PACIFIC JOURNAL OF MATHEMATICS
Vol. 77, No. 1, 1978

A NEW FAMILY OF PARTITION IDENTITIES

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The partition function $A(n; k)$ is the number of partitions of n with minimal difference k. Our principal result is that for all $k \geqq 1$, $A(n; k) \equiv B(n; k)$, where $B(n; k)$ is the number of partitions of n into distinct parts such that for $1 \leqq i \leqq k$, the smallest part $\equiv i \pmod{k}$ is $> k \sum_{j=1}^{i-1} r(j)$, where $r(j)$ is the number of parts $\equiv j \pmod{k}$. This arises as a corollary to a more general result.

The particular case $A(n; 2) = B(n; 2)$ was recently proved by Andrews and Askey [1]. It is known from the Rogers-Ramanujan identities (e.g., Harby and Wright [2], p. 291) that $A(n; 2)$ is equal to the number of partition of n into parts $\equiv \pm 1 \pmod{5}$. Andrews and Askey discovered a q-series identity due to Rogers which has the partition theoretic interpretation: $B(n; 2)$ is equal to the number of partitions of n into parts $\equiv \pm 1 \pmod{5}$.

The general identity. Given $k \geqq 1$, let $q(1), q(2), \cdots, q(k)$ be any complete residue system mod k. We define the following partition functions:

$A(n; k; q(1), \cdots, q(k); r(1), \cdots, r(k))$ = number of partitions of n with minimal difference k and such that for $1 \leqq i \leqq k$, there are $r(i)$ parts $\equiv q(i) \pmod{k}$.

$B(n; k; q(1), \cdots, q(k); r(1), \cdots, r(k))$ = number of partitions of n into distinct parts such that for $1 \leqq i \leqq k$, there are $r(i)$ parts $\equiv q(i) \pmod{k}$, and the smallest part $\equiv q(i) \pmod{k}$ is $> k \sum_{j=1}^{i-1} r(j)$.

$C(n; k; q(1), \cdots, q(k); r(1), \cdots, r(k))$ = number of partitions of n such that for $1 \leqq i \leqq k$, there are $r(i)$ parts $\equiv q(i) \pmod{k}$.

Given $r(1), \cdots, r(k)$, we set $S = \sum_{i=1}^{k} r(i)$ = number of parts in the partition.

LEMMA 1.

$$\begin{aligned} &A(n; k; q(1), \cdots, q(k); r(1), \cdots, r(k)) \\ &\qquad = C(n - kS(S-1)/2; k; q(1), \cdots, q(k); r(1), \cdots, r(k)) . \end{aligned}$$

Proof. Given a partition of n with minimal difference k and $r(i)$ parts $\equiv q(i) \pmod{k}$, subtract k from the second smallest part, $2k$ from the third smallest part, and, in general $k(j-1)$ from the jth smallest part. This gives us a partition of $n - kS(S-1)/2$ with $r(i)$ parts $\equiv q(i) \pmod{k}$ for all i, $1 \leqq i \leqq k$.

Similarly, given a partition of $n - kS(S-1)/2$ with $r(i)$ parts $\equiv q(i) \pmod k$, add $k(j-1)$ to the jth smallest part. This yields a partition of n with minimal difference k and $r(i)$ parts $\equiv q(i) \pmod k$.

LEMMA 2.

$$\begin{aligned} &B(n; k; q(1), \cdots, q(k); r(1), \cdots, r(k)) \\ &\qquad = C(n - kS(S-1)/2; k; q(1), \cdots, q(k); r(1), \cdots, r(k)) . \end{aligned}$$

Proof. Given a partition of n into distinct parts such that $r(i)$ parts are $\equiv q(i) \pmod k$ and the smallest part $\equiv q(i) \pmod k$ is $> k \sum_{l=1}^{i-1} r(l)$, we subtract k from the second smallest part $\equiv q(1) \pmod k$, $2k$ from the third smallest part $\equiv q(1) \pmod k$, and so on up to subtracting $k(r(1)-1)$ from the largest part $\equiv q(1) \pmod k$. We then subtract $kr(1)$ from the smallest part $\equiv q(2) \pmod k$, $k(r(1)+1)$ from the second smallest part $\equiv q(2) \pmod k$, and so on up to subtracting $k(r(1)+r(2)-1)$ from the largest part $\equiv q(2) \pmod k$.

We continue this process, in general subtracting $k(j-1+\sum_{l=1}^{i-1} r(l))$ from the jth smallest part $\equiv q(i) \pmod k$. Recall that the jth smallest part $\equiv q(i) \pmod k$ is

$$\begin{aligned} &\geqq k(j-1) + (\text{the smallest part} \equiv q(i) \pmod k) \\ &> k(j-1) + k \sum_{l=1}^{i-1} r(l) . \end{aligned}$$

Also note that:

$$\sum_{i=1}^{k} \sum_{j=1}^{r(i)} k\Big(j - 1 + \sum_{l=1}^{i-1} r(l)\Big) = kS(S-1)/2 .$$

Thus, this gives us a partition of $n - kS(S-1)/2$ with $r(i)$ parts $\equiv q(i) \pmod k$ for all i, $1 \leqq i \leqq k$.

Similarly, given a partition of $n - kS(S-1)/2$ with $r(i)$ parts $\equiv q(i) \pmod k$, add $k(j-1+\sum_{l=1}^{i-1} r(l))$ to the jth smallest part $\equiv q(i) \pmod k$. This yields a partition of n into distinct parts such that $r(i)$ parts are $\equiv q(i) \pmod k$ and the smallest part $\equiv q(i) \pmod k$ is $> k \sum_{l=1}^{i-1} r(l)$.

As an immediate consequence of Lemmas 1 and 2, we have:

THEOREM.

$$\begin{aligned} &A(n; k; q(1), \cdots, q(k); r(1), \cdots, r(k)) \\ &\qquad = B(n; k; q(1), \cdots, q(k); r(1), \cdots, r(k)) . \end{aligned}$$

COROLLARY 1. $A(n; k) = B(n; k)$.

Proof. For all i, $1 \leqq i \leqq k$, let $q(i) = i$. Then

$$A(n; k) = \sum_{r(1),\cdots,r(k)=0}^{\infty} A(n; k; 1, \cdots, k; r(1), \cdots, r(k)) ,$$

and

$$B(n; k) = \sum_{r(1),\cdots r,(k)=0}^{\infty} B(n; k; 1, \cdots, k; r(1), \cdots, r(k)) .$$

COROLLARY 2. *Let σ be a permutation function on the integers 1 through k. Given a function r defined on these integers, let $R(i) = r(\sigma^{-1}(i))$. Then:*

$$\begin{aligned} &B(n; k; \sigma(1), \cdots, \sigma(k); R(1), \cdots, R(k)) \\ &\quad = B(n; k; 1, 2, \cdots, k; r(1), \cdots, r(k)) . \end{aligned}$$

Proof.

$$\begin{aligned} &A(n; k; \sigma(1), \cdots, \sigma(k); R(1), \cdots, R(k)) \\ &\quad = A(n; k; 1, 2, \cdots, k; r(1), \cdots, r(k)) , \end{aligned}$$

since they count exactly the same partitions.

In particular, letting $k = 2$ in Corollary 2 gives us:

The number of partitions of n into distinct parts such that r of them are odd and s are even and the smallest even part is $>2r$.

=The number of partitions of n into distinct parts such that r of them are odd and s are even and the smallest odd part is $>2s$.

CONCLUSION. One corollary of Lemma 1 is of interest. Using Lemma 1, we have that:

$$\begin{aligned} &\text{the number of partitions of } n \text{ into parts} \equiv \pm 1 \pmod 5 \\ &\quad = \sum_{r,s=0}^{\infty} C(n; 5; 1, 2, 3, 4, 5; r, 0, 0, s, 0) \\ &\quad = \sum_{r,s=0}^{\infty} A(n + 5(r+s)(r+s-1)/2; 5; 1, 2, 3, 4, 5; r, 0, 0, s, 0) . \end{aligned}$$

By the first Rogers-Ramanujan identity:

$$\begin{aligned} &\text{the number of partitions of } n \text{ into parts} \equiv \pm 1 \pmod 5 \\ &\quad = A(n; 2) \\ &\quad = \sum_{R,S=0}^{\infty} A(n; 2; 1, 2; R, S) . \end{aligned}$$

Thus:

$$\begin{aligned} &\sum_{R,S=0}^{\infty} A(n; 2; 1, 2; R, S) \\ &\quad = \sum_{r,s=0}^{\infty} A(n + 5(r+s)(r+s-1)/2; 5; 1, 2, 3, 4, 5; r, 0, 0, s, 0) . \end{aligned} \tag{1}$$

The significance of equation (1) lies in the fact that if a purely combinatorial proof can be found for it, this will give us a purely combinatorial proof of the first Rogers-Ramanujan identity.

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Received December 28, 1976.

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