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FUNCTIONS THAT OPERATE ON THE ALGEBRA $B_0(G)$

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Let G be a locally compact group and let $B(G)$ be the algebra of linear combinations of positive definite continuous functions. We let $B_0(G) = B(G) \cap C_0(G)$ be the subalgebra consisting of the elements of $B(G)$ which vanish at infinity. A complex valued function F, defined on an open interval of the real line containing zero, is said to *operate* on $B_0(G)$ if for every $u \in B_0(G)$ whose range is contained in the domain of F, the composition $F \circ u$ is an element of $B_0(G)$. In this paper we prove that, if G is a separable group with noncompact center or a separable nilpotent group, then every function which operates on $B_0(G)$ can be extended to an entire function. This result follows directly from the corresponding theorem for noncompact commutative groups, which is well known, via a lemma which states that every function on the center Z of G which belongs to $B_0(Z)$ can be extended to an element of $B_0(G)$.

We refer of the paper of N. Th. Varopoulos [11] for the treatment of the commutative case and to the paper of P. Eymard [4] for the definition and properties of the algebra $B(G)$.

We remark that $B(G)$ may also be defined as the algebra of all coefficients of unitary representations of G. A natural subalgebra and ideal of $B(G)$ consists of the coefficients of the regular representation. This subalgebra is called $A(G)$ and is also studied in [4], where it is proved that $A(G) \subseteq B_0(G)$, and that the maximal ideal space of $A(G)$ is G itself. This implies that all functions which are defined and analytic on a neighborhood of zero and vanish at zero operate on $A(G)$. Thus the existence of suitably defined analytic functions which do not operate on $B_0(G)$ is an indication that $B_0(G)$ is quite different from $A(G)$.

To be precise, if some analytic function vanishing at zero and defined on an open interval fails to operate on $B_0(G)$, then the quotient algebra $B_0(G)/A(G)$ is not a radical algebra.

It is important to observe that for several noncompact noncommutative groups $B_0(G)$ coincides with $A(G)$. (The simplest example of such groups is the affine group of the real line, discussed in [6]; other examples are studied in [9], [10], and [12]; a sufficient, and probably necessary, condition to ensure that $B_0(G)$ be different from $A(G)$, for a unimodular group, is given in [5].) But even when $A(G) \neq B_0(G)$, the two algebras may have the same maximal ideal space and therefore all suitably defined analytic functions may operate on $B_0(G)$. In particular, J. R. Liukkonen and M. W. Mislove [7] have proved (under mild additional hypothesis) that if G is the semidirect product obtained from the action of a compact group on a finite dimensional vector space, and if the center of G is compact, then every element $u \in B_0(G)$ has an integral power u^n which belongs to $A(G)$. An analogous result was proved by M. Cowling [2] for the group $\mathrm{SL}(2, R)$. Furthermore, M. Cowling has communicated to us that, if G is any semisimple Lie group, then $B_0(G)/A(G)$ is a radical algebra, and G is the maximal ideal space of $B_0(G)$.

These results seem to support the following conjecture: if G is a connected Lie group with compact center and the maximal ideal space of $B_0(G)$ is larger than G, then G contains a nontrivial nilpotent direct factor. This conjecture does not extend to the case of nonconnected Lie groups: if G is the semidirect product defined by the action of the integers on $\boldsymbol{R}/\boldsymbol{Z}$ given by multiplication, it is not difficult to prove, using the corresponding result for the integers, that only entire functions operate on $B_0(G)$, and yet G has trivial center and is not nilpotent.

2. The basic lemma. Before stating the lemma we recall a few facts concerning induced representations. Let G be a separable group and N a closed normal subgroup. It is known that there exists a "smooth cross-section" for G/N in G, that is a Borel measurable function s mapping G/N into G in such a way that s carries the identity of G/N into the identity of G, and maps compact subsets of G/N into relatively compact subsets of G [8]. As a consequence, if $g \in G$, then g is uniquely decomposable, in a measurable fashion, as $g = n \cdot s(\dot{g})$, with $n \in N$ and $\dot{g} = gN \in G/N$. Let now π be a unitary representation of N on the separable Hilbert space $\mathscr{H}_\pi$; then it is easy to see that the induced representation $\mathrm{Ind}_N^G \pi$ is unitarily equivalent to the representation π^* on $L^2(G/N)$ defined by

$$\pi^*(g)f(\dot{g}_0) = \pi(s(\dot{g}_0)ns(\dot{g})s(\dot{g}_0\dot{g})^{-1})f(\dot{g}_0\dot{g}) \,, \tag{1}$$

where $g = n \cdot s(\dot{g})$, and $\dot{g} = gN \in G/N$. (See [1] for a proof.)

LEMMA. *Let G be a locally compact separable group and let Z be its center. Then every element of $B_0(Z)$ is the restriction to Z of an element of $B_0(G)$. Furthermore, if $u \in B_0(Z)$ is real valued and $-1 < u < 1$, there is an extension to G which is also real valued and satisfies the same inequalities.*

Proof. Since Z is commutative, every $u \in B_0(Z)$ can be written as

$$u(z) = \int_{\hat{Z}} \psi(z) d\mu(\psi) ,$$

where $\hat{Z}$ is the character group of Z, μ is a bounded regular measure on $\hat{Z}$, and $z \in Z$.

For $\psi \in \hat{Z}$, denote by π_ψ the representation induced by ψ from Z to G. Let

$$\pi = \int_{\hat{Z}}^{\oplus} \pi_\psi d\mu(\psi) ;$$

let f be a continuous real valued function with compact support in G/Z, such that $\|f\|_{L^2(G/Z)} = 1$, and define $\tilde{u}(g) = \langle \pi(g)f, f \rangle$. We shall prove the following: (i) $\tilde{u}(z) = u(z)$, for $z \in Z$; (ii) if u is real valued and $-1 < u < 1$, then $\tilde{u}$ is also real valued and satisfies the same inequalities; (iii) $\tilde{u} \in B_0(G)$. The first and second assertions follow readily from formula (1), which, in our case, yields (applying Fubini's theorem):

$$\tilde{u}(g) = \langle \pi(g)f, f \rangle = \langle \pi(z_g s(\dot{g}))f, f \rangle$$
$$(2) \qquad = \int_{\hat{Z}} \int_{G/Z} \psi(z_g s(\dot{g}_0) s(\dot{g}) s(\dot{g}_0 \dot{g})^{-1}) f(\dot{g}_0 \dot{g}) f(\dot{g}_0) d\dot{g}_0 d\mu(\psi) .$$

The third assertion will be proved if we show that the last expression in (2) vanishes at infinity. We observe that, since f has compact support in G/Z, $f(\dot{g}_0\dot{g})f(\dot{g}_0) = 0$ identically in $\dot{g}_0$, when $\dot{g}$ lies outside a fixed compact set K, and therefore $\tilde{u}(g) = 0$, unless $\dot{g}$ belongs to $K \subseteqq G/Z$. Since s maps compact subsets of G/Z into relatively compact subsets of G, when $\tilde{u}(g) \neq 0$, g can be uniquely decomposed as $g = z_g s(\dot{g})$, where $z_g \in Z$ and $s(\dot{g})$ belongs to a compact subset K' of G. If $\tilde{u}(g)$ does not vanish at infinity, there exists a sequence $\{g_k\} \subseteqq G$ such that g_k is eventually outside every fixed compact set and $|\tilde{u}(g_k)| \geqq \varepsilon > 0$. If $g_k = z_k s(\dot{g}_k)$, then the condition $|\tilde{u}(g_k)| \geqq \varepsilon$ implies that $\dot{g}_k \in K$ and $s(\dot{g}_k) \in K'$; therefore $g_k \in z_k K'$, and the sequence z_k belongs eventually to the complement of every fixed compact set. As a consequence, for every fixed element g_0 of G, the sequence $\zeta_k = z_k s(\dot{g}_0) s(\dot{g}_k) s(\dot{g}_0 \dot{g}_k)^{-1}$ tends to infinity: indeed $\dot{g}_0 \dot{g}_k \in \dot{g}_0 K$, as $\dot{g}_k \in K$, and both $s(\dot{g}_k)$ and $s(\dot{g}_0 \dot{g}_k)^{-1}$ must keep inside a fixed compact set because s is smooth and g_0 is fixed. Noticing that $\zeta_k \in Z$, we can define

$$I_k(\dot{g}_0) = \int_{\hat{Z}} \psi(\zeta_k) d\mu(\psi) .$$

Since $\zeta_k \rightarrow \infty$, and $I_k(\dot{g}_0) = u(\zeta_k)$, one has $\lim_k I_k(\dot{g}_0) = 0$ for each $\dot{g}_0 \in G/Z$. On the other hand, by (2),

$$\tilde{u}(g_k) = \int_{G/Z} f(\dot{g}_0 \dot{g}_k) f(\dot{g}_0) I_k(\dot{g}_0) d\dot{g}_0 .$$

The sequence of functions under the integral sign is uniformly bounded by $\sup|f(\dot{g})|\cdot||\mu||$; furthermore this sequence converges to zero for each $\dot{g}_0 \in G/Z$, and its elements all vanish outside the support of f, which is compact. We conclude, by the bounded convergence theorem, that $\lim_k \tilde{u}(g_k) = 0$, contradicting the assumption that $|\tilde{u}(g_k)| \geqq \varepsilon$.

REMARK. The fact that functions in $B(Z)$ may be extended to $B(G)$ is well known [7]. For an interesting generalization see [3].

3. The main theorem. We recall that, if G is a locally compact abelian group, then the only functions which operate on $B_0(G)$ are the entire functions [11]. We shall use this result and the lemma to prove our main theorem.

THEOREM. *Let G be a separable locally compact group with noncompact center. Suppose that F is a complex valued function defined on an open interval of the real line containing* 0, *and that* $F(0) = 0$. *If, for each $u \in B_0(G)$ with range contained in the domain of F, the composition $F \circ u$ belongs to $B_0(G)$, then F may be extended to an entire function.*

Proof. Without loss of generality we may suppose that F is defined on the interval $(-1, 1)$. (If not, replace $F(t)$ by $F(rt)$ for a sufficiently small positive real number r.) We shall prove that F operates on $B_0(Z)$. If $u \in B_0(Z)$, and $-1 < u < 1$, let $\tilde{u}$ be the extension of u to G, whose existence is asserted in the lemma. Then $F \circ \tilde{u} \in B_0(G)$ and its restriction to Z is $F \circ u$. Since the restriction map carries $B_0(G)$ into $B_0(Z)$, we have proved that F operates on $B_0(Z)$. This implies that F may be extended to an entire function, by the result of Varopoulos [11].

COROLLARY 1. *The conclusion of the theorem holds if G is a noncompact nilpotent group.*

Proof. Let $\{Z_j\}_{j=1}^k$ be an ascending central series for G. Since G is noncompact, there exists a positive integer $n < k$ such that Z_{n+1}/Z_n is noncompact and Z_n is compact; then Z_{n+1}/Z_n is the center of G/Z_n, and the theorem implies that every function that operates on $B_0(G/Z_n)$ is an entire function. As Z_n is compact, we may extend canonically an element $u \in B_0(G/Z_n)$ to an element $\tilde{u} \in B_0(G)$, letting $\tilde{u}(g) = u(gZ_n)$. Then it is immediate to see that, if F operates on $B_0(G)$, it operates also on $B_0(G/Z_n)$, and therefore it is extendable to an entire function.

COROLLARY 2. *If G satisfies the hypothesis of the theorem or of Corollary* 1, *then $B_0(G)$ is not symmetric on its maximal ideal space.*

Proof. This follows from the theorem and Corollary 1 by a standard argument. We sketch the argument here for completeness. Let $F(t) = t(t-i)^{-1}$ for $-1 < t < 1$; then F cannot be extended to an entire function, therefore there exists a real valued u in $B_0(G)$ such that $-1 < u(x) < 1$ and $F \circ u \in B_0(G)$. In other words $F(Z) = z(z-i)^{-1}$ cannot be analytic on the spectrum of u, and therefore the spectrum of u contains i; this implies that $B_0(G)$ is not symmetric.

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