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C^*-ALGEBRAS WITH APPROXIMATELY INNER FLIP

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A remarkable recent theorem of A. Connes shows that if N is a II_1 factor on a separable Hilbert space, the "flip automorphism" of $N \otimes N$ can be approximated pointwise by inner automorphisms if and only if N is hyperfinite. We have accordingly been led to consider the analogous question of when, for a C^*-algebra A, the "flip" on $A \otimes A$ (C^*-rather than W^*-tensor product) is "approximately inner." We find that under certain additional hypotheses, A must be UHF (if A has a unit) or matroid (if we allow nonunital algebras).

1. Introduction. Given C^*-algebras A and B, we shall let $A \otimes B$ denote the "minimal" or "spatial" C^*-algebraic tensor product. The *flip* for A is the automorphism

$$\sigma\colon A \otimes A \longrightarrow A \otimes A\colon a \otimes b \longmapsto b \otimes a .$$

In this paper we begin by investigating those C^*-algebras A for which σ is approximately inner, i.e., a point-norm limit of inner automorphisms. Our interest in these algebras was stimulated by a profound result of Connes concerning II_1 factors on a separable Hilbert space. He proved that for such an algebra N, the flip on the von Neumann tensor product $N \otimes N$ is a point-(weak operator) limit of inner automorphisms if and only if N is isomorphic to the hyperfinite II_1 factor R. This provided a key step in his characterization of the injective factors [8, V].

One might expect from Connes' result that very few C^*-algebras have an approximately inner flip (for the nonunital definition, see § 2). To the best of our knowledge, the only such C^*-algebras (resp., unital C^*-algebras) may be the matroid (resp., UHF) algebras. In § 2 we show that any C^*-algebra with approximately inner flip must be simple and nuclear. Then in § 3, we use a form of algebraic K-theory to show that such an AF algebra must be matroid.

In his proof, Connes made considerable use of "asymptotic" imbeddings of II_1 factors into the hyperfinite factor R (see, e.g., [18] for earlier results). This has led us to consider in § 4 those unital C^*-algebras which have analogous C^*-algebraic imbeddings into the "universal" $(2^\infty \cdot 3^\infty \cdot 5^\infty \cdot \cdots)$ UHF C^*-algebra $\mathscr{U}$ (as defined in [14]). We prove that they include the unital extensions of CCR algebras, the approximately finite (AF) C^*-algebras of Bratteli [3], and more generally quasi-diagonal algebras (such as the simple C^*-algebras of Bunce and Deddens [5]). On the other hand, type I C^*-algebras may be constructed without this property.

Following Connes' approach, we conclude in § 5 by showing that $\mathscr{U}$ is the only unital separable C^*-algebra A for which

(1) the flip is approximately inner,

(2) A has an asymptotic imbedding in $\mathscr{U}$,

(3) $A \cong A \otimes \mathscr{U}$.

It follows that for any separable A (not necessarily unital) with an approximate identity consisting of projections, with approximately inner flip, and with an asymptotic imbedding in $\mathscr{U}$, that $A \otimes \mathscr{U}$ is matroid. It is not clear to us, however, whether A itself must be matroid.

Throughout this paper, all maps, such as automorphisms, representations, etc., are assumed *-preserving. By "ideal" we shall mean norm-closed two-sided ideal, except in 2.9 and 2.10 where certain nonclosed ideals are needed. Given a complex Hilbert space $\mathscr{H}$, we shall let $\mathscr{B}(\mathscr{H})$ (resp., $\mathscr{K}(\mathscr{H})$) denote the bounded (resp., compact) linear operators on $\mathscr{H}$. If $\mathscr{H}$ is finite-dimensional, we may identify $\mathscr{B}(\mathscr{H}) = \mathscr{K}(\mathscr{H})$ with the complex $n \times n$ matrices M_n. An *approximate identity* $\{e_\gamma\}_{\gamma \in \Gamma}$ in a C^*-algebra A is a net of operators such that $0 \leqq e_\gamma \leqq 1$ and $\|e_\gamma a - a\| \to 0$ for all $a \in A$.

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2. Approximately inner flips. Given a C^*-algebra A, we let A^{**} and $\mathscr{B}(A)$ denote the second dual von Neumann algebra and the bounded linear operators on A, respectively. The *multiplier algebra* (or "double centralizer" algebra) $M(A)$ may be defined to be the set of elements $b \in A^{**}$ such that $bA + Ab \subseteq A$ (see [1]). The two natural maps $M(A) \to \mathscr{B}(A)$ defined by left and right multiplication are isometric isomorphisms. We let $A^\sim$ be the C^*-subalgebra of $M(A)$ generated by A and 1. If A has an identity $A^\sim = A$— otherwise $A^\sim$ is the usual unital extension of A by $\boldsymbol{C}$ (denoted by $A^\dagger$ in § 3).

If A is an ideal in a C^*-algebra B, we may identify A^{**} with $B^{**}e$, where e is a unique central projection in B^{**}. The map $b \mapsto be$ sends B into $M(A)$ since if $a \in A$,

$$(be)a = b(ea) = ba \in A ,$$

and a similar calculation applies to $b \mapsto eb$. In this sense, $M(A)$ may be regarded as the maximal "essential" unital extension of A.

If $\mathscr{H}$ is a Hilbert space and $\mathscr{H} \otimes \mathscr{H}$ is the Hilbert space tensor product, then the map $u: \mathscr{H} \otimes \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ determined by $x \otimes y \mapsto y \otimes x$ is unitary. Given a C^*-algebra A on $\mathscr{H}$, we may identify $A \otimes A$ with the closure of the algebraic tensor product of A with itself in $\mathscr{B}(\mathscr{H} \otimes \mathscr{H})$. Given $a, b \in A$, we have

$$u(a \otimes b)u^* = b \otimes a ,$$

hence the correspondence $a \otimes b \mapsto b \otimes a$ indeed extends to a *-automorphism $\sigma: A \otimes A \to A \otimes A$ that we called the flip in § 1.

If A is a C^*-algebra, each unitary $u \in M(A)$ defines an automorphism $\mathrm{Ad}\, u$ of A by

$$\mathrm{Ad}\, u(a) = uau^* ,$$

and we say that an automorphism of this form is *inner*. More generally, we say that an automorphism α of A is *approximately inner* if it is a point-norm limit of inner automorphisms. If A is nonunital, we may also call α *strongly* approximately inner if α is point-norm limit of automorphisms of the form $\mathrm{Ad}\, u$, u unitary in $A^\sim$. Although generally $A^\sim \subsetneqq M(A)$, often (and perhaps always) these two definitions agree (see also Theorem 3.8 below):

PROPOSITION 2.1. *If A is a C^*-algebra, α is an automorphism of A, and A has an approximate identity consisting of α-fixed projections, then α is approximately inner if and only if it is approximately inner in the stronger sense (approximable by* $\mathrm{Ad}\, u$*'s with $u \in A^\sim$).*

Proof. Suppose that α is approximately inner. Let $a_1, a_2, \cdots, a_n \in A$ and let $\varepsilon > 0$. Using the hypothesis, choose an α-fixed projection e such that $\|ea_j - a_j\| < \varepsilon$ and $\|a_j e - a_j\| < \varepsilon$ for all j. Choose u unitary in $M(A)$ with $\|ueu^* - e\| < \varepsilon$ and with $\|ua_ju^* - \alpha(a_j)\| < \varepsilon$ for $j = 1, \cdots, n$, using the fact that α is approximately inner. Then if $v = 1 - e + eu$, $v \in A^\sim$ and

$$\begin{aligned} \|vv^* - 1\| &= \|eu(1-e) + (1-e)u^*e\| \\ &\leqq 2\|eu - ue\| < 2\varepsilon , \\ \|v^*v - 1\| &= \|u^*eu - e\| < \varepsilon . \end{aligned}$$

Thus if ε is small enough, v^*v is invertible and $v(v^*v)^{-1/2} = w \in A^\sim$ is unitary and close to v in norm. Furthermore, for each j,

$$\begin{aligned} \|va_jv^* - \alpha(a_j)\| &= \|(1 - e + eu)a_j(1 - e + u^*e) - \alpha(a_j)\| \\ &\leqq \|(1-e)a_j(1 - e + u^*e)\| + \|eua_j(1-e)\| + \|e(ua_ju^*)e - \alpha(a_j)\| \\ &\leqq \|1 - e + u^*e\|\,\|a_j - ea_j\| + \|eu\|\,\|a_j - a_je\| \\ &\quad + \|e\{ua_ju^* - \alpha(a_j)\}e\| + \|e\alpha(a_j)e - \alpha(a_j)\| \\ &< 4\varepsilon + \|\alpha(ea_je - a_j)\| \\ &\leqq 4\varepsilon + \|e(a_je - a_j)\| + \|ea_j - a_j\| < 6\varepsilon . \end{aligned}$$

Thus $\|wa_jw^* - \alpha(a_j)\|$ will be small for ε small enough, and α is approximately inner in the stronger sense.

COROLLARY 2.2. *Suppose A is a C^*-algebra with an approximate identity consisting of projections. Then the flip α for A is approximately inner if and only if it is so in the stronger sense.*

Proof. If $\{e_\gamma\}$ is an approximate identity for A consisting of projections, then $\{e_\gamma \otimes e_\gamma\}$ is an approximate identity for $A \otimes A$ consisting of σ-fixed projections.

Keeping these results in mind, we shall work hereafter with our original definition, since $M(A)$ has better functorial properties than $A^\sim$. (For instance, the proof of the next lemma breaks down if we use $A^\sim$ in place of $M(A)$.)

Any inner automorphism α must *fix ideals*, i.e., if J is an ideal in A, then $\alpha(J) = J$, since J will again be an ideal in $M(A)$. It immediately follows that the same is true for approximately inner automorphisms. Another simple observation that we will need is:

LEMMA 2.3. *Given C^*-algebras A and B with approximately inner automorphisms α and β, respectively, the automorphism $\alpha \otimes \beta: A \otimes B \to A \otimes B$ determined by $a \otimes b \mapsto \alpha(a) \otimes \beta(b)$ is again approximately inner.*

Proof. We must show that given $c_1, \cdots, c_r \in A \otimes B$ and $\varepsilon > 0$, we may find a unitary $w \in M(A \otimes B)$ with

$$\|\alpha \otimes \beta(c_k) - wc_kw^*\| < \varepsilon\,, \quad k = 1, \cdots, r\,. \tag{2.1}$$

Since $\alpha \otimes \beta$ and $\mathrm{Ad}\, w$ are isometric and linear, it suffices to assume that each c_k has the form $a_k \otimes b_k$ with $\|a_k\|, \|b_k\| \leqq 1$. By hypothesis we may select unitaries $u \in M(A)$, $v \in M(B)$ with

$$\|\alpha(a_k) - ua_ku^*\| < \varepsilon/2$$
$$\|\beta(b_k) - vb_kv^*\| < \varepsilon/2\,, \quad \text{for} \quad k = 1, \cdots, r\,.$$

Regarding $M(A) \otimes M(B)$ as a unital subalgebra of $M(A \otimes B)$ (see [1, § 3]), $w = u \otimes v$ is a unitary in the latter algebra satisfying (2.1).

COROLLARY 2.4. *If A and B have approximately inner flips, then the same is true for $A \otimes B$.*

Proof. Under the natural identification

$$(A \otimes B) \otimes (A \otimes B) \cong (A \otimes A) \otimes (B \otimes B)\,,$$

$\sigma_{A\otimes B}$ corresponds to $\sigma_A \otimes \sigma_B$, hence we may apply Lemma 2.3.

Parallel to the "going-up Lemmas" 2.3 and 2.4 is the following "going-down lemma":

PROPOSITON 2.5. *Let A be a C^*-algebra, let α be an approximately inner automorphism of A, and let p be an α-fixed projection in A. Then $\alpha|_{pAp}$ is approximately inner.*

Proof. Let $a_1, \cdots, a_n \in pAp$ and let $\varepsilon > 0$. For convenience, let $a_0 = p$ and choose $\delta > 0$ small compared to ε ("how small" can be determined from Lemmas 1.8 and 1.10 of [14]). Since α is approximately inner, we may choose u unitary in $M(A)$ with

$$||ua_iu^* - \alpha(a_i)|| < \delta \quad \text{for} \quad i = 0, \cdots, n\ .$$

Since $e = upu^*$ and $p = \alpha(a_0)$ are projections in A close to each other in norm, Lemma 1.8 of [14] shows that there exists v unitary in $A^\sim$ (and thus in $M(A)$) with $vev^* = p$ and $||v - 1|| < \varepsilon/2$. Then vu commutes with p, hence $w = vup$ is unitary in pAp. Also

$$\begin{aligned} ||wa_iw^* - \alpha(a_i)|| &\leqq ||vua_iu^*v^* - ua_iu^*|| + ||ua_iu^* - \alpha(a_i)|| \\ &< \varepsilon||a_i|| + \delta \quad \text{for} \quad i = 1, \cdots, n\ , \end{aligned}$$

which can be made arbitrarily small, so $\alpha|_{pAp}$ is approximately inner.

COROLLARY 2.6. *If A is a unital C^*-algebra, B is a finite-dimensional C^*-algebra (or more generally any C^*-algebra with a minimal projection, such as any $\mathscr{K}(\mathscr{H})$), and $A \otimes B$ has an approximately inner flip, then so does A.*

Proof. If e is a minimal projection in B, then $p = (1 \otimes e) \otimes (1 \otimes e)$ in $(A \otimes B) \otimes (A \otimes B)$ is fixed by the flip $\sigma_{A\otimes B}$, and the pair $(p((A \otimes B) \otimes (A \otimes B))p$, rest. of $\sigma_{A\otimes B})$ is naturally isomorphic to $(A \otimes A, \sigma_A)$.

PROPOSITION 2.7. *If a C^*-algebra A has an approximately inner flip, then A is simple.*

Proof. Suppose that $J \neq 0$ is a proper ideal in A. We may regard $J_1 = J \otimes A$ and $J_2 = A \otimes J$ as ideals in $A \otimes A$, and it is evident that $\sigma(J_1) = J_2$. Assuming that σ is approximately inner, it follows that $J_1 = J_2$. But the latter is not the case. To see this, select $a \in J$, $b \notin J$ and $f, g \in A^*$ with $f(a) \neq 0$, $g|_J = 0$, $g(b) \neq 0$. Then $f \otimes g \in (A \otimes A)^*$ (see [15, 5.1]) and $f \otimes g(J_2) = \{0\}$, whereas

$$(f \otimes g)(a \otimes b) = f(a)g(b) \neq 0 ,$$

i.e., $a \otimes b \in J_1 \backslash J_2$.

A somewhat less obvious result is the following extension of [11, Theorem 3.1] to the nonunital case. (Simon Wassermann has kindly shown us a rather different proof for the unital case which might also generalize to this situation.)

PROPOSITION 2.8. *If A has an approximately inner flip, then A is nuclear.*

Proof. From [7, Theorem 3.1] it suffices to prove that given $a_1, \cdots, a_r \in A$ and $\varepsilon > 0$, we may find a finite rank completely positive map $\phi: A \to A$ such that

$$\|\phi(a_k) - a_k\| < \varepsilon , \quad k = 1, \cdots, r .$$

We may assume that $a_k \geqq 0$ and $\|a_k\| \leqq 1$. We select a positive linear functional p on A with $\|p\| = 1$ and an element $b_0 \in A^+$ with $p(b_0) \geqq 1 - (\varepsilon/2)$. By hypothesis there is a unitary $u \in M(A \otimes A)$ with

$$\|u(a_k \otimes b_0)u^* - b_0 \otimes a_k\| < \varepsilon/2 , \quad k = 1, \cdots, r . \tag{2.2}$$

Letting $\{e_\gamma\}$ be an approximate identity in $A \otimes A$, we have

$$(a_k^{1/2} \otimes b_0^{1/2})e_\gamma \longrightarrow a_k^{1/2} \otimes b_0^{1/2}$$

in norm, hence

$$e_\gamma(a_k \otimes b_0)e_\gamma \longrightarrow a_k \otimes b_0$$

in norm. Letting $v = ue_\gamma \in A \otimes A$ for some sufficiently large γ, we have from (2.2) that

$$\|v(a_k \otimes b_0)v^* - b_0 \otimes a_k\| < \varepsilon/2 , \quad k = 1, \cdots, r . \tag{2.3}$$

We may by continuity of the norm also assume that $v = \sum c_j \otimes d_j$, $c_j, d_j \in A$, $\|v\| \leqq 1$.

The map

$$p \otimes 1: A \odot A \longrightarrow A: a \otimes b \longmapsto p(a)b$$

(here $\odot$ denotes the algebraic tensor product)

extends to a contraction $A \otimes A \to A$ (this is Tomiyama's "slice map"—see [22]), and from (2.3) we conclude that

$$\|\sum p(c_i a_k c_j^*)d_i b_0 d_j^* - p(b_0)a_k\| < \varepsilon|2 .$$

The map

$$\phi\colon A \longrightarrow A\colon a \longmapsto \sum p(c_i a c_j^*)(d_i b_0^{1/2})(d_j b_0^{1/2})^*$$

is completely positive (see the argument for [11, Theorem 3.1]), and it is of finite rank. It is a contraction since

$$\|\phi(a)\| = \|(p \otimes 1)(v(a \otimes b_0)v^*)\| \leqq \|a\| .$$

Finally

$$\|\phi(a_k) - a_k\| \leqq \|\phi(a_k) - p(b_0)a_k\| + |1 - p(b_0)| < \varepsilon , \quad k = 1, \cdots, r ,$$

and we are done.

Propositions 2.7 and 2.8 show that C^*-algebras with approximately inner flip must be simple and nuclear, but in fact more is true. For instance, such algebras can have at most one trace:

LEMMA 2.9. *Let A be a C^*-algebra, let τ be a semi-finite, lower semi-continuous trace on A, and let $\mathfrak{m}$ be the ideal of definition of τ (see [9, 6.1.2]). (We use the same symbol for the trace τ and for the corresponding linear functional on $\mathfrak{m}$. Note that contrary to our usual convention regarding the term "ideal," $\mathfrak{m}$ need not be closed.) Then for any approximately inner automorphism α of A, $\alpha(\mathfrak{m}) = \mathfrak{m}$ and $\tau \circ \alpha = \tau$ (on $\mathfrak{m}$).*

Proof. Suppose $(\operatorname{Ad} u_\gamma)$ is a net of inner automorphisms converging to α. Then for $x \in \mathfrak{m}^+$, $\alpha(x) = \lim u_\gamma x u_\gamma^*$ and so $\tau(\alpha(x)) \leqq \limsup \tau(u_\gamma x^{1/2} x^{1/2} u_\gamma^*) = \limsup \tau(x^{1/2} u_\gamma^* u_\gamma x^{1/2}) = \tau(x) < \infty$ by lower semi-continuity of τ. Thus $\alpha(x) \in \mathfrak{m}^+$. Furthermore, $\tau(x) = \tau(\alpha^{-1}(\alpha(x))) \leqq \tau(\alpha(x))$ by the same argument, so $\alpha(\mathfrak{m}) = \mathfrak{m}$ and $\tau \circ \alpha = \tau$ on $\mathfrak{m}$.

PROPOSITION 2.10. *A C^*-algebra A with approximately inner flip can admit at most one (semi-finite, lower semi-continuous) trace (up to scalar multiplies).*

Proof. Let τ_1 and τ_2 be traces on A, and suppose the flip σ for A is approximately inner. There is a well-defined "product trace" $\tau = \tau_1 \otimes \tau_2$ on $A \otimes A$, which can be constructed by letting π_1 and π_2 be the traceable representations [9, 6.6] corresponding to τ_1 and τ_2 and then forming the tensor product representation $\pi = \pi_1 \otimes \pi_2$ on the tensor product Hilbert space. It is easy to see that π is traceable, and we can let τ be the corresponding trace. Note that since A is simple, τ_1 and τ_2 are defined on the Pedersen ideal K of A [19, Theorem 1.3] and clearly $\tau(a \otimes b) = \tau_1(a)\tau_2(b)$ for $a, b \in K$. By Lemma 2.9, τ is σ-invariant, and so $\tau_1(a)\tau_2(b) = \tau_1(b)\tau_2(a)$ for all $a, b \in K$. Taking $a \in K$ with $\tau_1(a) \neq 0$, we get $\tau_2 = (\tau_2(a)/\tau_1(a))\tau_1$ on K, hence everywhere [19, Corollary 3.2].

3. Approximately finite C^*-algebras and some algebraic K-theory.

We recall that a C^*-algebra A is said to be *approximately finite* or *AF* (respectively *matroid*) if there is a sequence of finite-dimensional (resp., matrix) algebras $A_1 \subseteqq A_2 \subseteqq \cdots$ with $A = (\bigcup A_p)^-$. (We consider here only separable algebras, although much of the discussion that follows is applicable to nonseparable AF algebras as well.) We call such a sequence (A_p) an *approximating system* for A. If A is unital, we say that (A_p) is a *unital approximating system* if, in addition, each A_p contains the identity of A.

We define an *equivalence* between approximating systems (A_p), (B_p) to be a sequence of unitaries $u_k \in A^\sim$ together with subsequences $(A_{p(k)})$, $(B_{q(k)})$ such that for all k,

(1) $\operatorname{Ad} u_k(A_{p(k)}) \subseteqq B_{q(k)}$,

(2) $\operatorname{Ad} u_n(A_{p(n)}) \supseteqq B_{q(k)}$ for large enough n,

(3) one has commutative diagrams

$$\begin{array}{ccc} A_{p(k)} & \xrightarrow[\operatorname{Ad} u_k]{} & B_{q(k)} \\ \big\downarrow & & \big\downarrow \\ A_{p(k+1)} & \xrightarrow[\operatorname{Ad} u_{k+1}]{} & B_{q(k+1)} \end{array} .$$

Extending results of Glimm [14, Theorem 1.12] and Dixmier [10], Bratteli proved

THEOREM 3.1. *Any two approximating systems for an AF algebra are equivalent* [3, *Theorem* 2.7].

Glimm and Dixmier had shown that Theorem 3.1 leads to a classification of the matroid algebras. Elliott has recently made the important observation [13] that in a sense this is also true for general AF algebras. More precisely, he showed that these algebras are classified by their "dimension groups" together with the ranges of their "dimension functions" (see Theorem 3.2 below). The "dimension group" is actually just the $\tilde{K}_0$ group of K-theory. Lacking a complete reference for this theory in the form that we need (it is well-known to the specialists), we have included a brief summary.

To begin with, we need a few algebraic notions. If S is an abelian monoid, i.e., an abelian semigroup with identity 0, an *enveloping* (or "Grothendieck") *group* G for S consists of a group G together with a (monoid) homomorphism $\theta: S \rightarrow G$ with the following properties:

(1) $\theta(S)$ generates G, and

(2) any homomorphism $S \to H$ of S into a group H must have a factorization $S \xrightarrow{\theta} G \to H$.

It easily follows that enveloping groups are essentially unique. An enveloping group $(G(S), \theta)$ may always be constructed by letting $G(S)$ be all equivalence classes $[(s, t)]$ of pairs (s, t), $s, t \in S$, under the equivalence relation

$$(s, t) \sim (s_1, t_1) \quad \text{if} \quad s + t_1 + u = t + s_1 + u \quad \text{for some } u \in S,$$

and by letting

$$\theta: S \longrightarrow G(S): s \longmapsto [(s, 0)] .$$

Given a monoid homomorphism $\phi: S \to T$, we obtain from the composition $S \to T \to G(T)$ a group homomorphism $G(\phi): G(S) \to G(T)$, and we may regard G as a functor. Given a direct product $S_1 \times S_2$ of abelian monoids with enveloping groups $(G_1, \theta_1), (G_2, \theta_2)$, then $(G_1 \times G_2, \theta_1 \times \theta_2)$ is an enveloping group for $S_1 \times S_2$, or in other words, one has a natural isomorphism

$$G(S_1 \times S_2) \cong G(S_1) \times G(S_2) . \tag{3.1}$$

By a *direct system* of abelian groups (G_p) we shall mean a sequence of abelian groups G_p together with homomorphisms $\phi_{pq}: G_p \to G_q$, $p < q$ such that $\phi_{qr}\phi_{pq} = \phi_{pr}$ for $p < q < r$. In particular, each abelian group G determines the *constant* system (G_p) where $G_p = G$ and $\phi_{pq} = id$ for all p and q. A homomorphism $\theta = (\theta_p)$ of direct systems (G_p), (H_p) is a sequence of homomorphisms $\theta_p: G_p \to H_p$ for which the diagrams

$$\begin{array}{ccc} G_p & \xrightarrow{\theta_p} & H_p \\ \downarrow & & \downarrow \\ G_q & \xrightarrow{\theta_q} & H_q \end{array}$$

are commutative (we shall write $\theta: (G_p) \to (H_p)$). In particular, if H is a group, by a homomorphism of (G_p) into H, denoted $(G_p) \to H$, we mean a homomorphism into the corresponding constant direct system. A *direct limit* for (G_p) is a pair (G, θ), where G is an abelian group and θ is a homomorphism of (G_p) into G such that any homomorphism λ of (G_p) into a group H has a unique factorization $(G_p) \xrightarrow{\theta} G \xrightarrow{\lambda_\infty} H$. Direct limits exist and are essentially unique. For the construction of a particular direct limit, which we shall denote by $(\varinjlim G_p, \phi_{p\infty})$, see [12, Ch. VIII]. If $\theta: (G_p) \to (H_p)$ is a homomorphism, then composition

$$(G_p) \longrightarrow (H_p) \longrightarrow \varinjlim H_p$$

determines a homomorphism

$$\theta_\infty\colon \varinjlim G_p \longrightarrow \varinjlim H_p .$$

Thus we obtain a functor from direct systems to groups.

Returning to K-theory, we shall restrict our attention to self-adjoint projections instead of to general idempotents, and to unitaries rather than to general invertible elements (both approaches give the same theory — see [17, pp. 24, 34–35]). Given a unital C^*-algebra A, we let $\mathscr{P}_n(A)$ be the set of projections in $M_n(A)$, and let $\mathscr{P}(A) = \bigcup_{n\geq 1} \mathscr{P}_n(A)$. Given projections $e \in \mathscr{P}_m(A)$ and $f \in \mathscr{P}_n(A)$, we have a corresponding projection $e \oplus f \in \mathscr{P}_{m+n}(A)$. We say that a projection of the form $e' = e \oplus 0_k$ is a *trivial extension* of e, and that projections e and f in $\mathscr{P}_m(A)$ are *unitarily equivalent* if there is a unitary $u \in M_m(A)$ with $\operatorname{Ad} u(e) = f$. We define an equivalence relation $\simeq$ on $\mathscr{P}(A)$ by $e \simeq f$ if there exist trivial extensions e' and f' of e and f, respectively, which are unitarily equivalent. If $e_1 \simeq e_2$ and $f_1 \simeq f_2$, then it is evident that $e_1 \oplus f_1 \simeq e_2 \oplus f_2$. Thus we may define a semigroup operation on the set of equivalence classes $\mathscr{P}(A)/\simeq$ by

$$[e] + [f] = [e \oplus f] .$$

Since it is clear that $e \oplus f \simeq f \oplus e$, $\mathscr{P}(A)/\simeq$ is abelian, and it has identity $[0]$. We let $K_0(A)$ be the enveloping group $G(\mathscr{P}(A)/\simeq)$, and we define the *dimension function* $\dim\colon \mathscr{P}(A) \to K_0(A)$ by

$$\dim e = \theta([e]) .$$

Given a unital homomorphism $\phi\colon A \to B$, it induces homomorphisms $\phi_n\colon M_n(A) \to M_n(B)$ by $[a_{ij}] \mapsto [\phi(a_{ij})]$, which in turn restrict to a map $\phi\colon \mathscr{P}(A) \to \mathscr{P}(B)$. Since ϕ preserves equivalence and direct sums, it defines a monoid homomorphism $[\phi]\colon \mathscr{P}(A)/\simeq \to \mathscr{P}(B)/\simeq$, and thus defines a group homomorphism $\phi_* = G([\phi])\colon K_0(A) \to K_0(B)$ satisfying

$$\phi_*(\dim e) = \dim \phi(e) .$$

In particular, K_0 may be regarded as a functor from unital C^*-algebras to abelian groups.

For any unital C^*-algebra A and any integer $m > 1$, the natural isomorphisms $M_n(M_m(A)) \cong M_{nm}(A)$ restrict to an injection of $\mathscr{P}(M_m(A))$ into $\mathscr{P}(A)$ that preserves equivalence. Since any projection in $\mathscr{P}(A)$ has a trivial extension in some $\mathscr{P}_{nm}(A)$, it is easy to see that this map induces a natural isomorphism of $K_0(M_m(A))$ onto $K_0(A)$. Taking $A = \boldsymbol{C}$, we observe in particular that $K_0(M_m) \cong K_0(\boldsymbol{C})$. But projections in $\mathscr{P}(\boldsymbol{C})$ are equivalent if and only if they have the same rank. It follows that the map

$$\rho: \mathscr{P}(\boldsymbol{C})/\simeq \longrightarrow \boldsymbol{Z}: [e] \longmapsto \operatorname{rank} e$$

is an identity-preserving homomorphism, which determines an isomorphism $K_0(\boldsymbol{C}) \cong \boldsymbol{Z}$ (and thus $K_0(M_m) \cong \boldsymbol{Z}$). Similarly,

$$K_0(M_{n_1} \oplus \cdots \oplus M_{n_r}) \cong \boldsymbol{Z}^r .$$

In order to study nonunital algebras, we must use reduced K-theory. Given a C^*-algebra A, consider the corresponding *unital extension*

$$0 \longrightarrow A \longrightarrow A^\dagger \xrightarrow{\chi} \boldsymbol{C} \longrightarrow 0 . \tag{3.2}$$

Algebraically, $A^\dagger$ is defined to be the pairs (a, α), $a \in A$, $\alpha \in \boldsymbol{C}$, with the multiplication

$$(a, \alpha)(b, \beta) = (ab + \beta a + \alpha b, \alpha\beta) ,$$

and $A \to A^\dagger$, $A^\dagger \xrightarrow{\chi} \boldsymbol{C}$ are defined by $a \mapsto (a, 0)$, $(a, \alpha) \mapsto \alpha$. If A does not have an identity, then the map

$$A^\dagger \longrightarrow A^\sim: (a, \alpha) \longrightarrow a + \alpha 1$$

is an isomorphism. If A has an identity, then

$$A^\dagger \longrightarrow A \oplus \boldsymbol{C}: (a, \alpha) \longmapsto (a + \alpha 1, \alpha) \tag{3.3}$$

is an isomorphism. In either case, we see that $A^\dagger$ may be regarded as a C^*-algebra. The map χ in (3.2) determines a homomorphism $K_0(A^\dagger) \to K_0(\boldsymbol{C}) \cong \boldsymbol{Z}$, and we let $\widetilde{K}_0(A)$ be the kernel of this map. It should be noted that if $e \in \mathscr{P}(A)$, then $\chi(e) = 0$, $\chi_*(\dim e) = 0$, and thus (if we define $\mathscr{P}$ and dim as for unital algebras) $\dim \mathscr{P}(A) \subseteq \widetilde{K}_0(A)$. It is not clear, however, that $\dim \mathscr{P}(A)$ must generate $\widetilde{K}_0(A)$. Each homomorphism $\phi: A \to B$ induces a unital homomorphism

$$\phi^\dagger: A^\dagger \longrightarrow B : (a, \alpha) \longrightarrow (\phi(a), \alpha) .$$

Since the diagram

$$\begin{array}{ccc} A^\dagger & \xrightarrow{\chi} & \boldsymbol{C} \\ {\scriptstyle \phi^\dagger}\big\downarrow & & \big\downarrow{\scriptstyle id} \\ B^\dagger & \xrightarrow{\chi} & \boldsymbol{C} \end{array}$$

is commutative, one has a commutative diagram

$$\begin{array}{ccc} K_0(A^\dagger) & \longrightarrow & K_0(\boldsymbol{C}) \\ {\scriptstyle \phi^\dagger_*}\big\downarrow & & \big\downarrow{\scriptstyle id} \\ K_0(B^\dagger) & \longrightarrow & K_0(\boldsymbol{C}) , \end{array}$$

hence $\phi_*^\dagger$ restricts to a homomorphism $\tilde{\phi}_*: \tilde{K}_0(A) \to \tilde{K}_0(B)$, and we obtain a functor $\tilde{K}_0$ for not necessarily unital algebras.

We have a natural homomorphism $\boldsymbol{C} \to A^\dagger$ defined by $\alpha \mapsto \alpha 1$, and the composition $\boldsymbol{C} \to A^\dagger \to \boldsymbol{C}$ is the identity (i.e., the unital extension is split). It follows that the corresponding composition

$$K_0(\boldsymbol{C}) \longrightarrow K_0(A^\dagger) \longrightarrow K_0(\boldsymbol{C})$$

is the identity map, hence we have an isomorphism

$$K_0(A^\dagger) \cong \tilde{K}_0(A) \oplus K_0(\boldsymbol{C}) . \tag{3.4}$$

Given a homomorphism $\phi: A \to B$, this isomorphism carries $\phi_*^\dagger$ into the map $(g, n) \mapsto (\tilde{\phi}_*(g), n)$. If A is unital, we may use (3.3) to identify $A^\dagger$ and $A \oplus \boldsymbol{C}$. Then one easily verifies that $K_0(A^\dagger) \cong K_0(A) \oplus K_0(\boldsymbol{C})$ (naturally), so that $K_0(A)$ and $\tilde{K}_0(A)$ are naturally isomorphic. Given a (not necessarily unital) homomorphism of unital algebras $\phi: A \to B$, one finds that the corresponding homomorphism $\tilde{\phi}_*: K_0(A) \to K_0(B)$ is just that induced by the map $[e] \mapsto [\phi(e)]$.

Next we state Elliott's classification theorem [13, 4.2]. Since we shall not use it, we have not included the proof, which is based on Theorem 3.1. For any C^*-algebra A we let

$$\Delta(A) = \{\dim e : e \in \mathscr{P}_1(A)\} \subseteq \tilde{K}_0(A) .$$

THEOREM 3.2. *Suppose that A and B are AF algebras. A and B are isomorphic if and only if there is an isomorphism of $\tilde{K}_0(A)$ onto $\tilde{K}_0(B)$ carrying $\Delta(A)$ onto $\Delta(B)$.*

Actually this may be regarded as a weakened form of Elliott's result, since in fact Elliott provided a faithful functor from the AF algebras into purely algebraic objects that he has characterized.

What is of interest to us is that for AF algebras A, $\tilde{K}_0(A)$ is easily computed. If A is an AF algebra with approximating system (A_p), the inclusions $A_p \hookrightarrow A_q$, $p < q$, determine a direct system of groups and homomorphisms $\phi_{pq}: \tilde{K}_0(A_p) \to \tilde{K}_0(A_q)$, whereas the inclusions $A_p \hookrightarrow A$ determine a homomorphism $(\lambda_p): (\tilde{K}_0(A_p)) \to \tilde{K}_0(A)$. Thus we have a unique homomorphism

$$\lambda_\infty : \varinjlim \tilde{K}_0(A_p) \longrightarrow \tilde{K}_0(A)$$

through which (λ_p) factors. In order to prove this is an isomorphism we need some information about AF algebras.

LEMMA 3.3. *Suppose that A is a unital AF algebra with a unital approximating system (A_p). Then given a projection $e \in A$,* *there is a unitary* $u \in A$ *with* $\operatorname{Ad} u(e) \in A_p$ *for some* p. *Given projections* $e, f \in A_p$ *and a unitary* $u \in A$ *with* $\operatorname{Ad} u(e) = f$, *there exists an integer* $q \geqq p$ *and a unitary* $v \in A_q$ *with* $\operatorname{Ad} v(e) = f$.

Proof. The first assertion is proved in [3, Lemma 2.3] (and elsewhere). Given e, f, and u as above, we may select an integer $q \geqq p$ and a unitary $v \in A_q$ with $\|u - v\| < 1/2$. It follows that $\|\operatorname{Ad} v(e) - f\| < 1$, and since $\operatorname{Ad} v(e)$ and f are both projections in A_q, there is by [14, Lemma 1.8] a unitary $w \in A_q$ with

$$\operatorname{Ad}(wv)(e) = \operatorname{Ad} w(\operatorname{Ad} v(e)) = f\ .$$

COROLLARY 3.4. *Suppose that* A *is a unital AF algebra and that* e, f, *and* g *are projections in* A *with* $e \perp g$, $f \perp g$, *and* $e + g$ *unitarily equivalent to* $f + g$. *Then* e *and* f *are unitarily equivalent.*

Proof. By Lemma 3.3, we may choose an integer p and unitaries $u, v \in A$ for which

$$\operatorname{Ad} u(e + g) \in A_p \quad \text{and} \quad \operatorname{Ad} v(f + g) \in A_p\ .$$

We let $e_1 = ueu^*$, $g_1 = ugu^*$, $f_1 = vfv^*$, $g_2 = vgv^*$. Since $e_1 + g_1$ and $f_1 + g_2$ are (unitarily) equivalent in A, Lemma 3.3 implies that (increasing p if necessary) we may assume that they are equivalent in A_p. Since equivalence is determined by rank in a matrix algebra, it is clear that e_1 and f_1 are equivalent in A_p, and thus e and f are equivalent in A.

LEMMA 3.5. *Given an AF algebra with approximating system* (A_p), *the map*

$$\lambda_\infty\colon \varinjlim \tilde{K}_0(A_p) \longrightarrow \tilde{K}_0(A)$$

is an isomorphism.

Proof. Let us first assume that A is unital, as are the maps $A_p \hookrightarrow A$. Then it suffices to prove that the map

$$\lambda_\infty\colon \varinjlim K_0(A_p) \longrightarrow K_0(A)$$

is an isomorphism. Suppose that e is a projection in $\mathscr{P}_n(A)$. Since $M_n(A)$ has the approximating system $(M_n(A_p))$, we have from Lemma 3.3 that $\operatorname{Ad} u(e) \in M_n(A_p)$ for some unitary $u \in A$ and integer p. We have

$$\dim_A e = \lambda_p(\dim_{A_p} e)\ ,$$

hence λ_∞ is a surjection. Given an element $g \in \varinjlim K_0(A_n)$ with

$\lambda_\infty(g) = 0$, let us suppose that $g = \phi_{p\infty}(g_p)$, where $g_p \in K_0(A_p)$. Then $\lambda_{p\infty}(g_p) = \lambda_\infty(g) = 0$. If we let g_p be the equivalence class of $([e], [f]) \in K_0(A_p)$, it follows that $([e], [f])$ is equivalent to $([0], [0])$ in $K_0(A)$, i.e., for suitable l and m and $h \in \mathscr{P}(A)$, $e \oplus 0_l \oplus h$ is unitarily equivalent to $f \oplus 0_m \oplus h$. Since $M_n(A)$ is an AF algebra for each n, we conclude from Lemma 3.3 and Corollary 3.4 that $e \oplus 0_l$ and $f \oplus 0_m$ are unitarily equivalent via some unitary matrix over A_q, $q \geqq p$. But this implies that $([e], [f])$ is equivalent to $([0], [0])$ in $K_0(A_q)$, i.e., $\phi_{pq}(g_p) = 0$ and $g = 0$.

In general, $(A_p^\dagger)$ is an approximating system for $A^\dagger$. We have exact sequences

$$0 \longrightarrow \tilde{K}_0(A_p) \longrightarrow K_0(A_p^\dagger) \longrightarrow K_0(\boldsymbol{C}) \longrightarrow 0$$

and

$$0 \longrightarrow \tilde{K}_0(A) \longrightarrow K_0(A) \longrightarrow K_0(\boldsymbol{C}) \longrightarrow 0 ,$$

and it is readily verified that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varinjlim K_0(A_p) & \longrightarrow & \varinjlim K_0(A_p) & \longrightarrow & K_0(\boldsymbol{C}) & \longrightarrow & 0 \\
 & & & & \Big\downarrow{\scriptstyle \lambda_\infty} & & \Big\downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(A) & \longrightarrow & K_0(\boldsymbol{C}) & \longrightarrow & 0
\end{array}$$

has exact rows and commutes. Hence λ_∞ restricts to an isomorphism $\tilde{\lambda}_\infty$ of $\varinjlim \tilde{K}_0(A_p)$ onto $\tilde{K}_0(A)$.

Let us now suppose that F is a finite-dimensional C^*-algebra. Then letting $\mathscr{C}(F)$ be the set of minimal central projections $c \in F$, we have from our preliminary discussion an isomorphism

$$\tilde{K}_0(F) \cong K_0(F) \cong \boldsymbol{Z}\mathscr{C}(F) ,$$

where the latter expression is the free abelian group on $\mathscr{C}(F)$. Given an element $e \in \mathscr{P}(F)$, this isomorphism carries $\dim e$ into $\sum_{c \in \mathscr{C}(F)} n_c c$, where n_c is the rank of ec in the matrix algebra Fc. Given a homomorphism $\phi: F \to G$, the corresponding homomorphism $\tilde{\phi}_*: \tilde{K}_0(F) \to \tilde{K}_0(G)$ is induced by

$$c \longmapsto \sum_d n_{cd} d ,$$

where n_{cd} is obtained by letting e be a minimal projection in Fc, and letting n_{cd} be the rank of $\phi(e)d$ in Gd.

Thus by Lemma 3.5 we conclude that if A is an AF algebra with approximating system (A_p), then

$$\tilde{K}_0(A) \cong \varinjlim \tilde{K}_0(A_p) = \varinjlim \boldsymbol{Z}\mathscr{C}(A_p) ,$$

where the homomorphisms $\boldsymbol{Z}\mathscr{C}(A_m) \to \boldsymbol{Z}\mathscr{C}(A_n)$ are determined by the inclusions $A_m \hookrightarrow A_n$ as above.

For unital AF algebras, the significance of the dimension function is particularly simple:

LEMMA 3.6. *Suppose that A is a unital AF algebra, and that e and f are projections in A. Then the following are equivalent:*

(1) $\dim e = \dim f$,

(2) *there exists a unitary* $u \in A$ *with* $\mathrm{Ad}\, u(e) = f$,

(3) *there is an element* $v \in A$ *with* $v^*v = e$, $vv^* = f$.

Proof. (1) $\Rightarrow$ (3). Suppose that $\dim e = \dim f$. Then there exist integers k and l and $g \in \mathscr{P}_k(A)$ such that $e \oplus g \oplus 0_l$ and $f \oplus g \oplus 0_l$ are unitarily equivalent in $M_n(A)$, $n = 1 + k + l$. But $M_n(A)$ is again a unital AF algebra and

$$e \oplus g \oplus 0_l = (e \oplus 0_{k+l}) + (0 \oplus g \oplus 0_l) \ ,$$

and similarly for f, hence from Corollary 3.4,

$$\mathrm{Ad}\, u(e \oplus 0_{k+l}) = f \oplus 0_{k+l}$$

for some unitary u. Letting $v = (f \oplus 0_{k+l})u(e \oplus 0_{k+l})$,

$$\begin{aligned} v^*v &= e \oplus 0_{k+l} \ , \\ vv^* &= f \oplus 0_{k+l} \ . \end{aligned}$$

But

$$v \in (1 \oplus 0_{k+l})M_n(A)(1 \oplus 0_{k+l}) \ ,$$

and using the map $a \oplus 0_{k+l} \mapsto a$ to identify the latter algebra with A, we have (3).

(3) $\Rightarrow$ (2). Let (A_p) be a unital approximating system for A. By Lemma 3.3, we may replace e and f by unitarily equivalent projections and assume that $e, f \in A_p$ for some p. Then by [14, Lemma 1.9], we can choose the v in some A_q, $q \geqq p$. Since (3) $\Rightarrow$ (2) for the finite-dimensional algebra A_q, e and f are unitarily equivalent.

(2) $\Rightarrow$ (1) is trivial.

REMARK. As is well-known, if A is an arbitrary unital C^*-algebra, then conditions (2) and (3) of Lemma 3.6 are in general not equivalent. However, Larry Brown has pointed out to us that if (3) holds for projections e and f in A, then (2) holds for $e \oplus 0$ and $f \oplus 0$ in $M_2(A)$. Thus von Neumann-Murray equivalence can be used in place of unitary equivalence in the definition of K_0.

The following proposition shows that $\widetilde{K}_0$ provides a simple necessary condition for an automorphism of a C^*-algebra to be approximately inner. What is more remarkable is that for AF algebras, this condition is also sufficient (Theorem 3.8 below). This latter fact was implicitly proved by Elliott in [13, Lemma 4.1].

PROPOSITION 3.7. *Let A be a C^*-algebra and let σ be an approximately inner automorphism of A. Assume further that either A is AF or that σ is strongly approximately inner. Then the automorphism $\tilde{\sigma}_*$ of $\tilde{K}_0(A)$ induced by σ is just the identity.*

Proof. If σ is strongly approximately inner, then obviously so is the induced automorphism $\sigma^\dagger$ of $A^\dagger$. Since $\tilde{\sigma}_*$ is a restriction of the automorphism $\sigma^\dagger_*$ of $K_0(A^\dagger)$, it is enough to show that $\sigma^\dagger_* = id$. Thus, in this case, we may replace A by $A^\dagger$ and assume A is unital. Having done this, let e be a projection in $B = M_n(A) \cong A \otimes M_n$ for some n. It is enough to show that e and $(\sigma \otimes 1)(e)$ are unitarily equivalent in B. Since σ is approximately inner, so is $\sigma \otimes 1$ by Lemma 2.3. Thus we may choose unitaries u_k in B with $u_k e u_k^* \to (\sigma \otimes 1)(e)$. For sufficiently large k, $u_k e u_k^*$ and $(\sigma \otimes 1)(e)$ are unitarily equivalent in B by [14, Lemma 1.8]. But e and $u_k e u_k^*$ are also unitarily equivalent, so we are done.

If A is AF, then by Lemma 3.5, every class in $\tilde{K}_0(A)$ is the image of a class in $\tilde{K}_0(A_p) \cong K_0(A_p)$ for some finite-dimensional subalgebra A_p of A. In particular, $\tilde{K}_0(A)$ is generated by the classes of elements $e \in \mathscr{P}(A)$. For such an e, say in $B = M_n(A)$, we may argue exactly as before, except that now our u_k's lie in $M(B)$ rather than in B. It is still true that $u_k e u_k^*$ and $(\sigma \otimes 1)(e)$ are close in norm, hence unitarily equivalent in $B^\sim$, for large k, and e and $u_k e u_k^*$ are unitarily equivalent in $B^\sim$ by Lemma 3.6 (let $v = u_k e$).

THEOREM 3.8. *Let A be an AF algebra and let σ be an automorphism of A. The following are equivalent:*

(1) *σ is approximately inner,*

(2) *σ is strongly approximately inner,*

and (3) *the automorphism $\tilde{\sigma}_*$ of $\tilde{K}_0(A)$ is the identity.*

Proof. $(2) \Rightarrow (1)$ is trivial, and $(1) \Rightarrow (3)$ is given by Proposition 3.7. For the proof of $(3) \Rightarrow (2)$, assume first that A is unital and that σ_* is the identity automorphism of $K_0(A)$. It suffices to show that if F is a finite-dimensional subalgebra of A with the same unit as A, then there exists a unitary $u \in A$ such that

$$\sigma_{|F} = \operatorname{Ad} u_{|F} .$$

We let e^k_{ij} be the matrix units for F. Since

$$\dim e_{11}^k = \dim \sigma(e_{11}^k) ,$$

Lemma 3.6 implies that we may find partial isometries v_k with

$$v_k^* v_k = e_{11}^k , \quad v_k v_k^* = \sigma(e_{11}^k) .$$

We define (following Elliott)

$$u = \sum_{k,i} \sigma(e_{i1}^k) v_k e_{1i}^k .$$

Then for any indices p, q, and r,

$$\begin{aligned} u e_{pq}^r u^* &= \sigma(e_{p1}^r) v_r e_{1q}^r u^* \\ &= \sigma(e_{p1}^r) v_r e_{11}^r v_r^* \sigma(e_{1q}^r) \\ &= \sigma(e_{pq}^r) . \end{aligned}$$

Hence for any element $a \in F$,

$$\sigma(a) = u a u^* .$$

In particular, $1 = \sigma(1) = uu^*$. A similar calculation shows that $u^*u = 1$, so u is unitary and we have the desired result.

Now we consider the general (nonunital) case. Given an automorphism σ of a C^*-algebra A,

$$\sigma^\dagger \colon A^\dagger \longrightarrow A^\dagger \colon (a, \alpha) \longmapsto (\sigma(a), \alpha)$$

is again an automorphism. By (3.4), $\sigma_*^\dagger \colon K_0(A^\dagger) \to K_0(A^\dagger)$ corresponds to the map

$$\tilde{K}_0(A) \oplus K_0(\boldsymbol{C}) \longrightarrow \tilde{K}_0(A) \oplus K_0(\boldsymbol{C}) \colon (g, n) \longmapsto (\tilde{\sigma}_*(g), n) .$$

Thus if A is AF and $\tilde{\sigma}_* = id$, $\sigma_*^\dagger$ also leaves elements fixed. It follows that $\sigma^\dagger$ and thus σ are strongly approximately inner.

THEOREM 3.9. *If A is an AF algebra, then the flip for A is approximately inner if and only if A is matroid.*

Proof. Let (A_p) be an approximating system for A, so that $(A_p \otimes A_p)$ is the same for $A \otimes A$. It is evident that

$$K_0(A_p \otimes A_p) \cong K_0(A_p) \otimes K_0(A_p) ,$$

and that the corresponding homomorphism

$$K_0(A_p) \otimes K_0(A_p) \longrightarrow K_0(A_q) \otimes K_0(A_q)$$

is just $\phi_{pq} \otimes \phi_{pq}$. It readily follows that

$$\begin{aligned}\tilde{K}_0(A \otimes A) &\cong \varinjlim K_0(A_p \otimes A_p)\\ &\cong \varinjlim K_0(A_p) \otimes K_0(A_p)\\ &\cong \varinjlim K_0(A_p) \otimes \varinjlim K_0(A_p)\\ &\cong \tilde{K}_0(A) \otimes \tilde{K}_0(A)\ ,\end{aligned}$$

and that under this isomorphism, $\tilde{\sigma}_*$ is transformed into the "algebraic flip"

$$S \in \mathrm{Aut}\,(\tilde{K}_0(A) \otimes \tilde{K}_0(A)) \colon g \otimes h \longmapsto h \otimes g\ .$$

By Theorem 3.8, σ is approximately inner if and only if S leaves points fixed. By the next lemma, this happens if and only if $\tilde{K}_0(A)$ is of rank 1, hence, by [13, 6.1], if and only if A is matroid.

LEMMA 3.10. *If G is an abelian torsion-free group, then the flip automorphism S of $G \otimes G$ coincides with the identity map if and only if G is of rank* 1 (i.e., *any two elements are dependent over $\boldsymbol{Z}$*).

Proof. If G is of rank 1, then given nonzero $g, h \in G$, there exist nonzero integers m and n with $mg = nh$. Then

$$m(g \otimes h) = mg \otimes h = nh \otimes h = h \otimes mg = m(h \otimes g)\ .$$

Since $G \otimes G$ is torsion free, $g \otimes h = h \otimes g$, hence S leaves all elements fixed.

Conversely suppose that $S \colon G \otimes G \to G \otimes G$ is trivial. If H is a finitely generated subgroup of G, $H \cong \boldsymbol{Z}^n$ for some n, and the map $H \otimes H \to G \otimes G$ is one-to-one since G is torsion-free. It follows that $S \colon H \otimes H \to H \otimes H$ leaves elements fixed, and this implies that $n = 1$. Since H was arbitrary, we conclude that any two elements of G are dependent, i.e., G is of rank 1.

4. **Asymptotic imbeddings.** The notion of an asymptotic imbedding of C^*-algebras is the C^*-algebraic analogue of a technique used by Connes in [8]. Since the main results of § 5 can be reduced immediately to the case of unital C^*-algebras, we shall only consider unital imbeddings.

Given a C^*-algebra B, we define the *asymptotic model* for B to be the C^*-algebraic quotient

$$B^\infty = B^N / J\ ,$$

where

$$B^N = B \oplus B \oplus \cdots$$

is the full C^*-algebraic (l^∞) direct sum, and J is the ideal

$$J = \{b = (b_n): \lim_{n\to\infty} \|b_n\| = 0\} .$$

We note that we are using the usual limit—we have not found a compelling reason to use the limit defined by a free ultrafilter on N.

Given unital C^*-algebras A and B, an *asymptotic imbedding* of A into B is a unital *-isomorphism $A \to B^\infty$. Throughout this section we shall assume that B is the *universal* UHF C^*-algebra $\mathscr{U}$, i.e., the UHF algebra with "generalized integer" $2^\infty 3^\infty 5^\infty \cdots$ [14, Definitions 1.1 and 1.3]. $\mathscr{U}$ may be simply described as the inductive limit of the finite-dimensional algebras

$$\mathscr{U}_n = M_1 \otimes M_2 \otimes \cdots \otimes M_n \tag{4.1}$$

where the injection $\mathscr{U}_n \to \mathscr{U}_{n+1}$ is just the diagonal map

$$a \longmapsto \begin{bmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{bmatrix} .$$

It is apparent that a unital C^*-algebra has an asymptotic imbedding in some UHF algebra if and only if it has such an imbedding in $\mathscr{U}$.

We recall that a collection of operators $\mathscr{S}$ on a separable Hilbert space $\mathscr{H}$ is *quasi-diagonal* if there is an increasing sequence of finite-dimensional projections e_k converging strongly to 1 such that

$$\|ae_k - e_k a\| \longrightarrow 0$$

for all $a \in \mathscr{S}$. Given a sequence of orthogonal finite-dimensional projections f_n with $\sum f_n = 1$, the corresponding *block diagonal* algebra $\mathscr{D}$ consists of the operators $b \in \mathscr{B}(\mathscr{H})$ commuting with the f_n. If $\mathscr{S}$ generates a separable C^*-algebra, it is quasi-diagonal if and only if $\mathscr{S} \subseteq \mathscr{D} + \mathscr{K}(\mathscr{H})$ for some block diagonal algebra $\mathscr{D}$ [21, Lemma 1], [16]. We will say that a representation π of a C^*-algebra A is quasi-diagonal if $\pi(A)$ has that property, and that A is quasi-diagonal if it has a *faithful* quasi-diagonal representation.

LEMMA 4.1. *Suppose that A is a quasi-diagonal unital C^*-algebra. Then A has an asymptotic imbedding in $\mathscr{U}$.*

Proof. Let e_k be a sequence of projections as above for $\mathscr{S} = \pi(A)$, π a faithful quasi-diagonal representation of A, and let

$$d_k = \dim e_k \mathscr{H} .$$

For each n we may fix an isomorphism

$$e_k\mathscr{B}(\mathscr{H})e_k \cong M_{d_k} .$$

For each n we have an injection

$$\theta_n: M_n \longrightarrow \mathscr{U}_n \hookrightarrow \mathscr{U}$$

determined by

$$\alpha \longmapsto 1 \otimes \cdots \otimes 1 \otimes \alpha .$$

We define a linear *-preserving contraction $\phi: A \to \mathscr{U}^N$ by

$$\phi(a) = (\theta_{d_1}(e_1\pi(a)e_1), \theta_{d_2}(e_2\pi(a)e_2), \cdots) .$$

Letting $\dot{\phi}: A \to \mathscr{U}^\infty$ be the corresponding composition with the quotient map, $\dot{\phi}$ is a homomorphism because given $a, b \in A$,

$$\begin{aligned} ||\dot{\phi}(ab) - \dot{\phi}(a)\dot{\phi}(b)|| &= \lim ||e_n\pi(ab)e_n - e_n\pi(a)e_n\pi(b)e_n|| \\ &\leqq \lim ||e_n\pi(a) - \pi(a)e_n||\,||b|| \\ &= 0 . \end{aligned}$$

$\dot{\phi}$ is an isomorphism since if $\dot{\phi}(a) = 0$, then $\lim ||e_n\pi(a)e_n|| = 0$, or since the e_n are increasing and converge to 1 strongly, $\pi(a) = 0$ and thus $a = 0$.

The separable C^*-algebras A such that $\pi(A) \subseteqq \mathscr{K}(\mathscr{H}) + \boldsymbol{C}1$ for each irreducible representation π, have only quasi-diagonal representations [21, Prop. 5]. This is also the case for Bratteli's AF algebras [20].

It should be noted that for simple separable C^*-algebras, one need only check one representation (this was pointed out to us by Larry Brown). In fact given a representation $\pi: A \to \mathscr{B}(\mathscr{H})$, let $\dot{\pi}: A \to \mathscr{A}(\mathscr{H}) = \mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$ be the corresponding homomorphism. We have

LEMMA 4.2. *Suppose that A is a separable unital C^*-algebra and that π_1 and π_2 are representations with* $\ker \pi_1 = \ker \pi_2 = \ker \dot{\pi}_1 = \ker \dot{\pi}_2$. *Then π_1 is quasi-diagonal if and only if π_2 is quasi-diagonal.*

Proof. Let H_i, $i = 1, 2$, be the underlying Hilbert spaces for π_i. From the Voiculescu theorem [23] (see also [2, Th. 5]), there exists a unitary operator $V: \mathscr{H}_2 \to \mathscr{H}_1$ such that for all $a \in A$

$$\pi_2(a) - V^*\pi_1(a)V \in \mathscr{K}(\mathscr{H}_2) .$$

If π_1 is quasi-diagonal, the same is also true for $V^*\pi_1 V$. But then

$$\pi_2(A) \subseteq V^*\pi_1(A)V + \mathscr{K}(\mathscr{H}_2) \subseteq \mathscr{D} + \mathscr{K}(\mathscr{H}_2)$$

for some block diagonal algebra $\mathscr{D}$, i.e., π_2 is quasi-diagonal.

It is useful to observe that if π is a representation of A with infinite multiplicity, then $\ker \dot{\pi} = \ker \pi$. On the other hand, it is easy to check that any (countable) multiple of a quasi-diagonal representation is quasi-diagonal. Therefore we immediately deduce from 4.2 the following

COROLLARY 4.3. *Suppose that A is a separable unital C^*-algebra. Then the following are equivalent*:

(1) *A is quasi-diagonal,*

(2) *some faithful representation of A of infinite multiplicity is quasi-diagonal,*

(3) *every faithful representation of A of infinite multiplicity is quasi-diagonal.*

One can also generalize this result to nonunital C^*-algebras. Still another convenient fact is that the class of quasi-diagonal algebras is closed under inductive limits. This was pointed out to us by Man-Duen Choi; perhaps it is known to others.

LEMMA 4.4. *Suppose that A is a separable C^*-algebra and that (A_n) is an ascending sequence of unital C^*-algebras with $A = \overline{\bigcup A_n}$. If each A_n is quasi-diagonal, so is A.*

Proof. We represent A on a separable Hilbert space $\mathscr{H}$ via a faithful representation with infinite multiplicity. Then by 4.3, the representations $A_n \hookrightarrow \mathscr{B}(\mathscr{H})$ are quasi-diagonal. Choose elements $a_n \in A_n$ with the sequence (a_n) dense in the unit ball of A. We inductively choose finite-dimensional projections e_n converging strongly to 1 such that

$$e_n \geqq e_{n-1}, \quad \|[e_n, a_j]\| < 1/n \quad \text{for} \quad 1 \leqq j \leqq n .$$

Then clearly $[e_n, a] \to 0$ in norm for each $a \in A$, which will prove the result. To construct the e_n, we first choose sequences $(f_n^{(m)})$ of finite-dimensional projections increasing to 1 such that $[f_n^{(m)}, a] \to 0$ as $n \to \infty$ for $a \in A_m$. (Such sequences exist by quasi-diagonality of the A_m's.) Assume e_{n-1} is constructed. We can choose k large enough so that

$$\|[f_k^{(n)}, a_j]\| < 1/(2n) \quad \text{for} \quad 1 \leqq j \leqq n ,$$

and

$$(*)\qquad \|f_k^{(n)}\xi - \xi\| < \|\xi\|/(30n) \quad \text{for} \quad \xi \in (e_{n-1} \vee f_n^{(1)})\mathscr{H}\ .$$

It is now sufficient to find e_n of finite rank with $\|e_n - f_k^{(n)}\| < 1/(4n)$ and with $e_n \geqq e = e_{n-1} \vee f_n^{(1)}$. (Then since $e_n \geqq f_n^{(1)}$ for all n, $e_n \to 1$ strongly; and

$$\|[e_n, a_j]\| < \|[f_k^{(n)}, a_j]\| + 2\|e_n - f_k^{(n)}\|\,\|a_j\| < 1/n$$

for $1 \leqq j \leqq n$.)

For this one can either adapt the argument in the proof of Theorem 2 of P. R. Halmos, "Quasitriangular operators," Acta Sci. Math., 29 (1968), 283–293, or else use the following simplification suggested by M.-D. Choi: We may assume e has a block matrix of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and that $f_k^{(n)}$ has block matrix

$$\begin{bmatrix} a & c \\ c^* & b \end{bmatrix}.$$

By (*), $\|a - 1\| < 1/(30n)$, and since $f_k^{(n)}$ is idempotent, $a^2 + cc^* = a$, $cc^* = a - a^2$, and $\|c\|^2 < 1/(30n)(1 + 1/(30n)) < 1/(24n)$. Since also $c^*c + b^2 = b$, $\|b^2 - b\| < 1/(24n)$ and there exists a projection q (obtained by functional calculus from b, hence of finite rank) with $\|q - b\| < 1/(12n)$. Now take

$$e_n = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix} \geqq e$$

and observe that $\|e_n - f_k^{(n)}\| < 2\sqrt{2}/(12n) < 1/(4n)$.

Note that the above proof shows incidentally that the projections e_n in the definition of quasi-diagonality can be chosen to majorize any fixed finite-dimensional projection.

A consequence of 4.4 is that the simple C^*-algebras of Bunce and Deddens [5] are quasi-diagonal and so have asymptotic imbeddings in $\mathscr{U}$. We also have a result concerning inductive limits of (not necessarily quasi-diagonal) algebras with asymptotic imbeddings:

LEMMA 4.5. *Suppose that A is a separable unital C^*-algebra and that (A_n) is an ascending sequence of unital C^*-algebras with $A = \overline{\bigcup A_n}$. If*

(†) *each A_n is nuclear and asymptotically imbeddable in $\mathscr{U}$, then A is asymptotically imbeddable in $\mathscr{U}$.*

Proof. Assuming (†), we have that for each n there is an isomorphism $\phi_n: A_n \to \mathscr{U}^\infty$. Since each A_n is nuclear, from the lifting theorem for completely positive maps [6, Th. 3.10] we may find for each n a completely positive unital map $\psi_n: A_n \to \mathscr{U}^N$ with $\phi_n = \dot{\psi}_n$. Letting $\psi_n(a) = (\psi_{nk}(a))_{k=1,2\cdots}$, we have

$$(3.1)\qquad \begin{aligned} &\|\psi_{nk}(ab) - \psi_{nk}(a)\psi_{nk}(b)\| \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty \quad \text{and} \\ &\|\psi_{nk}(a)\| \longrightarrow \|\phi_n(a)\| \qquad\qquad\quad \text{as} \quad k \longrightarrow \infty \end{aligned}$$

for each $a, b \in A_n$. Let (a_n) be a dense sequence in the unit ball of $\bigcup A_n$. Deleting some of the A_n we may assume that $a_n \in A_n$. From (3.1) we may choose a k for each n such that if $\theta_n = \psi_{nk}$,

$$\|\theta_n(a_i a_j) - \theta_n(a_i)\theta_n(a_j)\| < 1/n$$

and

$$|\|\theta_n(a_i)\| - \|\phi_n(a_i)\|| < 1/n$$

for $i, j \leqq n$. On the other hand, $\mathscr{U} = \overline{\bigcup \mathscr{U}_p}$, and for each p we have a conditional expectation Φ_p of $\mathscr{U}$ onto $\mathscr{U}_p$ since the latter is injective. It follows that if we let $\Theta_n = \Phi_p \circ \theta_n$ for a sufficiently large p, then we may assume that

$$(3.2)\qquad \begin{aligned} &\|\Theta_n(a_i a_j) - \Theta_n(a_i)\Theta_n(a_j)\| < 1/n \quad \text{and} \\ &|\|\Theta_n(a_i)\| - \|\phi_n(a_i)\|| < 1/n \end{aligned}$$

for $i, j \leqq n$. The reason that we have replaced θ_n by Θ_n is that since the latter has range in a matrix algebra, we may use injectivity to extend it to all of A—we will also write Θ_n for the extended map. From (3.2) it is evident that if we let

$$\Theta = (\Theta_n): A \longrightarrow \mathscr{U}^N$$

and let $\dot{\Theta}: A \to \mathscr{U}^\infty$ be the corresponding quotient map, the latter is an isomorphism.

We conclude this section with the observation that the algebraic significance of quasi-diagonality is only poorly understood. Choi has shown that for the free group on two generators, the full group C^*-algebra is quasi-diagonal, whereas its quotient, the regular group C^*-algebra, is not. On the other hand, the unital C^*-algebra generated by $S \oplus S \oplus S^*$, where S is the unilateral shift, has both nonquasi-diagonal and quasi-diagonal faithful representations.

5. The main results.

THEOREM 5.1. *Suppose that A is a separable unital C^*-algebra.*

Then A is isomorphic to the universal UHF algebra $\mathscr{U}$ if and only if

(1) *the flip for A is approximately inner,*

(2) *A has an asymptotic imbedding in $\mathscr{U}$, and*

(3) *A is isomorphic to $A \otimes \mathscr{U}$.*

Proof. The necessity of these conditions is evident (see [11] for (1)). Suppose that we have (1)-(3). By Propositions 2.7 and 2.8, A is a simple nuclear C^*-algebra. We let $\pi: A \to \mathscr{U}^\infty$ be a unital isomorphism, and we define isomorphisms

$$\pi_1: A \longrightarrow (A \otimes \mathscr{U})^\infty: a \longmapsto [(a \otimes 1)_n]$$
$$\pi_2: A \longrightarrow (A \otimes \mathscr{U})^\infty: a \longmapsto [(1 \otimes d_n)] ,$$

where given $a \in A$, $(d_n) \in \mathscr{U}^N$ is any representing sequence for $\pi(a) \in \mathscr{U}^\infty$. π_2 is well-defined because if $a = 0$, then $\pi(a) = 0$ implies

$$0 = \lim ||d_n|| = \lim ||1 \otimes d_n|| ,$$

and it follows that π_2 is an isomorphism. π_1 and π_2 are commuting isomorphisms. Since A is nuclear, they define a *-isomorphism

$$\theta = \pi_1 \otimes \pi_2: A \otimes A \longrightarrow (A \otimes \mathscr{U})^\infty: a \otimes b \longrightarrow \pi_1(a)\pi_2(b) .$$

Given $a_1, \cdots, a_r \in A$ we may by (1) choose a unitary operator $u \in A \otimes A$ for which

$$||a_j \otimes 1 - u^*(1 \otimes a_j)u|| < \varepsilon , \quad 1 \leqq j \leqq r .$$

Applying θ, we have

$$||\theta(a_j \otimes 1) - \theta(u)^*\theta(1 \otimes a_j)\theta(u)|| < \varepsilon .$$

Let $(d_{jn}) \in \mathscr{U}^N$ be a representing sequence for $\pi(a_j)$, $1 \leqq j \leqq r$, so that $(1 \otimes d_{jn})$ is a representing sequence for $\theta(1 \otimes a_j) = \pi_2(a_j)$. We also choose a representing sequence (u_n) in $(A \otimes \mathscr{U})^N$ for $\theta(u)$—since $\theta(u)$ is unitary, the argument in the proof of [14, Lemma 3.1] shows that we may require each u_n to be unitary as well. Then we have

$$\lim_{n\to\infty} ||a_j \otimes 1 - u_n^*(1 \otimes d_{jn})u_n||_{A\otimes\mathscr{U}} \leqq \varepsilon .$$

From hypothesis (3), we know that $A \cong A_1 \otimes \mathscr{U}$, where $A_1 \cong A$. Letting A_n, $n \geqq 1$ be the C^*-algebra generated by A_1 and $1 \otimes \mathscr{U}_n$, we have that $A = \overline{\bigcup A_n}$. We also have isomorphisms

$$A_n = A_1 \otimes \mathscr{U}_n \cong A$$

and for the relative commutant in A,

$$A_n^c \cap A = 1 \otimes (\mathscr{U}_n^c \cap \mathscr{U}) \cong \mathscr{U} .$$

The inverse isomorphisms $\alpha: A \to A_n$ and $\beta: \mathscr{U} \to A_n^c \cap A$ commute, and extend to an isomorphism

$$A \otimes \mathscr{U} \longrightarrow A_n A_n^c = A: a \otimes d \longmapsto \alpha(a)\beta(d) .$$

Thus the calculation above may be applied to any of the decompositions $A = A_n A_n^c$.

Given $a_1, \cdots, a_r \in A$ and $\varepsilon > 0$, we may choose an n and elements $a_1', \cdots, a_r' \in A_n$ with $\|a_k - a_k'\| < \varepsilon/3$. From the above we may select a unitary $u \in A$ and elements $d_k \in A_n^c \cong \mathscr{U}$ with

$$\|a_k' - u^* d_k u\| < \varepsilon/3 .$$

But we may find a (unital) matrix algebra $N \subseteq A_n^c$ and elements $e_k \in N$ with $\|d_k - e_k\| < \varepsilon/3$. We conclude that

$$\|a_k - u^* e_k u\| < \varepsilon ,$$

where the elements $u^* e_k u$ belong to the finite-dimensional algebra $u^* N u$. From [14, Th. 1.13], it follows that A is a UHF algebra. But from (3) it is apparent that the corresponding "generalized integer" is $2^\infty 3^\infty 5^\infty \cdots$, hence $A \cong \mathscr{U}$.

COROLLARY 5.2. *Suppose that A is a separable unital C^*-algebra which has an approximately inner flip, and may be asymptotically imbedded in $\mathscr{U}$. Then $A \otimes \mathscr{U} \cong \mathscr{U}$.*

Proof. By Corollary 2.4, $A \otimes \mathscr{U}$ has an approximately inner flip. On the other hand the asymptotic imbedding $A \to \mathscr{U}^\infty$ and the natural isomorphism $\mathscr{U} \to \mathscr{U}^\infty$ extend to an isomorphism

$$A \otimes \mathscr{U} \longrightarrow (\mathscr{U} \otimes \mathscr{U})^\infty = \mathscr{U}^\infty$$

(see the argument for π_2 and π_1 above). Since

$$(A \otimes \mathscr{U}) \otimes \mathscr{U} \cong A \otimes \mathscr{U}$$

we may apply Theorem 5.1 to $A \otimes \mathscr{U}$.

Given a possibly infinite family of distinct primes $p_1, p_2, \cdots$, let $\mathscr{U}(p_1, p_2, \cdots)$ be the UHF algebra with number $p_1^\infty p_2^\infty \cdots$. Precisely the same argument shows that Theorem 5.1 is also valid for this algebra rather than $\mathscr{U}$. In particular, it follows that $\mathscr{U}(p_1, p_2, \cdots)$ can be asymptotically imbedded in $\mathscr{U}(q_1, q_2, \cdots)$ if and only if the latter prime list is larger. Actually this can be proved more easily by a direct argument.

If enough projections are available, we can extend Corollary 5.2 to the nonunital case.

COROLLARY 5.3. *Suppose that A is a separable C^*-algebra. Then $A \otimes \mathscr{U}$ is matroid if A satisfies the following three conditions:*

(1) *A has an approximate identity consisting of projections,*

(2) *the flip for A is approximately inner,*

(3) *pAp has an approximate imbedding in $\mathscr{U}$ for each projection p in A.*

Proof. Because of (1) and because the property of being matroid is "local" [10, 1.1], it is enough to show that $pAp \otimes \mathscr{U}$ is UHF for each projection $p \in A$. By 2.5 (and the fact that $p \otimes p$ is fixed by the flip), pAp satisfies the hypotheses of 5.2. So this is an immediate consequence of 5.2.

Our results suggest a number of interesting questions which we have been unable to resolve. One of these is whether the restrictions on B in Corollary 2.6 are necessary, or in other words, whether A has an approximately inner flip if $A \otimes B$ does, for arbitrary B. More important is the question of whether A must itself be UHF under the hypotheses of Corollary 5.2 (or matroid under those of Corollary 5.3). To phrase the question somewhat differently, if A is a C^*-algebra and $A \otimes \mathscr{U}$ is AF, must A be AF? Ones first inclination would be to say "yes" (for lack of an obvious counterexample), but it seems to us that this problem must be difficult. There are two lines of evidence for this. One is that the corresponding problem for II_1 factors (does $N \otimes R \cong R$, R hyperfinite, imply $N \cong R$?) is quite hard even when one knows the von Neumann algebra analogue of 5.1 (see [8, Theorems 5.1 and 7.7]). On the other hand, if A is unital, $A \cong A \otimes 1$ is just the relative commutant in $A \otimes \mathscr{U}$ of $1 \otimes \mathscr{U}$. However, the relative commutant of an AF subalgebra of an AF algebra need not be AF, and in fact the center of an AF algebra need not be AF [4].

Finally, even if A must be matroid under the hypotheses of 5.3, we do not know if condition (3) (existence of asymptotic imbeddings) was actually needed. At the moment we know of no nonmatroid algebras with approximately inner flips, even among algebras not satisfying the conditions of Lemma 4.3. One can formulate additional necessary conditions for automorphisms to be approximately inner by using other C^*-algebraic invariants, such as $\tilde{K}_1$ and Ext. It is conceivable that for suitable classes of C^*-algebras, these conditions might also be sufficient.

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