PACIFIC JOURNAL OF MATHEMATICS
Vol. 73, No. 1, 1977

THE R-BOREL STRUCTURE ON A CHOQUET SIMPLEX

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The R-Borel structure on a Choquet simplex K is studied. It is shown that the central decomposition and maximal measures coincide, and this is used to improve the well-known theorem that maximal measures are pseudo-concentrated on the extreme boundary.

1. Introduction. Let K denote a compact convex subset of a locally convex Hausdorff topological vector space, and denote by $A^b(K)$ the Banach space of bounded real valued affine functions on K. The symbols $A(K)$, $A(K)^m$, and $A(K)_m$ denote respectively the sets of continuous, lower semi-continuous and upper semi-continuous functions in $A^b(K)$. Set $S(K) = A(K)^m + A(K)_m$, and let $S(K)^\mu$ be the smallest subset of $A^b(K)$ containing $S(K)$ and closed under the formation of pointwise limits of uniformly bounded monotone sequences. $S(K)^\mu$ is a Banach space, the following properties of which were obtained in [6].

THEOREM 1.1. *Consider* $a \in S(K)^\mu$.
(i) $\|a\| = \|a \mid \partial_e K\|$.
(ii) $a \geqq 0$ *if and only if* $a|\partial_e K \geqq 0$.

$S(K)^\mu$ is an order unit space and thus possesses a *centre* $Z(S(K)^\mu)$ defined in terms of order bounded operators [2]. However a more convenient formulation was obtained in [6]: $z \in S(K)^\mu$ is said to be a *central element* if and only if to each $a \in S(K)^\mu$ there corresponds $b \in S(K)^\mu$ satisfying $b(x) = a(x)z(x)$ for all $x \in \partial_e K$. $Z(S(K)^\mu)$ is then seen to be an algebra and a lattice with operations defined pointwise on $\partial_e K$.

Let π^s be the map which restricts elements of $S(K)^\mu$ to functions on $\partial_e K$. The following representation of $Z(S(K)^\mu)$ as an algebra of measurable functions on $\partial_e K$ was proved in [6]. The statement has been modified slightly to suit the purpose of this note.

THEOREM 1.2. *There exists a σ-algebra $\mathscr{R}$ of subsets of $\partial_e K$ such that π^s is an isometric algebraic isomorphism from $Z(S(K)^\mu)$ onto the algebra $F(\partial_e K, \mathscr{R})$ of bounded $\mathscr{R}$-measurable functions on $\partial_e K$. There exists a unique affine map $x \to \nu_x$ from K into the set of probability measures on $\mathscr{R}$ satisfying, for $z \in Z(S(K)^\mu)$,*

$$z(x) = \int_{\partial_e K} \pi^s(z)\,d\nu_x \,.$$

The σ-algebra $\mathscr{R}$ is termed the *R-Borel structure* on $\partial_e K$, while the measures ν_x constitute the *central representation* of points of K with respect to $Z(S(K)^\mu)$.

When K is a simplex, $S(K)^\mu$ is a lattice [5], and hence equal to its centre [2]. In this case $\mathscr{R}$ is large, and it is the purpose of this note to investigate further the R-Borel structure in this special situation. In particular the central decomposition measures ν_x are related to the unique maximal representing measures μ_x, and an extension is obtained of the well-known theorem that maximal representing measures vanish on every Baire set disjoint from $\partial_e K$.

[4] contains further information on Borel structures on compact convex sets, while [2] is the standard reference for convexity theory.

2. The main theorems. For the remainder of this note K is assumed to be a simplex. $S(K)^\mu$ is a lattice [5], and it follows, by the methods of [1], that the lattice operations are given, for $f, g \in S(K)^\mu$, $x \in K$, by

$$f \curlyvee g(x) = \int_K f \vee g d\mu_x$$

$$f \curlywedge g(x) = \int_K f \wedge g d\mu_x \, .$$

Let $\mathscr{B}_0$ and $\mathscr{B}$ denote the Baire and Borel structures on K respectively, and let their restrictions to $\partial_e K$ be denoted by $\overline{\mathscr{B}}_0$ and $\overline{\mathscr{B}}$.

THEOREM 2.1. $\overline{\mathscr{B}}_0 \subset \mathscr{R} \subset \overline{\mathscr{B}}$.

Proof. The second inclusion is clear since every function in $S(K)^\mu$ is Borel measurable.

Let E be an arbitrary compact G_δ subset of K. Then there exists a uniformly bounded decreasing sequence of continuous functions $(f_n)_{n=1}^\infty$ with pointwise limit χ_E. By [2, II.3.14] and Theorem 1.1, there exists a uniformly bounded decreasing sequence $(g_n)_{n=1}^\infty$ from $S(K)^\mu$ such that f_n and g_n agree on $\partial_e K$. This sequence has pointwise limit $g \in S(K)^\mu$, and clearly χ_E and g agree on $\partial_e K$. Hence $E \cap \partial_e K = g^{-1}(1) \cap \partial_e K \in \mathscr{R}$.

It is now clear that $\mathscr{R}$ contains $\overline{\mathscr{B}}_0$.

LEMMA 2.2. *Let f and g be nonnegative functions in $S(K)^\mu$ such that $f^{-1}(0) \cap \partial_e K = g^{-1}(0) \cap \partial_e K$. Then $f^{-1}(0) = g^{-1}(0)$.*

Proof. Let $E = f^{-1}(0) \cap \partial_e K$. Then $E \in \mathscr{R}$, and so let h be the unique element of $S(K)^{\mu}$ with the property that

$$E = h^{-1}(0) \cap \partial_e K , \qquad E^c = h^{-1}(1) \cap \partial_e K .$$

For $n \geqq 1$, $E_n = \{x \in \partial_e K : f(x) \geqq 1/n\}$ is an element of $\mathscr{R}$. Let $h_n \in S(K)^{\mu}$ be the corresponding function for which

$$E_n = h_n^{-1}(1) \cap \partial_e K , \qquad E_n^c = h_n^{-1}(0) \cap \partial_e K .$$

By Theorem 1.1, $(h_n)_{n=1}^{\infty}$ is a uniformly bounded increasing sequence from $S(K)^{\mu}$ with pointwise limit h. Hence $h^{-1}(0) = \bigcap_{n=1}^{\infty} h_n^{-1}(0)$. Now for each $n \geqq 1$, $nf \geqq h_n$, and thus $f^{-1}(0)$ is contained in $h_n^{-1}(0)$. It follows that $f^{-1}(0)$ is contained in $h^{-1}(0)$. Conversely, $f \leqq \|f\| h$ and so $h^{-1}(0)$ is contained in $f^{-1}(0)$.

In conclusion $f^{-1}(0) = h^{-1}(0)$, and the fact that $g^{-1}(0) = h^{-1}(0)$ is established by the same reasoning.

COROLLARY 2.3. *If $f \in S(K)^{\mu}$ attains its lower bound then it does so at an extreme point.*

Proof. It may be assumed that $f \geqq 0$ and that $f^{-1}(0)$ is nonempty. To derive a contradiction, suppose that $f^{-1}(0)$ does not contain an extreme point. Now apply Lemma 2.2 to the functions f and 1_K.

Let $\mathscr{S}$ denote the smallest σ-algebra of subsets of K with respect to which every function in $S(K)^{\mu}$ is measurable. [2, I.1.1, I.1.3] together imply that every continuous function is $\mathscr{S}$-measurable. Since every function in $S(K)^{\mu}$ is Borel measurable, it follows that

$$\mathscr{B}_0 \subset \mathscr{S} \subset \mathscr{B} .$$

The following theorem relates the maximal representing measures to the central decomposition measures.

THEOREM 2.4. *For each $x \in K$ the maximal representing measure μ_x may be restricted to a measure $\bar{\mu}_x$ on $\mathscr{S} \cap \partial_e K$. $\mathscr{S} \cap \partial_e K = \mathscr{R}$ and $\bar{\mu}_x = \nu_x$.*

Proof. Define an equivalence relation on the algebra of bounded Borel measurable functions on K by setting $f \sim g$ if and only if, for all $x \in K$,

$$\int_K |f - g| \, d\mu_x = 0 .$$

If $f, g \in S(K)^{\mu}$ then the relations

$$f \curlyvee g \sim f \vee g , \qquad f \curlywedge g \sim f \wedge g$$

are an easy consequence of the fact that functions in $S(K)^\mu$ satisfy the barycentric calculus (see [2]).

Let $\mathscr{H}$ be the set of Borel sets E for which there exists $h \in S(K)^\mu$ such that $\chi_E \sim h$. The proof now proceeds in several stages.

(i) Suppose that $E, F \in \mathscr{H}$ with associated functions $f, g \in S(K)^\mu$ respectively. Then

$$\chi_{E\cap F} = \chi_E \wedge \chi_F \sim f \wedge g \sim f \curlywedge g \in S(K)^\mu$$

and

$$\chi_{E^c} = 1_E - \chi_F \sim 1_K - f \in S(K)^\mu .$$

Hence $E \cap F$ and E^c are members of $\mathscr{H}$.

Suppose that $(E_n)_{n=1}^\infty$ is an increasing sequence from $\mathscr{H}$ with associated sequence $(h_n)_{n=1}^\infty$ from $S(K)^\mu$. Theorem 1.1 implies that the latter sequence is uniformly bounded and increasing with pointwise limit $h \in S(K)^\mu$. Let $E = \bigcup_{n=1}^\infty E_n$. Then the dominated convergence theorem implies that $\chi_E \sim h$. Since $K \in \mathscr{H}$, it follows that $\mathscr{H}$ is a σ-algebra.

(ii) Suppose that f is a nonnegative element of $S(K)^\mu$, and write $E = f^{-1}(0)$. Then $E \cap \partial_e K \in \mathscr{R}$, and there exists a unique element $g \in S(K)^\mu$ such that

$$g^{-1}(1) \cap \partial_e K = E \cap \partial_e K , \qquad g^{-1}(0) \cap \partial_e K = E^c \cap \partial_e K .$$

By Lemma 2.2, $E = g^{-1}(1)$ and hence $g \geqq \chi_E$. If $x \in E$ then

$$\int_K (1_K - g) d\mu_x = 1 - g(x) = 0 .$$

$g \leqq 1_K$ and thus μ_x is supported by $g^{-1}(1)$. Hence $\mu_x(E) = 1$. Similar arguments applied to 1_K and $1_K - g$ yield $\mu_x(E) = 0$ for $x \in g^{-1}(0)$. $g^{-1}(0)$ and $g^{-1}(1)$ are complementary split faces [1, 3]. Each $x \in K$ then has decomposition $g(x)y + (1-g(x))z$ where $y \in g^{-1}(1)$, and $z \in g^{-1}(0)$, and

$$\int_K \chi_E d\mu_x = \mu_x(E) = g(x)\mu_y(E) + (1 - g(x))\mu_z(E) = g(x) = \int_K g(x) d\mu_x .$$

Since $g \geqq \chi_E$ it follows that $\chi_E \sim g$ and $E \in \mathscr{H}$.

(iii) Suppose $f \in S(K)^\mu$ and $\alpha \in \boldsymbol{R}$. Write $g = f \wedge \alpha 1_K$ and $h = f \curlywedge \alpha 1_K$, and denote $g^{-1}(\alpha)$ and $h^{-1}(\alpha)$ by G and H respectively. For $x \in K$,

$$\int_K (g - h) d\mu_x = 0 .$$

$g \geqq h$ and thus the set on which $g > h$ has μ_x-measure zero. This set contains $G\backslash H$, hence $\mu_x(G) = \mu_x(H)$, and it follows that $\chi_H \sim \chi_G$. However by (ii), $H \in \mathscr{H}$, and therefore $G \in \mathscr{H}$.

$G = \{x \in K: f(x) \geqq \alpha\}$ and, since f and α were arbitrary, every function in $S(K)^\mu$ is $\mathscr{H}$-measurable.

(iv) Suppose that $E \in \mathscr{H}$ and that E is disjoint from $\partial_e K$. If $\chi_E \sim h \in S(K)^\mu$ then $h | \partial_e K = 0$. By Theorem 1.1 $h = 0$, and, for all $x \in K$, $\mu_x(E) = 0$.

(v) From (iii), $\mathscr{H}$ contains $\mathscr{S}$, and it is not difficult to show that $\mathscr{S} \cap \partial_e K = \mathscr{R}$. Thus, by the methods of [2, I.4.13, I.4.14], each μ_x may be restricted to a measure $\bar{\mu}_x$ on $\mathscr{R}$ satisfying, for $f \in S(K)^\mu$,

$$f(x) = \int_{\partial_e K} f d\bar{\mu}_x .$$

The uniqueness of the central decomposition measures (Theorem 1.2) now implies that, for each $x \in K$, $\bar{\mu}_x = \nu_x$. The proof is complete.

REMARKS. (i) Since $\mathscr{B}_0 \subset \mathscr{S} \subset \mathscr{B}$, it is clear that Theorem 2.4 extends the result that maximal measures are pseudo-concentrated on $\partial_e K$. For a nonmetrizable Bauer simplex it is clear that $\mathscr{S}$ strictly contains $\mathscr{B}_0$. On the other hand there exists a simplex K and a maximal measure μ such that $\partial_e K \in \mathscr{B}$ and $\mu(\partial_e K) = 0$. In this case $\mathscr{S}$ and $\mathscr{B}$ cannot be equal. This example is discussed in [2, II.3.17].

(ii) The set of step functions is dense in $F(K, \mathscr{S})$ and hence any $\mathscr{S}$-measurable function is equivalent to an element of $S(K)^\mu$. In particular if $f, g \in S(K)^\mu$ then fg is $\mathscr{S}$-measurable, and there exists $h \in S(K)^\mu$ such that $fg \sim h$. Denote the product of f and g in $S(K)^\mu$ by $f \circ g$. Then, for $x \in \partial_e K$, $f \circ g(x) = f(x)g(x)$, and thus h and $f \circ g$ agree on $\partial_e K$. It follows from Theorem 1.1 that $h = f \circ g$, and thus, for $x \in K$,

$$f \circ g(x) = \int_K fg d\mu_x .$$

Direct approaches do not seem to yield this formula.

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Received February 1, 1977.

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