PACIFIC JOURNAL OF MATHEMATICS
Vol. 73, No. 1, 1977

ON THE COMPACTNESS OF THE HYPERSPACE OF FACES

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Let $X \subset R^d$, $d > 2$, be compact and convex. It is shown that the space of proper faces (poonems) of X is compact if and only if the space of k-exposed (extreme) points of X is compact, $0 \leqq k \leqq d-2$.

By a *flat* we mean a translate of a subspace of R^d and by a *hyperplane*, a flat of dimension $d-1$. If X is a compact convex subset of R^d the symbols $\dim(X)$, $\operatorname{rel\,int}(X)$, and $\operatorname{rel\,bd}(X)$ denote respectively, the dimension of the flat generated by X, the interior, and the boundary of X with respect to the flat X generates. A hyperplane H is called a *supporting hyperplane* of X if $H \cap X \neq \varnothing$ and $H \cap \operatorname{relint}(X) = \varnothing$. A set A is called a *face* of X if $A = X$, $A = \varnothing$ or if there exists a supporting hyperplane H of X such that $A = H \cap X$. The set of proper faces (those not X or $\varnothing$) is denoted by $\mathscr{F}(X)$. A set B is called a *poonem* of X if there exists sets $B_0, B_1, \cdots, B_m$ such that $B_m = X$ and $B_{i-1} \in \mathscr{F}(B_i)$ for $i = 1, \cdots, m$. The set of poonems of X is denoted by $\mathscr{P}(X)$. A point x in X is called a k-exposed [k-extreme] point if for some $j \leqq k$, x belongs to a j-dimensional face [j-dimensional poonem] of X. The symbols $\exp_k(X)$ and $\operatorname{ext}_k(X)$ denote the set of k-exposed and k-extreme points of X respectively. A point x of X is called an *exposed point* of X if $\{x\} \in \mathscr{F}(X)$ and x is called an *extreme point* if whenever $x \in [a, b] \subset X$, we have $x = a$ or $x = b$, where $[a, b]$ denotes the closed line segment from a to b. The symbols $\operatorname{ext}(X)$ and $\exp(X)$ denote the extreme points and exposed points of X, respectively. Note that $\exp(X) = \exp_0(X)$ and $\operatorname{ext}(X) = \operatorname{ext}_0(X)$. Also, let q denote the Hausdorff metric. Finally, if $D \subset R^d$, the symbols $\operatorname{cl}(D)$ and $\operatorname{conv}(D)$ denote the closure of D and the convex hull of D respectively.

We require the following results

PROPOSITION. *Let X, A and B be nonempty compact convex subsets of R^d, $d \geqq 2$, with $A \subset X$ and $B \subset X$.*

(a) *If $A, B \in F(X)$ $(P(X))$ and $A \cap B \neq \varnothing$, then $A \cap B \in F(X)$ $(P(X))$.*

(b) *If $A \in F(X)$ $(P(X))$ with relint $(B) \cap A \neq \varnothing$ then $B \subset A$.*

(c) *If $A \in F(X)$ $(P(X))$ and $A \subsetneqq B \subset X$ then $A \in F(B)$ $(P(B))$.*

(d) $\operatorname{ext}_k(X) \subset \operatorname{cl}(\exp_k(X))$.

(e) *If $A \in F(X)$ then* $\operatorname{ext}_k(A) \subset \operatorname{ext}_k(X)$.

Propositions (b), (c) *and* (e) *follow quickly from definitions while* (a) *is proved in Grunbaum* [2]. *Proposition* (d) *is due to Asplund* [1]. *The following theorem is the main communication of this note.*

THEOREM. *Let $X \subset R^d$, $d > 2$, be compact and convex. Then*

(i) *The metric space $(P(X), q)$ is compact if and only if $\mathrm{ext}_k(X)$ is compact, $0 \leqq k \leqq d-2$.*

(ii) *The metric space $(F(X), q)$ is compact if and only if $\exp_k(X)$ is compact, $0 \leqq k \leqq d-2$.*

Proof. We consider part (i). The only if part is obvious. Suppose $\{A_n\} \rightarrow A$ with $A_n \in P(X)$ for all n. Clearly if $\dim(A) = 0$, A is a limit point of $\mathrm{ext}(X)$ and we are done. We proceed now by induction on the dimension of A. Let $\dim(A) = k+1$. We may suppose $k+1 \leqq d-2$ since if $k+1 = d-1$ the sequence $\{A_n\}$ is eventually constant and we are done. We consider cases:

Case 1. For infinitely many n, $\dim(A_n) = k+1$. Without loss of generality we may suppose $\dim(A_n) = k+1$ (otherwise we pass to an appropriate subsequence). Let $x \in \mathrm{relint}(A)$. Since x is a limit point of the set $\mathrm{ext}_{k+1}(X)$ and $k+1 \leqq d-2$ our hypotheses imply that there exists $Q \in P(X)$ with $\dim(Q) = k+1$ and $x \in Q$. By Proposition (b) $A \subset Q$. We claim $A = Q$. Suppose not. Then there exists $y \in \mathrm{rel\,bd}(A) \cap \mathrm{relint}(Q)$ and there exists a sequence $\{y_n\} \rightarrow y$ with $y_n \in \mathrm{rel\,bd}(A_n)$ for all n. Let $F_n \in P(A_n)$ with $y_n \in F_n$ and $F_n \neq A_n$ for all n. Without loss of generality we may suppose $\{F_n\} \rightarrow F$ for some F. Since $\dim(F_n) \leqq k$ for all n we have $\dim(F) \leqq k$. Since we have that $F_n \in P(X)$ for all n and since $\dim(F) \leqq k$ the induction hypothesis implies $F \in P(X)$. Since $F \cap \mathrm{rel\,int}(Q) \neq \varnothing$, Proposition (b) implies $Q \subset F$, a contradiction since $\dim(F) < \dim(Q)$. Thus $A = Q$ and we are done.

Case 2. There is no subsequence of $\{A_n\}$ each element of which has dimension $k+1$. Then there exists $\{x_n\} \rightarrow x$ with $x_n \in \mathrm{rel\,bd}(A_n)$ for all n. Let $F_n \in P(A_n)$, $F_n \neq A_n$, with $x_n \in F_n$ for all n. Without loss of generality we may suppose $\{F_n\} \rightarrow F$ for some F. Note $F_n \in P(X)$ and $\dim(F_n) < \dim(A_n)$ for all n. If there is a subsequence of $\{F_n\}$ each element of which has dimension $k+1$ then x is a limit point of $\mathrm{ext}_{k+1}(X)$ and we have returned to the argument of the first case. If not, we repeat the latter procedure as many times as needed to return.

We turn now to the proof of part (ii). The only if part is trivial and is omitted. In view of (i), we need only show "a face of a face is a face." Let $A \in \mathscr{F}(X)$, $\dim(A) \geqq 1$ and let $B \in \mathscr{F}(A)$ and suppose $\dim(B) = k$. We may suppose B is a maximal proper face of A with respect to set inclusion. By definition $B \subset \mathrm{ext}_k(X)$ and by hypothesis and Propositions (e) and (d) $B \subset \exp_k(X)$. Let $x \in \mathrm{rel\,int}(B)$ and let $Q \in \mathscr{F}(X)$ with $x \in Q$ and $\dim(Q) = j \leqq k$. Since $\mathrm{rel\,int}(B) \cup (A \cap Q) \neq \varnothing$, by Propositions (a) and (b), $B \subset A \cap Q$. Since $\dim(A \cap Q) \leqq \dim(Q) \leqq j \leqq k < \dim(A)$, we have $A \cap Q \neq A$. Since we have that $A \cap Q \in \mathscr{F}(X)$ Proposition (c) implies that $A \cap Q \in \mathscr{F}(A)$. We must now have $B = A \cap Q$ since B is a maximal proper face of A. This completes the proof of (ii).

If one defines $f\colon \mathrm{rel\,bd}(X) \to P(X)$ where $f(x)$ is the smallest poonem of X containing x, then one has an example of the face function introduced by V. Klee and M. Martin [3]. Part (i) provides a characterization of those compact convex sets X in R^d for which f is continuous at each point of $\mathrm{rel\,bd}(X)$.

We wish to consider some examples from R^3 and R^4. Let D be a compact convex subset of R^3 with nonempty interior such that $\exp(D)$ is compact but $(\mathscr{F}(D), q)$ is not compact. It is not difficult to show that there is a 1-poonem B of D which is not a face of D. On the basis of the latter example, one might conjecture that for a compact subset X of R^3 $(\mathscr{F}(X), q)$ is compact if and only if $\exp(X)$ is compact and for each $A \in \mathscr{F}(X)$, $\mathscr{F}(A) \subset \mathscr{F}(X)$ i.e., "a face of a face is a face." While this is true in R^3 it is not in R^4 as the following example shows.

Let $C = \{(x, y) \mid x^2 + y^2 = 1, x \geqq 0, y \geqq 0\}$ and let $E = \mathrm{conv}\{(1, 0), (-1, 0), (0, 1), (0, -1)\}$. Let l be the closed line segment from $(0, -1)$ to $(0, 1)$ in E. Let $\{x_n\}$ be a sequence of distinct points in C convergent to $(1, 0)$. Regarding R^4 as $R^2 \times R^2$ let $F_i = \{x_i\} \times l$ for $i = 1, 2, \cdots$ and let $F_0 = \{(1, 0)\} \times E$. Define X as $X = \mathrm{conv}(\bigcup_{i=0}^{\infty} F_i)$. One may check that $\exp(X)$ is compact and that "a face of a face is a face." Note also for each i, $F_i \in \mathscr{F}(X)$ and $\{F_i\} \to F$ where $F = \{(1, 0)\} \times l$. However, $F \notin \mathscr{F}(X)$ so that $(\mathscr{F}(X), q)$ is not compact. (This particular example was pointed out by V. Klee; the authors had a more cumbersome one.) Returning to the set D in R^3 mentioned earlier, perhaps the more important observation about D (as far as higher dimensions are concerned) is not that we have $B \in \mathscr{P}(D)$ and $B \notin \mathscr{F}(D)$ but rather that $\exp_1(D)$ is not compact since the theorem in R^3 that generalizes is that $(\mathscr{F}(X), q)$ is compact if and only if $\exp(X)$ and $\exp_1(X)$ are compact.

References

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Received March 1, 1976 and in revised form April 20, 1977.

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