && aR \dotplus bR \cong aR \qquad\qquad\qquad\qquad \Longrightarrow bR = (0)\,.
\end{aligned}$$

There are two common ways of weakening these laws. We may define for a fixed $g \in R$,

$$\begin{aligned}
&C_{0'}^{\text{in}}(g)\colon && aR \dotplus bR = gR = cR \dotplus dR,\ aR \cong cR && \Longrightarrow bR \cong dR\\
&C_{1'}^{\text{in}}(g)\colon && aR \dotplus bR \cong gR \cong cR \dotplus dR,\ aR \cong cR && \Longrightarrow bR \cong dR\\
&C_{2'}^{\text{in}}(g)\colon && aR \dotplus bR \cong gR \cong aR \dotplus dR && \Longrightarrow bR \cong dR\\
&C_{3'}^{\text{in}}(g)\colon && aR \dotplus bR \cong gR \cong aR && \Longrightarrow bR = (0)\,,
\end{aligned}$$

and

$$\begin{aligned}
&C_{0''}^{\text{in}}(g)\colon && aR \dotplus bR = cR \dotplus dR,\ aR \cong gR \cong cR && \Longrightarrow bR \cong dR\\
&C_{1''}^{\text{in}}(g)\colon && aR \dotplus bR \cong cR \dotplus dR,\ aR \cong gR \cong cR && \Longrightarrow bR \cong dR\\
&C_{2''}^{\text{in}}(g)\colon && aR \dotplus bR \cong aR \dotplus dR,\ aR \cong gR && \Longrightarrow bR \cong dR\\
&C_{3''}^{\text{in}}(g)\colon && aR \dotplus bR \cong aR\,, \qquad aR \cong gR && \Longrightarrow bR = (0)\,.
\end{aligned}$$

These laws may be considered as *local* cancellation laws, specifying a property of element a, or element g or be considered as a *global* cancellation law, (holding for all a) specifying a property of the ring R. The case $g = 1$, of course, being of special interest. For any fixed g it is clear that locally, and hence globally, $C_{3'}^{\text{in}}$ and $C_{3''}^{\text{in}}$ coincide, while $C_{k''}^{\text{in}} \Leftarrow C_{k''}^{\text{in}} \Rightarrow C_{k'}^{\text{in}}$ for $k = 0, 1, 2, 3$. Moreover,

$$C_0^{\text{in}} \Longleftrightarrow C_1^{\text{in}} \Longrightarrow C_2^{\text{in}} \Longrightarrow C_3^{\text{in}}\,. \tag{4.1}$$

In fact, if C_0^{in} holds and $\phi\colon aR \dotplus bR \to cR \dotplus dR$ is an isomorphism then $\phi(a)R \dotplus \phi(b)R = cR \dotplus dR$, and so if $aR \cong cR$ then $\phi(a)R \cong cR$ ensuring that $bR \cong \phi(b)R \cong dR$. Analogous implications hold for the weaker cancellation laws.

We may similarly define the corresponding local or global cancellation laws for *external* direct sums of principal right ideals:

$$\begin{aligned}
&C_1^{\text{ex}}\colon aR \oplus bR \cong cR \oplus dR,\ aR \cong cR && \Longrightarrow bR \cong dR\\
&C_2^{\text{ex}}\colon aR \oplus bR \cong aR \oplus dR\,, && \Longrightarrow bR \cong dR\\
&C_3^{\text{ex}}\colon aR \oplus bR \cong aR && \Longrightarrow bR = (0)
\end{aligned}$$

$$\begin{aligned}
&C_{1'}^{\text{ex}}(g)\colon aR \oplus bR \cong gR \cong cR \oplus dR,\ aR \cong cR && \Longrightarrow bR \cong dR\\
&C_{2'}^{\text{ex}}(g)\colon aR \oplus bR \cong gR \cong aR \oplus dR\,, && \Longrightarrow bR \cong dR\\
&C_{3'}^{\text{ex}}(g)\colon aR \oplus bR \cong gR \cong aR && \Longrightarrow bR = (0)
\end{aligned}$$

and

$$\begin{aligned}
&C_{1''}^{\text{ex}}(g)\colon aR \oplus bR \cong cR \oplus dR,\ aR \cong gR \cong cR && \Longrightarrow bR \cong dR\\
&C_{2''}^{\text{ex}}(g)\colon aR \oplus bR \cong aR \oplus dR,\ aR \cong gR && \Longrightarrow bR \cong dR\\
&C_{3''}^{\text{ex}}(g)\colon aR \oplus bR \cong aR\,, \qquad aR \cong gR && \Longrightarrow bR = (0)\,.
\end{aligned}$$

We note that C_0^{ex} and $C_{0''}^{\text{ex}}$ are trivial, while $C_{0'}^{\text{ex}}$ is impossible. Indeed, if $aR \oplus bR = cR \oplus dR$ then $aR = cR$ and $bR = dR$. It is easily seen that locally, and hence globally,

$$C_1^{\mathrm{ex}} \Longleftrightarrow C_2^{\mathrm{ex}} \Longrightarrow C_3^{\mathrm{ex}} , \tag{4.2}$$

with an analogous result for the weak external cancellation laws.

It is always true that the *external* cancellation laws imply the corresponding *internal* cancellation laws, but the converse may not be true in general. When R is regular we may of course replace the ring elements in these laws by idempotents. By analogy to the above, we may define local and global internal and external cancellation laws for right ideals, and for right R-modules. For example,

$$\begin{aligned} &\mathfrak{C}_1^{\mathrm{ex}}\colon A \oplus B \cong C \oplus D,\, A \cong C &&\Longrightarrow B \cong D\,, \\ &\mathfrak{C}_{0''}^{\mathrm{in}}(M)\colon A \dotplus B = C \dotplus D,\, A \cong M \cong C &&\Longrightarrow B \cong D \end{aligned}$$

and

$$\mathfrak{C}_{0'}^{\mathrm{in}}(M)\colon A \dotplus B = M = C \dotplus D,\, A \cong C \Longrightarrow B \cong D\,.$$

The latter type of cancellation law was used by Ehrlich [4] and reduces to $C_{0'}^{\mathrm{in}}(1)$, when applied to complemented (and hence principal) ideals in R.

If we consider $\mathfrak{C}_{0''}^{\mathrm{in}}(M)$ as a condition *on* the module M then it is implied by Fuchs' substitution property for right R-modules, which states that the right R-module M obeys the substitution property if

$$A \dotplus B = C \dotplus D \quad \text{and} \quad A \cong M \cong C\,,$$

implies that for suitable module E,

$$A \dotplus B = E \dotplus B = E \dotplus D\,.$$

Also, when considered as local conditions on $A = M$, it follows that

$$\mathfrak{C}_i^{\mathrm{ex}} \Longleftrightarrow \mathfrak{C}_{i''}^{\mathrm{ex}}(M) \quad i = 1, 2, 3\,. \tag{4.3}$$

Before turning to our main comparison between finite and unit regular rings, let us first examine some of the interdependence between the above cancellation laws. First, it is clear that each cancellation law for modules implies the corresponding one for ideals which in turn implies the one for principal ideals.

By analogy to the above, it is easily seen that for right R-modules

$$\mathfrak{C}_{i'} \Longleftarrow \mathfrak{C}_i \Longrightarrow \mathfrak{C}_{i''}\,, \tag{4.4}$$

for internal as well as external cancellation laws for all possible i. Also,

$$\mathfrak{C}_0^{\mathrm{in}} \Longleftrightarrow \mathfrak{C}_1^{\mathrm{in}} \Longrightarrow \mathfrak{C}_2^{\mathrm{in}} \Longrightarrow \mathfrak{C}_3^{\mathrm{in}} \tag{4.5}$$

and

$$(4.6) \qquad \mathfrak{C}_1^{\mathrm{ex}} \Longleftrightarrow \mathfrak{C}_2^{\mathrm{ex}} \Longleftrightarrow \mathfrak{C}_3^{\mathrm{ex}} ,$$

with analogous implications for $\mathfrak{C}_{i'}$ and $\mathfrak{C}_{i''}$, and again $\mathfrak{C}_{3'}$ and $\mathfrak{C}_{3''}$ being identical.

As mentioned earlier the internal laws for, say (principal) ideals, may not imply the corresponding external law. We do have however that for modules,

$$(.47) \qquad \mathfrak{C}_1^{\mathrm{in}} \Longleftrightarrow \mathfrak{C}_1^{\mathrm{ex}} ,$$

with analogous results for $\mathfrak{C}_{i'}$ and $\mathfrak{C}_{i''}$. It is exactly the equivalence of the internal and external laws which appears naturally in the study of regular rings. Lastly, if we take $M = gR$ in the cancellation laws $\mathfrak{C}_{i'}$, $i = 1, 2, 3$, then because of the much stronger condition we may conclude that

$$(4.8) \qquad \mathfrak{C}_{i'}^{\mathrm{in}}(gR) \Longleftrightarrow \mathfrak{C}_{i'}^{\mathrm{ex}}(gR) \Longleftrightarrow C_{i'}^{\mathrm{ex}}(g) \Longleftrightarrow C_{i'}^{\mathrm{in}}(g) \quad i = 1, 2, 3 .$$

Indeed if, say, $i = 1$ all that is needed is that $C_{1'}^{\mathrm{in}}(g) \Rightarrow \mathfrak{C}_{1'}^{\mathrm{ex}}(gR)$.

Therefore, let $A \oplus B \cong gR \cong C \oplus D$, $A \cong C$, where A, B, C, D are right R-modules. If $\phi: A \oplus B \to gR$ and $\psi: C \oplus D \to gR$ are the isomorphism, then there is $a \in A$, $b \in B$, such that $\phi(a, 0) + \phi(0, b) = \phi(a, b) = g$ and hence $gR = \phi(a, 0)R + \phi(0, b)R$ with $\phi(a, 0)R \cong A$, $\phi(0, b)R \cong B$. Likewise, there are $c \in C$, $d \in D$ such that

$$gR = \psi(c, 0)R + \psi(0, d)R \qquad \psi(c, 0)R \cong C, \psi(0, d)R \cong D .$$

Since $\phi(a, 0)R \cong \psi(c, 0)R$, it follows by $C_{1'}^{\mathrm{in}}(g)$ that $C \cong \phi(0, b)R \cong \psi(0, d)R \cong D$.

By analogy to the substitution property of [5] it can be shown that the cancellation law $\mathfrak{C}_1$ is inherited by internal as well as external direct sums of modules obeying these laws.

We have now arrived at the following relationship between finiteness, unit regularity and these cancellation laws.

THEOREM 3. *Let R be a regular ring with unity* 1, *and let e_i, f_i, $i = 1, 2$ be idempotents in R.*

(α) *The following are equivalent.*
1. *R is unit regular.*
2. C_1^{ex}
3. $\begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} \sim \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}, e_1 \sim f_1 \Longrightarrow e_2 \sim f_2 .$
4. C_1^{in}
5. $C_{1'}^{\mathrm{in}}(1)$
6. $C_{0'}^{\mathrm{in}}(1)$
7. $C_{1'}^{\mathrm{ex}}(1)$

8. $\mathfrak{C}^{\mathrm{ex}}_{1'}(R)$
9. $\mathfrak{C}^{\mathrm{in}}_{1'}(R)$.

(β) *The following are equivalent.*
1. *R is finite.*
2. C^{ex}_{3}
3. $\begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} \sim \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix},\ e_1 \sim f_1 \Longrightarrow e_2 = 0$
4. C^{in}_{3}
5. $C^{\mathrm{in}}_{3'}(1)$
6. $C^{\mathrm{ex}}_{3'}(1)$
7. $\mathfrak{C}^{\mathrm{ex}}_{3'}(R)$
8. $\mathfrak{C}^{\mathrm{in}}_{3'}(R)$.

Proof. (α). (1) $\Leftrightarrow$ (2). This has been proven by Handelman [8]. Alternatively, the unit regularity of $R_{2\times 2}$ could be used to prove this.

(2) $\Leftrightarrow$ (3). This follows from (1.3) on rewriting C^{ex}_{1} using idempotents.

(2) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Leftrightarrow$ (6). This always holds.

(6) $\Rightarrow$ (1). This follows from the result of Ehrlich [4] Theorem 2, which states that if $R = \mathrm{End}_A(M)$ is a regular ring, where M is a right A-module, and $1 \in A$, then R is unit regular exactly when

$$M = A_1 \dotplus H = A_2 \dotplus K,\ A_1 \cong A_2 \Longrightarrow H \cong K\,.$$

Since regular rings are faithful, this includes the cancellation law $\mathfrak{C}^{\mathrm{in}}_{0'}(R)$:

$$R = I_1 \dotplus J_1 = I_2 \dotplus J_2,\ I_1 \cong I_2 \Longrightarrow J_1 \cong J_2$$

for complemented right ideals, which reduces to $C^{\mathrm{in}}_{0'}(1)$. An alternative proof is obtained from Theorem 4 and Corollary 1 of [5]. Indeed, if $R/bR \cong R/dR$ and we pick a, c such that $aR \dotplus bR = R = cR \dotplus dR$, then $aR \cong cR$. Hence by $C^{\mathrm{in}}_{0'}(1)$, $bR \cong dR$, which by Corollary 1 of [5] ensures that R is unit regular.

The equivalence of the latter four parts was established in (4.8).

(β) The equivalence of (1)-(4) is contained in Theorem 2, while the equivalence of (5)-(8) follows from (4.8). Lastly, if $C^{\mathrm{in}}_{3'}$ holds and $eR \cong R = eR \dotplus (1-e)R$, then $e = 1$ ensuring that R is finite.

REMARK. In his paper [8], Handelman actually showed that unit regularity is equivalent to the external cancellation law $\mathfrak{C}^{\mathrm{ex}}_{1}$ for finitely generated projective modules. It is an open question whether *finite* regularity is equivalent to $\mathfrak{C}^{\mathrm{ex}}_{3}$ for modules of this type.

We close this section with a counterexample showing that the isomorphism inclusion law (3.4) as well as the external cancellation law $\mathfrak{C}_1^{ex}$ may *not* be valid for non-principal right ideals in a unit regular ring.

EXAMPLE 2. Let R be a unit regular ring and let

$$S = \left\{ M \in R_{\infty\times\infty};\ M = \left[\begin{array}{c|ccc} A & & 0 & \\ \hline & d & & \\ & & d & \\ 0 & & & \ddots \end{array}\right],\ A \in R_{n\times n},\ n \geqq 0 \right\},$$

be the ring of all infinite matrices over R which consist of a direct sum of a finite matrix A and an infinite scalar matrix $D = \begin{bmatrix} d & & 0 \\ & d & \\ & & \ddots \\ 0 & & & \ddots \end{bmatrix}$. It is easily verified that S is again a unit regular ring. Now let $J = \left\{ M \in S;\ M = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix},\ A \in R_{n\times n},\ n \geqq 0 \right\}$, be the ideal of all matrices in S that have zero tail and let $a = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 0 & \\ \vdots & & \ddots \end{bmatrix}$ and $I = HJ$, where $H = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & \ddots & \\ & & 1 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}$ is the infinite row-shift matrix. Then I and J are right ideals in S such that $I \subsetneqq J$, $I \cong J$(as S-modules) $I \neq J$ and $aR + I = J$, $I \cong J$, $aR \neq (0)$.

5. Matrices over regular rings. In this section we shall give necessary and sufficient conditions for the matrix ring $R_{2\times 2}$ to be finite regular or unit regular. We shall need some preliminary results dealing with 2×2 matrices over a regular ring R. If $b = paq$, with p and q invertible, then the *sets* of reflexive inverses $\{b^+\}$, $\{a^+\}$ are related through $\{b^+\} = q^{-1}\{a^+\}p^{-1}$. Thus each reflexive inverse b^+ is of the form $q^{-1}a^+p^{-1}$ for *some* $a^+ \in \{a^+\}$. Hence $bb^+ = paa^+p^{-1}$ for *some* a^+, but this is not necessarily true for *all* aa^+. We do have, however, that $b = paq \Rightarrow bR = paR \Rightarrow bb^+R = bR \cong aR = aa^+R \Rightarrow bb^+ \sim aa^+$, for *all* choices of a^+ and b^+. It is this "decoupling" of the idempotent bb^+ from aa^+, this allows us to select suitable b^+ and a^+ *independently* from each other.

If $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is an arbitrary matrix in the regular ring $R_{2\times 2}$, then it can be shown that [10] there exists a reflexive inverse M^+ of M such that *simultaneously*

$$MM^+ = \begin{bmatrix} e_1 & 0 \\ \alpha & e_2 \end{bmatrix},\ M^+M = \begin{bmatrix} f_1 & \beta \\ 0 & f_2 \end{bmatrix}, \tag{5.1}$$

where

$$\begin{aligned} e_1 &= aa^+ + uu^+(1 - aa^+),\ e_2 = ss^+(1 - vv^+) + vv^+ \\ f_1 &= a^+a + (1 - a^+a)v^+v,\ f_2 = u^+u + (1 - u^+u)s^+s \\ \alpha &= (1 - ss^+)(1 - vv^+)[ba^+ + zu^+(1 - aa^+)] \\ \beta &= [a^+c + (1 - a^+a)v^+z](1 - u^+u)(1 - s^+s) \end{aligned} \tag{5.2}$$

and

$$u = (1 - aa^+)c,\ v = b(1 - a^+a),\ z = d - ba^+c$$

with $s = (1 - vv^+)z(1 - u^+u)$. In these expressions a^+ is arbitrary but the same choice of a^+ must be used throughtout. Moreover (5.1)-(5.2) remain valid if $(\cdot)^+$ is replaced by $(\cdot)^-$ throughout. Since MM^+ and M^+M are triangular idempotents they are similar to diagonal matrices. Indeed, for example

$$\begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix} = \begin{bmatrix} 1 & f_1\beta - \beta f_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} f_1 & \beta \\ 0 & f_2 \end{bmatrix}\begin{bmatrix} 1 & \beta f_2 - f_1\beta \\ 0 & 1 \end{bmatrix}. \tag{5.3}$$

We thus may state

PROPOSITION 10. *If* $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in R_{2\times 2}$, *where* R *is a regular ring, then*

$$MR_{2\times 2} \cong \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} R_{2\times 2} \tag{5.4}$$

for suitable idempotents e_1, e_2 *given by* (5.2).

COROLLARY 10. *If* $E^2 = E \in R_{2\times 2}$, *where* R *is a regular ring, then* $E \sim \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}$ *for suitable idempotents* e_1 *and* e_2 *in* R.

Because (5.1)-(5.3) also hold for matrices of the form $M = \begin{bmatrix} A & C \\ B & D \end{bmatrix} \in R_{n\times n}$, where $A \in R_{p\times p}$ and $D \in R_{q\times q}$, with $p + q = n$, we may extend these results by induction to $R_{n\times n}$. In particular, if $M = \left[\begin{array}{c:c} a & \boldsymbol{c}^T \\ \hdashline \boldsymbol{b} & D \end{array}\right]$, where $a \in R$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are columns in R^{n-1} and $D \in R_{(n-1)\times(n-1)}$, then there exists an M^+ such that $MM^+ \approx \begin{bmatrix} e_1 & 0 \\ 0 & E_1 \end{bmatrix}$ for suitable idempotents $e_1 \in R$ and $E_1 \in R_{(n-1)\times(n-1)}$. Hence by Lemma 1, $MR^n \cong$

$\begin{bmatrix} e_1R \\ E_1R^{n-1} \end{bmatrix} \cong e_1R \oplus E_1R^{n-1}$, and so we may now repeat the reduction with E_1. By induction we obtain $MR^n \cong e_1R \oplus \cdots \oplus e_nR$ for suitable idempotents $e_i \in R$, $i = 1, \cdots, n$, and thus we have

COROLLARY 11. *If $M \in R_{n\times n}$, where R is a regular ring, then*

$$MR_{n\times n} \cong \begin{bmatrix} e_1 & & & 0 \\ & e_2 & & \\ & & \ddots & \\ 0 & & & e_n \end{bmatrix} R_{n\times n} ,$$

for suitable idempotents $e_i \in R$, $i = 1, \cdots, n$. In particular if $E^2 = E \in R_{n\times n}$ then $E \sim \begin{bmatrix} e_1 & & 0 \\ & \ddots & \\ 0 & & e_n \end{bmatrix}$ for some $e_i^2 = e_i \in R$, $i = 1, \cdots, n$.

We now recall from Theorem 2, that if R is regular then $R_{2\times 2}$ is finite regular if and only if $MR_{2\times 2} \cong R_{2\times 2} \Rightarrow MR_{2\times 2} = R_{2\times 2}$. So if we let $E = MM^+$ be given by (5.1), then $ER_{2\times 2} = MR_{2\times 2} \cong I_2R_{2\times 2}$ where I_n is the identity matrix in $R_{n\times n}$. From (1.2) we see that this holds exactly when $\begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} R^2 \cong R^2$, or

$$e_1R \oplus e_2R \cong R \oplus R . \tag{5.5}$$

Since $R_{2\times 2}$ is regular exactly when R is, we have proven the following.

THEOREM 4. *If R is a regular ring with unity* 1, *then $R_{2\times 2}$ is finite if and only if for any two idempotents $e_1, e_2 \in R$,*

$$e_1R \oplus e_2R \cong R \oplus R \Longrightarrow e_1 = e_2 = 1 . \tag{5.6}$$

It is well-known that finiteness for $R_{2\times 2}$ implies that of R. Let us now examine the converse question for regular rings. We begin by noting that

$$\begin{aligned} MX = I \text{ for some } X \in R_{2\times 2} &\Longleftrightarrow MM^- = I \text{ for all } M^- \\ &\Longleftrightarrow MM^+ = I \text{ for some } M^+ , \end{aligned} \tag{5.7}$$

and hence using (5.1)-(5.2) we see that

$$\begin{aligned} MX = I \text{ for some } X \in R_{2\times 2} \Longleftrightarrow (1 - uu^+)(1 - aa^+) &= 0 , \\ (1 - ss^+)(1 - vv^+) &= 0 \end{aligned} \tag{5.8}$$

while

$$\begin{aligned} YM = I \text{ for some } Y \in R_{2\times 2} \Longleftrightarrow (1 - a^+a)(1 - v^+v) &= 0 , \\ (1 - u^+u)(1 - s^+s) &= 0 . \end{aligned} \tag{5.9}$$

Thus $R_{2\times 2}$ is finite if and only if these are equivalent. We shall show now that if R is regular, then $R_{2\times 2}$ will not be finite if there exist elements a and x in R, with a not unit regular, such that

$$R = aR + x(a^\circ)\ . \tag{5.10}$$

We begin by noting that (5.8) and (5.9) are, with aid of regularity, equivalent to

$$\begin{array}{ll} \text{(i)} & uR = (1 - aa^+)R \\ \text{(ii)} & sR = (1 - vv^+)R \\ \text{(iii)} & Rv = R(1 - a^+a) \\ \text{(iv)} & Rs = R(1 - u^+u)\ . \end{array} \tag{5.11}$$

Suppose we select $c = 1$, $d = 1 + ba^+c$ and $b = (1 - aa^+)x$. Then $z = 1$ and $u = 1 - aa^+$ is idempotent so that (i) holds. Now if (5.10) holds, then by (3.2)

$$uR = (1 - aa^+)R = (1 - aa^+)x(1 - a^+a)R = vR\ . \tag{5.12}$$

Hence, by regularity $(1 - vv^+)u = 0$ so that $s = (1 - vv^+)(1 - u) = 1 - vv^+$, if we pick $u^+ = u$. It now follows that (ii) and (iv) also hold. Indeed $Rs = R(1 - vv^+) = {}^\circ v = {}^\circ u = R(1 - u)$. Now consider the last identity $Rv = R(1 - a^+a)$. This implies that

$$(1 - aa^+)R = vR \cong (1 - a^+a)R\ ,$$

and so $aa^+ \approx a^+a$ since always $aa^+ \sim a^+a$. But the latter is locally equivalent to a being unit regular [12], which is excluded by assumption. The structure of (5.10) should be compared with the representation $R = aR \dotplus u(a^\circ)$, u a unit which is equivalent to a being unit regular [12]. We note in passing that none of the "obvious" choices for x seem to work. For example, $x = 1$ implies that $a^2R = aR$, which if R is finite, ensures that $a^\#$ exists so that a is unit regular. Let us now turn to unit regular rings.

THEOREM 5. *If R is a regular ring with unity then $R_{2\times 2}$ is unit regular exactly when for any idempotents $e_1, e_2, f_1, f_2 \in R$,*

$$\begin{aligned} e_1R \oplus e_2R \cong f_1R \oplus f_2R \Longrightarrow & (1 - e_1)R \oplus (1 - e_2)R \\ & \cong (1 - f_1)R \oplus (1 - f_2)R\ . \end{aligned} \tag{5.13}$$

Proof. That (5.13) is necessary is an immediate consequence of a result of Handelman (see [8, Theorem 2]).

Now suppose (5.13) holds, and that $E^2 = E \sim F = F^2$. By (5.1) there exist $1 - 2$ inverses E^+, F^+ such that

$$EE^+ \approx \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}, \; FF^+ \approx \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}.$$

Hence by (1.1)-ii,

$$I - EE^+ \sim \begin{bmatrix} 1 - e_1 & 0 \\ 0 & 1 - e_2 \end{bmatrix} \quad \text{and} \quad I - FF^+ \sim \begin{bmatrix} 1 - f_1 & 0 \\ 0 & 1 - f_2 \end{bmatrix}.$$

But now we may apply (3.1)-ii with $E^- = E$ and $E^= = E^+$ to show that

$$I - E \sim I - EE^+ \sim \begin{bmatrix} 1 - e_1 & 0 \\ 0 & 1 - e_2 \end{bmatrix}, \; I - F \sim \begin{bmatrix} 1 - f_1 & 0 \\ 0 & 1 - f_2 \end{bmatrix}.$$

Consequently if $E \sim F$, then

$$ER_{2\times 2} = EE^+R_{2\times 2} \cong \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} R_{2\times 2} \cong \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix} R_{2\times 2} \cong FR_{2\times 2},$$

and so by (1.3) $e_1R \oplus e_2R \cong f_1R \oplus f_2R$. Using (5.13) and (1.3) once more yields $\begin{bmatrix} 1 - e_1 & 0 \\ 0 & 1 - e_2 \end{bmatrix} \sim \begin{bmatrix} 1 - f_1 & 0 \\ 0 & 1 - f_2 \end{bmatrix}$ as desired. This result is easily seen to generalize to $R_{n\times n}$, with aid of Corollary 8.

COROLLARY 12. *If R is unit regular then* (5.13) *holds.*

We may also use Theorem 3α to characterize the unit regularity of $R_{2\times 2}$.

THEOREM 6. *If R is a regular ring then $R_{2\times 2}$ is unit regular if and only if for any idempotents $e_i, f_i, g_i, h_i \in R$, $i = 1, 2$,*

$$\left.\begin{array}{l} e_1R \oplus e_2R \oplus f_1R \oplus f_2R \cong g_1R \oplus g_2R \oplus h_1R \oplus h_2R \\ e_1R \oplus e_2R \cong g_1R \oplus g_2R \end{array}\right\} \Longrightarrow f_1R \oplus f_2R \cong h_1R \oplus h_2R. \tag{5.14}$$

Proof. By Theorem 3α, $R_{2\times 2}$ is unit regular if and only if for all $A, B, C, D \in R_{2\times 2}$ such that

$$AR_{2\times 2} \oplus BR_{2\times 2} \cong CR_{2\times 2} \oplus DR_{2\times 2} \quad \text{and} \quad AR_{2\times 2} \cong CR_{2\times 2}, \tag{5.15}$$

it follows that $BR_{2\times 2} \cong DR_{2\times 2}$. Suppose now that (5.14) and (5.15) hold. Then by (5.4) there exist e_i, f_i, g_i and h_i, $i = 1, 2$ such that

$$AR_{2\times 2} \cong \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} R_{2\times 2} \cong \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} R_{2\times 2} \cong CR_{2\times 2}, \quad \text{and}$$

$$\begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} R_{2\times 2} \oplus \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix} R_{2\times 2} \cong \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} R_{2\times 2} \oplus \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} R_{2\times 2}.$$

With aid of Lemma 1 this implies that $e_1R \oplus e_2R \cong g_1R \oplus g_2R$ and $e_1R \oplus e_2R \oplus f_1R \oplus f_2R \cong g_1R \oplus g_2R \oplus h_1R \oplus h_2R$. Hence by (5.14) $\begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix} R^2 \cong \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} R^2$ which by Lemma 1 ensures that

$$BR_{2\times 2} \cong \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix} R_{2\times 2} \cong \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} R_{2\times 2} \cong DR_{2\times 2} .$$

Conversely, if $R_{2\times 2}$ is unit regular, it suffices to take

$$A = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}, B = \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}, C = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \text{ and } D = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix}$$

and apply Theorem 3α and Lemma 1.

REMARK. We may replace the idempotents in Theorem 6 by arbitrary regular ring elements.

Our last result will be the converse of a theorem by Henriksen [13].

THEOREM 7. *If $R_{n\times n}$ is unit-regular for some $n \geqq 1$, then R is unit regular.*

Proof. It is well-known that if $R_{n\times n}$ is regular with identity for some $n \geqq 1$, then so is R. So let $e \sim f$ with $e = p\hat{p}$, $f = p\hat{p}$ and $p \in eRf$, $\hat{p} \in fRe$. Then $E = \begin{bmatrix} e & 0 \\ 0 & I_{n-1} \end{bmatrix} = \begin{bmatrix} p & 0 \\ 0 & I_{n-1} \end{bmatrix} \begin{bmatrix} \hat{p} & 0 \\ 0 & I_{n-1} \end{bmatrix} = P\hat{P}$ and $F = \begin{bmatrix} f & 0 \\ 0 & I_{n-1} \end{bmatrix} = \begin{bmatrix} \hat{p} & 0 \\ 0 & I_{n-1} \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & I_{n-1} \end{bmatrix} = \hat{P}P$ where $P \in ER_{n\times n}F = \begin{bmatrix} eRf & ? \\ ? & ? \end{bmatrix}$ and $\hat{P} \in FR_{n\times n}E = \begin{bmatrix} fRe & ? \\ ? & ? \end{bmatrix}$. Hence $E \sim F$ and so by the unit regularity of $R_{n\times n}$, $I - E \sim I - F$. That is,

$$\begin{bmatrix} 1-e & 0 \\ 0 & 0 \end{bmatrix} R_{n\times n} \cong \begin{bmatrix} 1-f & 0 \\ 0 & 0 \end{bmatrix} R_{n\times n}$$

as $R_{n\times n}$-modules, which implies by (1.2) that

$$\begin{bmatrix} (1-e)R \\ 0 \end{bmatrix} = \begin{bmatrix} 1-e & 0 \\ 0 & 0 \end{bmatrix} R^n \cong \begin{bmatrix} 1-f & 0 \\ 0 & 0 \end{bmatrix} R^n = \begin{bmatrix} (1-f)R \\ 0 \end{bmatrix}$$

as R-modules.

Hence $(1-e) \sim (1-f)$ as desired.

We remark that this result may also be obtained from Theorem 5 extended to $R_{n\times n}$, which is far less transparent however.

6. **Conclusions.** We have compared some of the properties of finite regular and unit regular rings, and have shown that both are

closely related to the study of generalised inverses on the one hand, and a study of cancellation laws on the other. We have seen that the finiteness property may often replace the concept of rank (cf. Prop. 6) in matrix calculations, and that the finiteness of $R_{2\times 2}$ depends, for a regular ring on the existence of certain nonunit regular elements.

It will be of interest to study the semigroup and subgroup structure of finite regular and unit regular rings.

We conclude with several open problems.

1. If R is finite does $a^2R = aR \Rightarrow Ra^2 = Ra$?
2. Does C_1^{in} imply regularity?
3. Does $C_1^{\mathrm{in}} \Rightarrow C_1^{\mathrm{ex}}$ in arbitrary rings? If the answer to 2 is affirmative, then the answer is yes.
4. Is a finite *regular* subring of a unit regular ring also unit regular?
5. Is finite regularity equivalent to C_3^{ex} for finitely *generated projective* modules?
6. When does $I \subseteqq J$, $I \cong J \Rightarrow I = J$ for right ideals?
7. If R is finite regular can $R \oplus R \cong R \oplus R \oplus R$ as R-modules hold?
8. Is the (k, l) property inherited by $R_{2\times 2}$?
9. What sort of finite regular ring satisfy C_2^{in} or C_2^{ex}?

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