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CONNECTIVE COVERINGS OF BO AND IMMERSIONS OF PROJECTIVE SPACES

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New immersions and nonimmersions of real projective space RP^n in Euclidean space are proved when the number of 1's in the binary expansion of n is 7. The method is obstruction theory, utilizing the connective coverings of BO.

1. **Introduction.** Let $BO[j]$ (resp. $BO_N[j]$) denote the space obtained from BO (resp. BO_N) by killing π_i for $i < j$. In [4] Mahowald and the author computed the cohomology and stable homotopy groups of $BO[8]/BO_N[8]$ through degree $N+16$ and used these results to prove some new immersion and nonimmersion results for real projective spaces P^n. In this paper similar computations are performed when $j > 8$ and used to obtain some more new immersion and nonimmersion results.

Let $\alpha(n)$ denote the number of 1's in the binary expansion of n and $\nu(2^a(2b+1)) = a$.

THEOREM 1.1. *If $\nu(n+1) \geqq \alpha(n) - 4 \geqq 3$, then P^n cannot be immersed in $R^{2n-2^{\alpha(n)-3}-4}$.*

THEOREM 1.2. *If $\alpha(n) = 7$, $\nu(n+1) = 4$ or 5, then P^n can be immersed in R^{2n-16}.*

Theorem 1.2 is within 3 dimensions of best possible (by Theorem 1.1). It provides another counterexample to the previously conjectured nonimmersions ([6], [4]). For $\alpha(n) \geqq 8$, Theorem 1.1 is probably not very close to best possible. It gives the densest set of metastable nonimmersion results known to the author. The number of $n < 2^k$ satisfying the condition of 1.1 is $\binom{k-2}{5}$.

Let $\mathscr{A}$ denote the mod 2 Steenrod algebra. For $j \equiv 0, 1, 2$, or $4(8)$ let I_j denote the left ideal generated by

$$\begin{cases} Sq^2 & \text{if } j \equiv 1(8) \\ Sq^3 & \text{if } j \equiv 2(8) \\ Sq^1 \text{ and } Sq^5 & \text{if } j \equiv 0(8) \\ Sq^1 \text{ and } Sq^2 & \text{if } j \equiv 4(8) \end{cases}$$

Let $\mathscr{A}(g_0, g_9)$ be a free $\mathscr{A}$-module with generators of degree 0 and 9, and I the left ideal generated by $Sq^1 g_0$, $Sq^2 g_0$, $Sq^4 g_0$, $Sq^8 g_0$, $Sq^2 g_9$, and $Sq^{16}g_0 + Sq^7g_9 + Sq^4Sq^2Sq^1g_9$. Let $P_N = RP^\infty/RP^{N-1}$. All cohomology groups have Z_2-coefficients. Our other main result is

THEOREM 1.3. (i) *There is an isomorphism of $\mathscr{A}$-modules through degree $N + 18$*

$$H^*(BO[9], BO_N[9]) \approx \tilde{H}^*(\Sigma P_N) \otimes \mathscr{A}(g_0, g_9)/I$$

(ii) *For $j = 0, 1, 2, 4(8)$ and $j \geqq 10$, there is an isomorphism of $\mathscr{A}$-modules through degree $N + 2j$*

$$H^*(BO[j], BO_N[j]) \approx \tilde{H}^*(\Sigma P_N) \oplus \tilde{H}^*(\Sigma P_N) \otimes \Sigma^j \mathscr{A}/I_j \ .$$

In Proposition 2.1 we show how 1.3 can be used to compute the Adams spectral sequence (ASS) for $BO[j]/BO_N[j]$ through degree $N + 2j$.

This work owes a heavy debt to Mark Mahowald, who devised this approach to immersions and suggested the validity of 1.3 (ii) and the case $\alpha(n) = 7$ of 1.1.

2. The spaces $BO[j]/BO_N[j]$. In this section we study the Z_2-cohomology and stable homotopy groups of the spaces $BO[j]/BO_N[j]$ through degree $N + 2j$.

Proof of Theorem 1.3(ii) Let $k\colon BO[j] \to BO$ and $\bar{k}\colon BO[j]/BO_N[j] \to BO/BO_N$. Let $i\colon \Sigma P_N = CP_N/P_N \to BO[j]/BO_N[j]$ be induced from the $2N$-equivalence $P_N \to V_N$ and the fibration $V_N \to BO_N[j] \to BO[j]$. The Serre spectral sequence ([9], [10]) of the relative fibration $(CV_N, V_N) \to (BO[j], BO_N[j]) \to BO[j]$ is trivial through degree $2N$ because it is mapped onto by that of $(BO, BO_N) \to BO$. Thus as a vector space $H^*(BO[j], BO_N[j])$ is isomorphic to $\langle\{\bar{k}^*w_m\colon m > N\}\rangle \otimes H^*(BO[j])$, where $\langle S\rangle$ is the vector space spanned by S. Here we use the external cup product and the fact that $i^*\bar{k}^*w_m = s\alpha^{m-1}$, the nonzero element in $H^m(\Sigma P_N)$.

Stong ([11]) showed $k^* = 0\colon H^i(BO) \to H^i(BO[j])$ for $i < 2^{\phi(j)-1}$, where $\phi(j)$ is the number of positive integers $\leqq j$ which are $\equiv 0, 1, 2, 4(8)$. Thus for $j \geqq 10$ $k^* = 0\colon H^i(BO) \to H^i(BO[j])$ for $i \leqq 2j$. By the Wu formula

$$S^iq(\bar{k}^*w_m) = \binom{m-1}{i}\bar{k}^*w_{m+1} = \sum_{a=0}^{i}\binom{m-1-i+a}{a}k^*w_a \cup \bar{k}^*w_{m+i-a} \ ,$$

so that for $i \leqq 2j$ $S^iq(\bar{k}^*w_m) = \binom{m-1}{i}\bar{k}^*w_{m+i}$. Hence through degree $N + 2j$ $\langle\bar{k}^*w_m\colon N < m \leqq N + 2j\rangle$ is an $\mathscr{A}$-submodule of $H^*(BO[j], BO_N[j])$ isomorphic to $\tilde{H}^*(\Sigma P_N^{N+2j-1})$. Thus by the Cartan formula the vector space splitting of the previous paragraph gives an isomorphism of $\mathscr{A}$-modules through degree $N+2j$ $H^*(BO[j], BO_N[j]) \approx \tilde{H}^*(\Sigma P_N) \otimes H^*(BO[j])$. Through degree $2j-1$ $H^*(BO[j]) \approx Z_2 \oplus \Sigma^j \mathscr{A}/I_j$ as $\mathscr{A}$-modules ([11]).

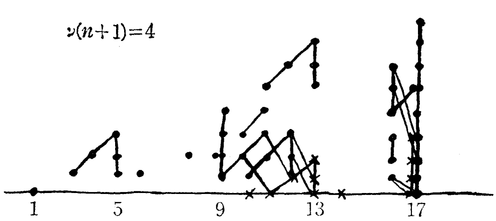
Proof of Theorem 1.3(i) This follows the same outline as the previous proof with a few modifications due to the fact that $2^{\phi(9)-1} < 2 \cdot 9$. This time

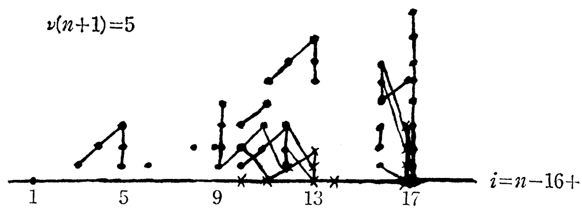
$$Sq^i(\bar{k}^* w_m) = \binom{m-1}{i} \bar{k}^* w_{m+i} + \begin{cases} \bar{k}^* w_{m+i-16} \cup k^* w_{16} & \text{if } i = 16 \text{ or } m \text{ is even and } i = 17 \\ 0 & \text{otherwise.} \end{cases}$$

Through degree 17, $H^*(BO[\mathbf{9}]) \approx \boldsymbol{Z}_2 \oplus \Sigma^9 \mathscr{A}/\mathscr{A}(Sq^2)$. Let u_9 denote the nonzero element of $H^9(BO[\mathbf{9}])$. Then $k^* w_{16} = (Sq^7 + Sq^4 Sq^2 Sq^1) u_9$ because $(Sq^7 + Sq^4 Sq^2 Sq^1) u_9$ is the only nonzero element of $H^{16}(BO[\mathbf{9}])$ annihilated by Sq^1 and Sq^2, which is true of $k^* w_{16}$. The homomorphism $\psi \colon \tilde{H}^*(\Sigma P_N) \otimes \mathscr{A}(g_0, g_9)/I \to \tilde{H}^*(BO[\mathbf{9}]/BO_N[\mathbf{9}])$ defined by $\psi(s\alpha^m \otimes g_0) = \bar{k}^* w_{m+1}$, $\psi(s\alpha^m \otimes Sq^I g_9) = \bar{k}^* w_{m+1} \cup Sq^I u_9$ is easily seen to be an $\mathscr{A}$-module isomorphism in the desired range. For example, $\psi(Sq^{16}(s\alpha^m \otimes g_0)) = \psi\left(\binom{m}{16} s\alpha^{m+16} \otimes g_0 + s\alpha^m \otimes (Sq^7 + Sq^4 Sq^2 Sq^1) g_9\right) = \binom{m}{16} \bar{k}^* w_{m+17} + \bar{k}^* w_{m+1} \cup (Sq^7 + Sq^4 Sq^2 Sq^1) u_9 = Sq^{16}(\bar{k}^* w_{m+1}) = Sq^{16}(\psi(s\alpha^m \otimes g_0))$.

Theorem 1.3 enables one to compute $\operatorname{Ext}_{\mathscr{A}}(\tilde{H}^*(BO[j]/BO_N[j]), \boldsymbol{Z}_2)$, the E_2-term of the ASS converging to the stable homotopy groups of $BO[j]/BO_N[j]$. We exemplify with the case which will be used in proving Theorem 1.2. Other cases are treated similarly. Ext groups are graphed as in [2]-[8], with vertical lines indicating multiplication by $h_0 \in \operatorname{Ext}_{\mathscr{A}}^{1,1}(\boldsymbol{Z}_2, \boldsymbol{Z}_2)$ which corresponds (up to elements of higher filtration) to multiplication by 2 in homotopy groups, and diagonal (/) lines indicating multiplication by $h_1 \in \operatorname{Ext}_{\mathscr{A}}^{1,2}(\boldsymbol{Z}_2, \boldsymbol{Z}_2)$, which corresponds to $\eta \in \pi_{n+1}(S^n)$. Differentials in the ASS are indicated by diagonal (\) lines.

PROPOSITION 2.1. *Suppose* $n \equiv 7(8)$. *The ASS chart for* $BO[\mathbf{9}]/BO_{n-16}[\mathbf{9}]$ *is given by*





with some differentials omitted in the top degree.

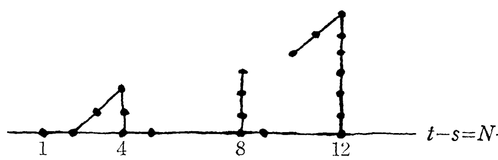
Proof. Let $N = n - 16$. By 1.3 (i) there is a short exact sequence of $\mathscr{A}$-modules

$$0 \longrightarrow \tilde{H}^*(\Sigma P_N) \otimes \Sigma^9 \mathscr{A}/\mathscr{A}(Sq^2) \longrightarrow H^*(BO[9], BO_N[9]) \longrightarrow \tilde{H}^*(\Sigma P_N) \longrightarrow 0 ,$$

inducing a long exact sequence in $\mathrm{Ext}_{\mathscr{A}}(\ , \boldsymbol{Z}_2)$. $\mathrm{Ext}_{\mathscr{A}}(\tilde{H}^*(\Sigma P_N), \boldsymbol{Z}_2)$ is given in [7; 8.16]. Let $\mathscr{A}_1$ denote the subalgebra of $\mathscr{A}$ generated by Sq^1 and Sq^2. By the method of [1; § 6] $\mathscr{A}/\mathscr{A}(Sq^2) \approx \mathscr{A}//\mathscr{A}_1 \otimes M$, where M is the $\mathscr{A}_1$-module with nonzero element Sq^0, Sq^1, Sq^2Sq^1. By the change-of-rings theorem ([2; 3.1])

$$\mathrm{Ext}_{\mathscr{A}}(\tilde{H}^*(\Sigma P_N) \otimes \mathscr{A}//\mathscr{A}_1 \otimes M, \boldsymbol{Z}_2) \approx \mathrm{Ext}_{\mathscr{A}_1}(\tilde{H}^*(\Sigma P_N) \otimes M, \boldsymbol{Z}_2) .$$

This is computed as in [2; Ch. 3] or [8, Ch. 4] to begin



There is a nonzero boundary homomorphism in the Ext-sequence, which we picture as a d_1-differential in the ASS. The d_2-differentials are deduced by applying $\pi_*(\ )$ to the diagram

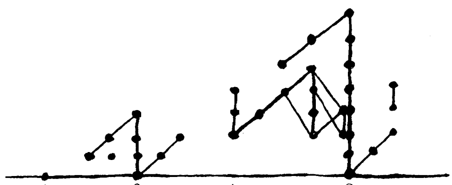
$$\begin{tikzcd}
\Sigma^{-1}(BO[9]/BO_{n-16}[9]/\Sigma P_{n-16}) \arrow[d] \arrow[dr] & \\
 & \Sigma P_{n-16} \\
\Sigma^{-1}(BO[8]/BO_{n-16}[8]/\Sigma P_{n-16}) \arrow[ur] &
\end{tikzcd}$$

using the results of [4; Ch. 3]. The higher differentials in the top degree which are present in ΣP_{n-16} are deduced by considering the map of ASS induced by $P_{n-16} \rightarrow P_{n-16} \wedge bJ$ (see [5]). We have also used $d_2(h_0({}_{16}1)) = 0$ in ASS (P_{n-16}), which is proved by going back to P_{n-24}. When $\nu(n+1) = 6$, the element in $s = 5$, $t - s = n$ may not be killed, which is the reason no immersion result is stated in this case.

3. Proof of nonimmersions (Theorem 1.1). This proof is very similar to that in [4; Ch.5]. Let $f\colon P^{n-2^{\alpha(n)-3}+11} \to BO[8]$ classify the restriction of the stable normal bundle $(2^L - n - 1)\xi$ of P^n. We will prove the composite $P^{n-2^{\alpha(n)-3}+11} \xrightarrow{f} BO[8] \xrightarrow{k} BO[8]/BO_{n-2^{\alpha(n)-3}-4}[8]$ is essential.

Let $a = \alpha(n) - 4$ and $b = (n+1)/2^a$. Then $\alpha(b-1) = 4$ and $a \geqq 3$. From [4; 3.2] the ASS chart for $\pi_i(BO[8]/BO_{(b-2)2^a-5}[8])$ is

(3.1)



In order to use the main result of [3], we "factor" f through QP by going into $BSpin$ as in [4; 4.1]. Thus we have

(3.2)
$$\begin{tikzcd} BO[8] \arrow[r] & BSpin & BSp \arrow[l] \\ P^{(b-2)2^a+10} \arrow[u, "f"] \arrow[rr] & & QP^{(b-2)2^{a-2}+2} \arrow[u, "g"'] \end{tikzcd}$$

where g classifies $(2^{L-2} - b2^{a-2})H$.

LEMMA 3.3. *If $\alpha(b-1) = 4$ and $a \geqq 3$, then $\nu\binom{2^{L-2} - b2^{a-2}}{(b-2)2^{a-2} + \Delta} = 3$ if $\Delta = 0$ and is greater than 3 if $\Delta = -1, 1,$ or 2.*

Proof. By [3; 4.1]

$$\begin{aligned}\nu\binom{2^{L-2} - b2^{a-2}}{(b-2)2^{a-2}} &= \alpha((b-2)2^{a-2}) + \alpha(2^{L-2} - 2^{a-2}(2b-2)) - \alpha(2^{L-2} - b2^{a-2}) \\ &= \alpha(b-2) + L - a - \alpha(2b-3) - (L - a - \alpha(b-1)) \\ &= \alpha(b-2) - (\alpha(2b-4) + 1) + \alpha(b-1) = -1 + 4 = 3\,.\end{aligned}$$

The case $\Delta \neq 0$ is handled by similar techniques.

Let $8l = (b-2)2^a$ and $Q = BO[8]/BO_{8l-5}[8]$. Let $Q\langle 3\rangle$ be the space formed from Q by killing Ext^s for $s < 3$. ([4; 2.1]). $Q\langle 3\rangle$ has cohomology generators $k_{8l}, k_{8l+4}, k_{8l+5}, k_{8l+7}, k_{8l+8}, k'_{8l+8},$ and k_{8l+10} corresponding to the elements of (3.1) of filtration 3.

LEMMA 3.4. *kf lifts to a map $P^{8l+10} \to Q\langle 3\rangle$ which sends only k_{8l} nontrivially.*

Proof. By the method of [4; Ch. 2 and 4.2], which is based upon [3; 1.8], and using Lemma 3.3 and the fact that $\pi^s_{4i}(\Sigma P_{8l-5}) \to \pi^s_{4i}(\Sigma P_{8l-5} \wedge bo)$ is injective for $i \leqq 2l + 2$ (by [7; 8.4 and 8.12]), there exists a lifting of g to E_3 in the modified Postnikov tower (MPT) of the fibration $\widetilde{BSp}_{8l-5} \to BSp$ which sends only the $8l$-dimensional k-invariant nontrivially. Thus, since (3.2) effectively gives a factorization through QP, as in the proof of [4; 5.1] f lifts to a map $\tilde{f}$: $P^{8l+10} \to E_3''$, where E_3'' is the third stage of the MPT of $BO_{8l-5}[8] \to BO[8]$, sending only the $8l$-dimensional k-invariant $\bar{k}_{8l}$ nontrivially. There is a map $E_3'' \xrightarrow{j} Q\langle 3\rangle$ and its behaviour on k-invariants is computed by computing the induced morphism of minimal resolutions. In particular, for the k-invariant $k'_{8l+8} \in H^{8l+8}(Q\langle 3\rangle)$ corresponing to the element in coker $(\mathrm{Ext}^{3,8l+11}_{\mathscr{A}}(\tilde{H}^*(\Sigma P_{8l-5}), \boldsymbol{Z}_2) \to \mathrm{Ext}^{3,8l+11}_{\mathscr{A}}(\tilde{H}^*(Q), \boldsymbol{Z}_2))$, if l is odd, $j^*(k'_{8l+8}) = (Sq^8 + w_8)\bar{k}_{8l}$ so that $\tilde{f}^*j^*k_{8l+8} = Sq^8\alpha^{8l} + w_8((2^L - 8l - 2^{a+1})\xi) \cup \alpha^{8l} = \alpha^{8l+8} + \alpha^{8l+8} = 0$, while if l is even, $\mathscr{A}(BO[8])$ annihilates $\bar{k}_{8l}$ in the range under consideration.

Theorem 1.1 follows from Lemma 3.4 together with

PROPOSITION 3.5. *If $P^{8l+10} \to Q\langle 3\rangle$ sends only k_{8l} nontrivially, then $P^{8l+10} \to Q\langle 3\rangle \to Q$ is essential.*

Proof. We show that such a map represents a nontrivial class of $\mathrm{Ext}^{3,3}_{\mathscr{A}}(\tilde{H}^*(Q), \tilde{H}^*(P^{8l+10}))$ which is not hit by a differential in the ASS converging to $[P^{8l+10}, Q]$.

A minimal $\mathscr{A}$-resolution of $\tilde{H}^*(Q)$ corresponds to a minimal $\mathscr{A}_2$-resolution of $\tilde{H}^*(\Sigma P_{8l-5})$. (See [4; 3.1].) This is listed in the Appendix. $\mathrm{Ext}^{3,3}_{\mathscr{A}}(\tilde{H}^*(Q), \tilde{H}^*(P^{8l+10})) \approx \ker d_3^*/\mathrm{im}\, d_2^*$ in

$$\mathrm{Hom}(\Sigma^{-3}C_2, \tilde{H}^*P^{8l+10}) \xrightarrow{d_2^*} \mathrm{Hom}(\Sigma^{-3}C_3, \tilde{H}^*P^{8l+10}) \xrightarrow{d_3^*} \mathrm{Hom}(\Sigma^{-3}C_4, \tilde{H}^*P^{8l+10}) .$$

If we denote by $\hat{k}_i$ the $\mathscr{A}_2$-homomorphism $\Sigma^{-3}C_3 \to \tilde{H}^*(P^{8l+10})$ sending only $\Sigma^{-3}k_i$ nontrivially, then $d_3^*(\hat{k}_0) = 0$ because, for example $d_3^*\hat{k}_0(\Sigma^{-3}l_4) = \hat{k}_0 d_3(\Sigma^{-3}l_4) = \hat{k}_0(\Sigma^{-3}(Sq^1k_4 + Sq^2Sq^3k_0)) = Sq^2Sq^3\alpha^{8l} = 0$. The image of d_2^* is generated by $\hat{k}_8 = d_2^*(\hat{h}_8)$, $\hat{k}'_8 + \hat{k}_{10} = d_2^*(\hat{h}_7)$, $\hat{k}_0 + \hat{k}'_8 = d_2^*(\hat{h}_{-1})$ and $\hat{k}_4 + \hat{k}_5 = d_2^*(\hat{h}_4)$. Thus $\hat{k}_0$ gives a nonzero element of Ext.

Similarly $\mathrm{Ext}^{j,j+1}_{\mathscr{A}}(\tilde{H}^*Q, \tilde{H}P^{8l+10}) = Z_2$ for $j = 0$ and 1. The nonzero elements in these groups survive to give the nontrivial elements $2^j[f_0]$, where f_0 is the map defined after (4.6). Thus there are no elements which could support a differential hitting $\hat{k}_0$.

4. **Proof of immersions (Theorem 1.2).** The proof is very similar to that of [4; 1.1]. We let $g: P^n \to BO[9]$ classify the stable normal

bundle and let $\mathscr{E}$ denote the fibre of $k_0\colon BO[9] \to C$, where $C = BO[9]/BO_{n-16}[9]$. We consider the diagram

$$\begin{tikzcd} & BO_{n-16}[9] \arrow[d] & \\ \Omega C \times \mathscr{E} \arrow[r, "\mu"] & \mathscr{E} \arrow[r, "k_1"] \arrow[d] & \mathscr{E}/BO_{n-16}[9] \\ P^n \arrow[r, "g"] & BO[9] \arrow[r, "k_0"] & C \end{tikzcd}$$

As in [4; 1.4(c)] the fibre of k_1 has the same n-type as $BO_{n-16}[9]$. (This is the main reason for using $BO[9]$ instead of $BO[8]$.) It suffices to prove

(4.1) $k_0 g$ is null-homotopic, so that there is a lifting l of g, and

(4.2) there is a map $P^n \to \Omega C$ such that $k_1\mu(f \times l)$ is null-homotopic.

Proof of 4.1. We use the charts of $\pi_*(C)$ given in 2.1.

Similarly to [4; 4.1] one shows that $k_0 g$ has filtration $\geqq 5$. This is accomplished by noting that if $n = 16l + 15$, then $\nu\binom{2^L - 16l - 16}{4l + \varepsilon} = \begin{cases} 5 & \varepsilon = 1 \text{ or } 3 \\ 4 & \varepsilon = 2 \end{cases}$, so that QP^{4l+3} lifts to E_4 sending only k_{16l+8} nontrivially. (QP gets past the irregular element in $s = 2$, $t - s = 16l + 8$ as in [4; 4.2]. It gets by the $x'd$ tower in $t - s = 16l + 12$ because they are not present in the MPT for $\widetilde{BSp_{n-16}} \to BSp$, where the liftings are first performed. (See [4; 4.1, 4.2].).) Thus P^n lifts to E_5 since primary indeterminacy enables one to vary k_{16l+8} without varying the other k^4-invariants.

Finally we show that any filtration 5 map $P^n \to C$ is null-homotopic. The only possible map not trivial by Ext-considerations (or by the differentials in the top degree) is an extension over P^n of the map $P^{n-3} \xrightarrow{k} S^{n-3} \xrightarrow{f} C$, where k is the collapse and $[f]$ the filtration 5 generator. But $[f]$ is divisible by 2 by [4; 3.5] since $BO[9]/BO_{n-16}[9] \to BO[8]/BO_{n-16}[8]$ sends the filtration 2 class in π_{n-3} to the filtration 3 class. Thus $P_{n-4}^{n-3} \to S^{n-3} \xrightarrow{f} C$ is trivial and hence so is fk. But there is a unique filtration 5 extension of fk over P^n since for $i = 0, 1, 2$, $\pi_{n-i}(C)$ has no elements of filtration $\geqq 5$. Hence the extension over P^n is trivial.

Proof of 4.2. This is very similar to [4; Ch. 4 beginning with 4.3]. $[k_1\mu(f \times l)]$ is considered as the homotopy sum of three stable maps

$$P^n \xrightarrow{l} \mathscr{E} \longrightarrow \mathscr{E}/BO_{n-16}[9] \tag{4.4}$$

$$P^n \xrightarrow{f} \Omega C \longrightarrow \mathscr{E}/BO_{n-16}[9] \tag{4.5}$$

$$P^n \xrightarrow{f \wedge l} \Omega C \wedge \mathscr{E} \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}/BO_{n-16}[9]\,. \tag{4.6}$$

Let f_0 denote the composite $P^n \to V_{n-16} \to \Omega\Sigma V_{n-16} \to \Omega C$ and f_1 the composite $P^n \to S^n \xrightarrow{u} \Omega C$ where $[u]$ has smallest possible filtration ($\leqq 2$). As in [4] we have the following results.

PROPOSITION 4.7. *(4.5) with $f = f_0$ and (4.6) with $f = f_1$ are null-homotopic.*

PROPOSITION 4.8. $[P^n, \mathscr{E}/BO_{n-16}[9]] \approx Z_2 \oplus Z_8 \oplus Z_2 \oplus Z_{16}$.

Rroof. As in [4; 4.3] $H^*(\mathscr{E}, BO_{n-16}[9]) \approx \tilde{H}^*(\Sigma^9 P_{n-16}) \otimes \mathscr{A}/I(Sq^2)$, so that its homotopy groups are as in (2.2, $j \equiv 1$) reindexed. $[P^n, \mathscr{E}/BO_{n-16}[9]]$ is computed as in [4; 4.9] or by computing $\mathrm{Ext}^{s,s}_{\mathscr{A}}(H^*(\mathscr{E}, BO_{n-16}[9]), P^n)$ as in 3.5. The Z_2's are due to the Z_2 homotopy groups in degrees $n-3$ and $n-7$.

Let G_1, G_2, G_3, and G_4 denote generators under a splitting of Proposition 4.8.

PROPOSITION 4.9. *Some multiple of (4.5) with $f = f_1$ equals $4G_2$.*

PROPOSITION 4.10. *If $\nu(n+1) = 4$, then (4.6) with $f = f_0$ has odd coefficient of G_1. If $\nu(n+1) = 5$, then (4.6) with $f = f_0$ is $2aG_1 + 2bG_2$ where a is an odd integer and b is an integer.*

Proof. This can be seen by using the map $\mathscr{E}/BO_{n-16}[9] \to \bar{\mathscr{E}}/BO_{n-16}[8]$, where $\bar{\mathscr{E}} = \mathrm{fibre}\,(BO[8] \to BO[8]/BO_{n-16}[8])$, and [4; 4.11, 4.12].

To deduce 1.2, we use the fact ([4; 1.2]) that P^n immerses in R^{2n-14} and argue as in the last paragraphs of [4; Ch. 4]. We consider the diagram

$$\begin{array}{ccccc}
\Omega(BO[9]/BO_{n-16}[9]) \times \mathscr{E} & \xrightarrow{\mu} & \mathscr{E} & \xrightarrow{k_1} & \mathscr{E}/BO_{n-16}[9] \\
\big\downarrow{\scriptstyle j_1 \times j_2} & & \big\downarrow{\scriptstyle j_2} & & \big\downarrow{\scriptstyle j_3} \\
\Omega(BO[9]/BO_{n-14}[9]) \times \mathscr{E}' & \xrightarrow{\mu'} & \mathscr{E}' & \xrightarrow{k_1'} & \mathscr{E}'/BO_{n-14}[9]
\end{array}$$

$[P^n, j_3]$ is onto with kernel $\langle 4G_2, 8G_4\rangle$. $[P^n, j_1]$ is onto all elements except a filtration 1 map f_2 trivial on P^{n-2}. The analogue of (4.6) with $f = f_2$ is null-homotopic and the analogue of (4.5) with $f = f_2$ is 0 or $4G_2'$.

Let $l: P^n \to \mathscr{E}$ be some lifting of g. There exists $f': P^n \to \Omega(BO[9]/BO_{n-14}[9])$ such that $k_1'\mu'(f' \times j_2 l) = 0$. Either f' or $f' - f_2$ factors as $P^n \xrightarrow{f} \Omega C \xrightarrow{j_1} \Omega(BO[9]/BO_{n-14}[9])$. Then $k_1'\mu'(j_1 \times j_2)(f \times l) = 0$ or $4G_2'$. Hence $k_1\mu(f \times l)$ is $4aG_1 + 4bG_2$ for some $a \in \boldsymbol{Z}_2$, $b \in \boldsymbol{Z}_4$. If $a=0$, by 4.7 and 4.9 there is some multiple df_1 such that $k_1\mu((f-df_1)\times l)=0$. If $a=1$, there exists d such that $k_1\mu((f+2^{6-\nu(n+1)}f_0-df_1)\times l)=0$.

5. **Appendix.** A minimal $\mathscr{A}_2$-resolution through degree $8l+10$ of $\tilde{H}^*(\Sigma P_{8l-5})$ is given by $C_0 \xleftarrow{d_0} C_1 \xleftarrow{d_1} C_2 \xleftarrow{d_2} C_3 \xleftarrow{d_3} C_4 \leftarrow$, where C_s is a free $\mathscr{A}_2$-module generated by elements x_i, g_i, h_i, k_i, or l_i for $s=0,1,2,3$, or 4 with subscripts indicating $t-s-8l$, where t is the degree of the generator. (See [4; Ch. 6].) We omit Sq for Steenrod squares; thus, $62g$ denotes Sq^6Sq^2g. This resolution corresponds to (3.1).

C_0 has generators x_{-4}, x_0, and x_8

g_{-2}: $21x_{-4}$ (This means $d_0(g_{-2}) = Sq^2Sq^1x_{-4}$)
g_{-1}: $4x_{-4}$
g_0 : $1x_0 + 41x_{-4}$
g_1 : $2x_0 + 42x_{-4}$
g_8 : $1x_8 + 27x_0$
g_9 : $2x_8 + 424x_0$
h_{-1}: $2g_{-2}$
h_0 : $1g_0 + 21g_{-2}$
h_2 : $2g_1 + 3g_0 + 4g_{-1}$
h_4 : $51g_{-1} + (7 + 421)g_{-2}$
h_7 : $621g_{-1} + (91 + 46)g_{-2}$
h_8 : $1g_8 + 521g_1 + 54g_0 + 46g_{-1}$
h_8' : $(46 + 73 + 631)g_{-1} + 461g_{-2}$
h_{10} : $2g_9 + 3g_8 + (46 + 91)g_1 + 47g_0$
k_0 : $1h_0 + 2h_{-1}$
k_4 : $1h_4 + 41h_0$
k_5 : $2h_4 + (7 + 421)h_{-1}$
k_7 : $1h_7 + 4h_4 + 51h_2 + 72h_{-1}$
k_8 : $1h_8 + 43h_2 + (27 + 72)h_0 + 631h_{-1}$
k_8' : $1h_8' + 2h_7 + 43h_2 + (27 + 72)h_0 + 46h_{-1}$
k_{10} : $3h_8' + 4h_7 + 423h_2 + (66 + 75)h_{-1}$
l_4 : $1k_4 + 23k_0$
l_6 : $2k_5 + 3k_4 + (7 + 421)k_0$
l_7 : $1k_7 + 21k_5 + 4k_4 + 62k_0$
l_8 : $1k_8 + 72k_0 + 41k_4$
l_{10} : $1k_{10} + 21k_8' + 4k_7 + (6 + 51)k_5 + 7k_4$

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