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A FORMULA FOR THE NORMAL PART OF THE LAPLACE-BELTRAMI OPERATOR ON THE FOLIATED MANIFOLD

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In this paper, we give a formula for the normal part of the Laplace-Beltrami operator with respect to the second connection on a foliated manifold with a bundle-like metric. This formula is analogous to the formula obtained by S. Helgason.

1. Itroduction. We shall be in C^∞-category and manifolds are supposed to be paracompact, connected Hausdorff spaces.

Let M be a complete $(p+q)$-dimensional Riemannian manifold and H a compact subgroup of the Lie group of all isometries of M. We suppose that all orbits of H have the same dimension p. Then H defines a p-dimensional foliation F whose leaves are orbits of H, and the Riemannian metric is a bundle-like metric with respect to the foliation F. A quotient space $B = M/F$ is a Riemannian V-manifold [5]. Let L_D be the Laplace-Beltrami operator on M with respect to the second connection D[8], and let $\varDelta(L_D)$ denote the operator defined by (*) in § 4. Our goal in this paper is the following theorem:

THEOREM. *Let L_D be the Laplace-Beltrami operator on M with respect to the second connection D and L_B the Laplace-Beltrami operator on B with respect to the Levi-Civita connection associated with the Riemannian metric defined by the normal component of the metric on M. Then*

$$\varDelta(L_D) = \delta^{-1/2} L_B \circ \delta^{1/2} - \delta^{-1/2} L_B(\delta^{1/2})$$

where δ is the function given by (**) *below.*

This theorem is analogous to the following result obtained by S. Helgason [2]: *Suppose V is a Riemannian manifold, H a closed unimodular subgroup of the Lie group of all isometries of V* (with the compact open topology). *Let $W \subset V$ be a submanifold satisfying the condition: For each $w \in W$,*

$$(H \cdot w) \cap W = \{w\}\ , \quad V_w = (H \cdot w)_w \oplus W_w\ ,$$

where $\oplus$ denotes orthogonal direct sum. Let L_V and L_W denote the Laplace-Beltrami operators on V and W, respectively. Then

$$\varDelta(L_V) = \delta^{-1/2} L_W \circ \delta^{1/2} - \delta^{-1/2} L_W(\delta^{1/2})$$

where $\Delta(L_V)$ *denotes the operator called the radial part of* L_V *and* δ *is the function given by* $d\sigma_w = \delta(w)dh$ ($d\sigma_W$ is the Riemannian volume element on the orbit $H\cdot w$ and dh is an H-invariant measure on each orbit $H\cdot w = H/\{$the isotropy subgroup of H at $w\}$).

2. **Definition of V-manifold [1, 6, 7].** The concept of V-manifold is defined by I. Satake. Let M be a Hausdorff space. *A* C^∞-local uniformizing system $\{\tilde{U}, G, \varphi\}$ for an open set U in M is a collection of the following objects:

$\tilde{U}$: a connected open set in the m-dimensional Euclidean space (or C^∞-manifold).

G: a finite group of C^∞-transformations of $\tilde{U}$.

φ: a continuous map from $\tilde{U}$ onto U such that $\varphi\circ\sigma = \varphi$ for all $\sigma \in G$, inducing a homeomorphism from the quotient space $\tilde{U}/G$ onto U.

Let $\{\tilde{U}, G, \varphi\}$, $\{\tilde{U}', G', \varphi'\}$ be local uniformizing systems for U, U' respectively, and let $U \subset U'$. By a C^∞-injection $\lambda: \{\tilde{U}, G, \varphi\} \to \{\tilde{U}', G', \varphi'\}$ we mean a C^∞-isomorphism from $\tilde{U}$ onto an open subset of $\tilde{U}'$ such that for any $\sigma \in G$ there exists $\sigma' \in G'$ satisfying relations $\varphi = \varphi'\circ\lambda$ and $\lambda\circ\sigma = \sigma'\circ\lambda$.

A C^∞-V-manifold consists of a connected Hausdorff space M and a family $\mathscr{F}$ of C^∞-local uniformizing systems for open subsets in M satisfying the following conditions:

(I) If $\{\tilde{U}, G, \varphi\}$, $\{\tilde{U}', G', \varphi'\} \in \mathscr{F}$ and $U \subset U'$, then there exists a C^∞-injection $\lambda: \{\tilde{U}, G, \varphi\} \to \{\tilde{U}', G', \varphi'\}$.

(II) The open sets U, for which there exists a local uniformizing system $\{\tilde{U}, G, \varphi\} \in \mathscr{F}$, form a basis of open sets in M.

The set R of all real numbers is regarded as a V-manifold defined by a single local uniformizing system $\{R, \{1\}, 1\}$, then a C^∞-function on a V-manifold $(M, \mathscr{F})$ is defined as a C^∞-map $M \to R$ defined by a C^∞-V-manifold map $(M, \mathscr{F}) \to (R, \{R, \{1\}, 1\})$.

A C^∞-V-bundle over C^∞-V-manifold is also defined, and in particular the tangent bundle $(TM, \mathscr{F}^*)$ of a C^∞-V-manifold $(M, \mathscr{F})$ is defined. Let $(M, \mathscr{F})$ be a C^∞-V-manifold, then an h-form ω on $(M, \mathscr{F})$ is a collection of h-forms $\{\omega_{\tilde{U}}\}$, where $\omega_{\tilde{U}}$ is a G-invariant h-form on $\tilde{U}$ such that $\omega_{\tilde{U}} = \omega_{\tilde{U}'}\circ\lambda$ for any injection $\lambda: \{\tilde{U}, G, \varphi\} \to \{\tilde{U}', G', \varphi'\}(\{\tilde{U}, G, \varphi\}, \{\tilde{U}', G', \varphi'\} \in \mathscr{F})$, and if the support of ω is contained in $U = \varphi(\tilde{U})$,

$$\int_M \omega := \frac{1}{N_G}\int_{\tilde{U}} \omega_{\tilde{U}} ,$$

where N_G denotes the order of G. A Riemannian metric g on $(M, \mathscr{F})$ is a collection of Riemannian metrices $\{g_{\tilde{U}}\}$, where $g_{\tilde{U}}$ is a G-invariant Riemannian metric on $\tilde{U}$ satisfying some condition with any injection $\lambda: \{\tilde{U}, G, \varphi\} \rightarrow \{\tilde{U}', G', \varphi'\}$.

3. Review of the results from [4, 5]. Let M be a complete $(p+q)$-dimensional manifold with a "bundle-like matric" with respect to a p-dimensional foliation F. We suppose that each leaf of the foliation F is closed.

The quotient space $B = M/F$ is the space formed from M by identifying each leaf to a point, and let $\pi: M \rightarrow B$ denote the identification map. $H(S)$ denotes the holonomy group of a leaf S. Since M has the bundle-like metric with respect to F and all leaves are closed, $H(S)$ is a finite group for any S and B is a metric space defining the distance between two points of B to be the minimum distance between them considered as leaves is M. B is a connected Hausdorff space, since it is metric space and is the continuous image of M under π. Given any point $b \in B$, let $S = \pi^{-1}(b)$. Let U be a flat coordinate neighborhood of some point of S. Since $H(S)$ may be considered as a group of isometries of the sphere of unit vectors orthogonal to the leaf S at some arbitrary point of S, $H(S)$ operates the q-ball orthogonal to S. Thus we may consider that $H(S)$ operates on U such a manner that $\{U, H(S), \pi\}$ is a local uniformizing system for the neighborhood $\pi(U)$ in B. The natural injection map of two such local uniformizing systems are of C^∞. Thus B is a C^∞-V-manifold. Since $H(S)$ is an isometry on the normal vectors at a point of S, the normal component of the metric of M defines a Riemannian structure on B. Thus B is a Riemannian V-manifold.

4. Laplace-Beltrami operator with respect to the second connection. Let M be a $(p+q)$-dimensional manifold with a Riemannian metric $\langle\ ,\ \rangle$ and a p-dimensional foliation F. Let $(U, (x^1, \cdots, x^p, y^1, \cdots, y^p))$ be a flat coordinate neighborhood system, that is, in U, the foliation F is defined by $dy^\alpha = 0$ for $1 \leqq \alpha \leqq q$. Hereafter we will agree on the following ranges of indices: $1 \leqq i, j, k \leqq p$, $1 \leqq \alpha, \beta, \gamma, \delta \leqq q$.

We may choose in each flat coordinate neighborhood system $(U, (x^1, \cdots, x^p, y^1, \cdots, y^q))$ 1-forms $w^1, \cdots, w^p$ such that $\{w^1, \cdots, w^p, dy^1, \cdots, dy^q\}$ is a basis for the cotangent space, and vectors $v_1, \cdots, v_q$ such that $\{\partial/\partial x^1, \cdots, \partial/\partial x^p, v_1, \cdots, v_q\}$ is the dual base for the tangent space. Then we may get

$$w^i := dx^i + A^i_\alpha dy^\alpha , \quad v_\alpha := \frac{\partial}{\partial y^\alpha} - A^i_\alpha \frac{\partial}{\partial x^i} .$$

We may choose A^i_α such that $\langle \partial/\partial x^i, v_\alpha \rangle = 0$, then the metric has the local expression

$$ds^2 = g_{ij}(x, y)w^i w^j + g_{\alpha\beta}(x, y)dy^\alpha dy^\beta$$

where

$$g_{ij} := \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle, \quad g_{\alpha\beta} := \langle v_\alpha, v_\beta \rangle$$

and $x := (x^1, \cdots, x^p)$, $y := (y^1, \cdots, y^q)$.

We may uniquely define the "second connection" D on M as follows (cf. [8]);

(a)
$$D_{\partial/\partial x^i}\frac{\partial}{\partial x^j} = \Gamma^k_{ji}\frac{\partial}{\partial x^k}, \quad D_{v_\alpha}\frac{\partial}{\partial x^j} = \Gamma^k_{\alpha j}\frac{\partial}{\partial x^k},$$
$$D_{\partial/\partial x^i}v_\beta = \Gamma^\gamma_{i\beta}v_\gamma, \quad D_{v_\alpha}v_\beta = \Gamma^\gamma_{\alpha\beta}v_\gamma,$$

(b)
$$\frac{\partial}{\partial x^i}\left\langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right\rangle = \left\langle D_{\partial/\partial x^i}\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right\rangle + \left\langle \frac{\partial}{\partial x^j}, D_{\partial/\partial x^i}\frac{\partial}{\partial x^k} \right\rangle,$$
$$v_\alpha\langle v_\beta, v_\gamma \rangle = \langle D_{v_\alpha}v_\beta, v_\gamma \rangle + \langle v_\beta, D_{v_\alpha}v_\gamma \rangle,$$

(c)
$$T\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = T^\gamma_{ij}v_\gamma, \quad T\left(\frac{\partial}{\partial x^i}, v_\beta\right) = 0,$$
$$T\left(v_\alpha, \frac{\partial}{\partial x^j}\right) = 0, \quad T(v_\alpha, v_\beta) = T^k_{\alpha\beta}\frac{\partial}{\partial x^k},$$

where T denotes the torsion of D, that is, for any vector fields X, Y on M, $T(X, Y) := D_X Y - D_Y X - [X, Y]$ ($[\ ,\ ]$ denotes the usual bracket operator). Note that, in general, the torsion of D doesn't vanish. If the metric has the local expression

$$ds^2 = g_{ij}(x, y)w^i w^j + g_{\alpha\beta}(y)dy^\alpha dy^\beta,$$

the metric is called a "bundle-like metric" with respect to the foliation F. Hereafter we suppose that M has a bundle-like metric with respect to F. Then we get

$$\frac{\partial}{\partial x^i}\langle v_\alpha, v_\beta \rangle = \langle D_{\partial/\partial x^i}v_\alpha, v_\beta \rangle + \langle v_\alpha, D_{\partial/\partial x^i}v_\beta \rangle.$$

For a vector field X on M, $\mathrm{div}_D X$ is defined by

$$\mathrm{div}_D X := \mathrm{Trace}\ (Y \longrightarrow D_Y X),$$

for any vector field Y on M. For a function f on M, $\mathrm{grad}_D f$ is defined by

$$\begin{aligned}\mathrm{grad}_D f &:= (\tilde{g}^{ij}D_{\partial/\partial x^j}f)\frac{\partial}{\partial x^i} + (\tilde{g}^{\alpha\beta}D_{v_\beta}f)v_\alpha \\ &= \left(\tilde{g}^{ij}\frac{\partial}{\partial x^j}(f)\right)\frac{\partial}{\partial x^i} + (\tilde{g}^{\alpha\beta}v_\beta(f))v_\alpha\end{aligned}$$

where $(\tilde{g}^{ij})$ and $(\tilde{g}^{\alpha\beta})$ are inverse matrices of (g_{ij}) and $(g_{\alpha\beta})$ respectively. We define the Laplace-Beltrami operator L_D with respect to the second connection D by

$$L_D(f) := \mathrm{div}_D \, \mathrm{grad}_D \, f \, ,$$

that is,

$$L_D(f) = \tilde{g}^{ij}\frac{\partial}{\partial x^i}\Big(\frac{\partial}{\partial x_j}(f)\Big) - \tilde{g}^{ij}\Gamma^k_{ij}\frac{\partial}{\partial x^k}(f) + \tilde{g}^{\alpha\beta}v_\alpha(v_\beta(f)) - \tilde{g}^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}v_\gamma(f) \, .$$

Let B be the C^∞-V-manifold M/F. Let $\mathscr{E}(B)$ (resp. $\mathscr{D}(B)$ be the space of C^∞-functions (resp. C^∞-functions of compact support) on B, and let $\mathscr{E}_S(M)$ be the space of C^∞-functions on M which are constants on leaves. We may define a map $\Phi\colon \mathscr{E}_S(M) \to \mathscr{E}(B)$ by $\Phi(f)(\pi(m)) := f(m)$ where $f \in \mathscr{E}_S(M)$, $m \in M$ and $\pi\colon M \to B$, then Φ is of one-to-one. Let $\mathscr{E}^0_S(M) := \Phi^{-1}(\mathscr{D}(B))$.

It is clear that $f \in \mathscr{E}_S(M)$ if and only if $\partial/\partial x^i(f)=0$ for $1 \leqq i \leqq p$.

LEMMA. *If $f \in \mathscr{E}_S(M)$, then $L_D(f) \in \mathscr{E}_S(M)$.*

Proof. For $f \in \mathscr{E}_S(M)$, we get

$$L_D(f) = \tilde{g}^{\alpha\beta}v_\alpha(v_\beta(f)) - \tilde{g}^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}v_\gamma(f) \, .$$

Since $g_{\alpha\beta} = g_{\alpha\beta}(y)$ and $\Gamma^\gamma_{\alpha\beta} = (1/2)\tilde{g}^{\gamma\delta}\{v_\alpha(g_{\delta\beta}) + v_\beta(g_{\alpha\delta}) - v_\delta(g_{\alpha\beta})\}$, we get $\tilde{g}^{\alpha\beta} = \tilde{g}^{\alpha\beta}(y)$ and so $\partial/\partial x^i(L_D(f)) = 0$. Thus we get $L_D(f) \in \mathscr{E}_S(M)$.

REMARK. Let L be the Laplace-Beltrami operator with respect to the Levi-Civita connection associated with the bundle-like metric. In general $L(f) \notin \mathscr{E}_S(M)$ for $f \in \mathscr{E}_S(M)$.

For L_D and $\underline{f} \in \mathscr{E}(B)$, we define $\Delta(L_D)$ by

$$(*) \qquad \Delta(L_D)(\underline{f})(b) := L_D(\Phi^{-1}(\underline{f}))(\pi^{-}(b)) \, , \quad b \in B \, .$$

This is well-defined by lemma. Roughly speaking, $\Delta(L_D)$ seems to be an operator projected on B of the normal part of L_D.

5. Proof of theorem. Using the same notations as above sections, we give a proof of our theorem.

The isotropy subgroup H_m at each point $m \in M$ is compact and the orbit $H\cdot m$ is compact. We fix a Haar measure on H and a Haar measure on H_m, we get an H-invariant measure $d\dot{h}$ on each orbit $H\cdot m = H/H_m$. Since M has the bundle-like metric, $ds^2 = g_{ij}(x, y)w^iw^j + g_{\alpha\beta}(y)dy^\alpha dy^\beta$, the volume element dM of M is given by

$$dM = G(x, y)dx^1 \wedge \cdots \wedge dx^p \wedge dy^1 \wedge \cdots \wedge dy^q$$
$$(= G(x, y)w^1 \wedge \cdots \wedge w^p \wedge dy^1 \wedge \cdots \wedge dy^q)$$

where

$$G(x, y) := \sqrt{\det\begin{pmatrix} g_{ij} & 0 \\ 0 & g_{\alpha\beta} \end{pmatrix}} .$$

For a flat coordinate system $(U, (x^1, \cdots, x^p, y^1, \cdots, y^q))$ and the projection $\pi: M \to B$,

$$d\sigma = G'(y)dy^1 \wedge \cdots \wedge dy^q ,$$

where $G'(y) := \sqrt{|\det(g_{\alpha\beta})|}$, is regarded as the volume element dB of B, since $\{U, H(S), \pi\}$ is a local uniformizing system for $\pi(U)$ in B. Also we get

$$G(x, y) = \sqrt{|\det(g_{ij}(x, y))|} \cdot G'(y) .$$

However

$$\sqrt{|\det(g_{ij}(x, y))|}\, w^1 \wedge \cdots \wedge w^p$$

is the volume element dS_m on the leaf S_m through a point $m=(x, y)$ (that is, on the orbit $H\cdot m$). Thus, if $f \in \mathscr{E}^0_S(M)$ we get from the Fubini's theorem that

$$\int_M f dM = \int_B \left[\underline{\int_{H\cdot m} f dS_m} \right] dB(\pi(m))$$

where "__" denotes the image under Φ. dS_m is invariant under H, so it must be a scalar multiple of $d\dot{h}$,

$$dS_m = \bar{\delta}(m)d\dot{h} .$$

Then the function $\bar{\delta}$ belongs to $\mathscr{E}_S(M)$. We put

$$(**) \qquad \delta := \Phi(\bar{\delta}) .$$

Thus we get

$$\int_K f dM = \int_B \left[\underline{\int_{H\cdot m} f(h\cdot m)\, d\dot{h}} \right] \delta(\pi(m)) dB(\pi(m)) .$$

The normal component of the bundle-like metric $ds^2 = g_{ij}(x, y)w^i w^j + g_{\alpha\beta}(y)dy^\alpha dy^\beta$ is $ds^2_N = g_{\alpha\beta}(y)dy^\alpha dy^\beta$, thus L_B is defined by the Levi-Civita connection associated with the metric defined from dS^2_N. Thus we observe that

$$\Delta(L_D) = L_B + \text{lower order terms} .$$

The operator L_D restricted to $\mathscr{E}^0_S(M)$ is symmetric with respect to dM(cf. [8]), that is,

$$(***) \qquad \int_M L_D(f_1)f_2 dM = \int_M f_1 L_D(f_2) dM$$

for $f_1,\ f_2 \in \mathscr{E}^0_S(M)$.

For $f \in \mathscr{E}_S(M)$ and $m \in M$, we get

$$\int_{H\cdot m} f d\dot{h} = \underline{f}(\pi(m))c$$

where c denotes a nonzero constant $\int_{H\cdot m} d\dot{h}$. Putting $\underline{f}_1 = \Phi(f_1)$, $\underline{f}_2 = \Phi(f_2)$ for $f_1,\ f_2 \in \mathscr{E}^0_S(M)$, we get

$$\begin{aligned}\int_M L_D(f_1)f_2 dM &= \int_B \left[\int_{H\cdot m} L_D(f_1)f_2 d\dot{h}\right]\delta dB \\ &= \int_B \left[\underline{\int_{H\cdot m} L_D(f_1)\, d\dot{h}}\right] c\delta \underline{f}_2 dB \\ &= c^2 \int_B \underline{L_D(f_1)}\underline{f}_2 \delta dB\,.\end{aligned}$$

Thus we get from (***)

$$\int_B \underline{L_D(f_1)f_2}\delta dB = \int_B \underline{f}_1 \underline{L_D(f_2)}\delta dB$$

for $f_1,\ f_2 \in \mathscr{E}^0_S(M)$. By the definition of $\Delta(L_D)$ we get $\underline{L_D(f)} = \Delta(L_D)(\underline{f})$ for $f \in \mathscr{E}_S(M)$, so

$$\int_B \Delta(L_D)(\underline{f}_1)\underline{f}_2 \delta dB = \int_B \underline{f}_1 \Delta(L_D)(\underline{f}_2)\delta dB\,.$$

This expression implies that $\Delta(L_D)$ is symmetric with respect to δdB. Since L_B is symmetric with respect to dB, $\delta^{-1/2}L_B\circ\delta^{1/2}$ is symmetric with respect to $\delta\ dB$ and it clearly agrees with L_B up to lower order terms. The symmetric operators $\Delta(L_D)$ and $\delta^{-1/2}L_B\circ\delta^{1/2}$ agree up to an operator of order $\leqq 1$, thus this operator, being symmetric, must be a function. By applying the operators to the constant function 1, we get

$$\Delta(L_D)(1) - \delta^{-1/2}L_B\circ\delta^{1/2}(1) = -\delta^{-1/2}L_B(\delta^{1/2})\,.$$

Thus

$$\Delta(L_D) = \delta^{-1/2}L_B\circ\delta^{1/2} - \delta^{-1/2}L_B(\delta^{1/2})\,.$$

This completes the proof of our theorem.

REMARK. The example of "RS-manifold of almost fibered type" given by S. Kashiwabara (Apendix 5 in [3]) is a foliated manifold with a 1-dimensional foliation and bundle-like metric. Each leaf of the foliation is a "S-geodesic." This example is constructed from the space D which consists of all points $x_1e_1 + x_2e_2 + x_3e_3 + te_4$ such that $|x_i| \leqq 1 (i = 1, 2, 3)$, $0 \leqq t \leqq 1$, where (e_1, e_2, e_3, e_4) denotes an orthonormal frame with origin o in Euclidean 4-space. If S-geodesics are of direction of e_4, a leaf through the origin o has nontrivial holonomy group. Then $\delta = 1$.

REMARK. The semi-reducible Riemannian space are a special class of foliated manifolds with bundle-like metrices. The metric of such a space has the local expression

$$d\ s^2 = \sigma(y) q_{ij}(x) dx^i dx^j + g_{\alpha\beta}(y) dy^\alpha dy^\beta$$

(cf. [4]). Then δ is defined from σ.

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