

ON SINGULAR INDICES OF ROTATION FREE DENSITIES

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The properties of singular indices of nonnegative rotation free densities on $\Omega(\lambda) = \{\lambda < |z| \leq 1\} (\lambda \geq 0)$ will be studied. The relation between the singular index $\alpha(P)$ of a nonnegative rotation free density P on $\Omega(\lambda) (\lambda > 0)$ and the Martin compactification $\Omega(\lambda)^*$ of $\Omega(\lambda)$ with respect to the elliptic equation $\Delta u = Pu$ will be established.

A nonnegative locally Hölder continuous function $P(|z|) (|z| = r)$ on $\Omega(\lambda)$ is called a *nonnegative rotation free density* on $\Omega(\lambda)$. The *singular index* $\alpha(P)$ of a density P at $r = \lambda$ is the quantity given by

$$\alpha(P) = \lim_{r \rightarrow \lambda} e_1(r)/e_0(r),$$

where $e_j(r) (j = 1, 2)$ is a unique bounded solution of the equation

$$\frac{d^2}{dr^2} e(r) + \frac{1}{r} \frac{d}{dr} e(r) - (P(r) + j^2/r^2) e(r) = 0 \quad (j = 0, 1)$$

on $(\lambda, 1]$ with $e_j(1) = 1$ for $\lambda = 0$, and furthermore with $\lim_{r \rightarrow \lambda} e_j(r) = 0$ for $\lambda > 0$. In particular, $\alpha(0)$ with $P \equiv 0$ on $\Omega(\lambda) (\lambda \geq 0)$ will be referred to as the *harmonic index* at $r = \lambda$. The *elliptic dimension*, including the harmonic dimension with $P \equiv 0$, of a density P at the ideal boundary $|z| = \lambda$, $\dim \mathcal{S}(\lambda)$ in notation, is the dimension of the half module $\mathcal{S}(\lambda)$ of the positive solutions u of $\Delta u = Pu$ on $\Omega(\lambda)$ with $u = 0$ on $|z| = 1$. The elliptic dimension is an ideal boundary property (M. Heins [2], K. Hayashi [1], and M. Ozawa [12], [13]). The *Picard principle* is said to be valid for a density P at $|z| = \lambda$ if $\dim \mathcal{S}(\lambda) = 1$ or equivalently $\alpha(P) = 0$ (M. Nakai [8]).

In [8] M. Nakai had shown that the *singular index* $\alpha(P)$ at $\lambda = 0$ determines the Martin compactification in the following ways:

$$\Omega(0)_P^* \approx \{\alpha(P) \leq |z| \leq 1\}, \quad \alpha(P) = 0 \text{ or } \alpha(P) > 0$$

in the sense of homeomorphism for any nonnegative rotation free density P on $\Omega(0)$ and each ideal boundary point is minimal.

It will be shown that the *singular indices* are linearly ordered in the sense that $\alpha(P) \leq \alpha(P_1)$ if $P \leq P_1$ on $\Omega(\lambda) (\lambda \geq 0)$. In particular, the *harmonic index* which is equal to $(-2\lambda \log \lambda)(1 - \lambda^2)^{-1}$ for $\lambda > 0$ minimizes the singular indices at $r = \lambda$ among the nonnegative rotation free densities on $\Omega(\lambda)$. As a counterpart of

the result of M. Nakai listed above, we will have that

$$\Omega(\lambda)_P^* \approx \{\alpha(P) \leq |z| \leq 1\} \quad (\lambda > 0)$$

and each boundary point is minimal for any rotation free density $P \geq 0$ on $\Omega(\lambda)$. Thus, the Picard principle is not valid on a hyperbolic end $\lambda < |z| < 1$ ($\lambda > 0$) for any rotation free density $P(r) \geq 0$ on $\Omega(\lambda)$.

From the other aspect, a net $\{\Omega(\lambda)\}_{(\lambda > 0)}$ may be viewed as a regular exhaustion of a parabolic end $\Omega(0)$ (precisely, $0 < |z| < 1$). Consider a density $P(r) \geq 0$ on $\Omega(0)$. Then, $\alpha_\lambda(P)$ at $r = \lambda > 0$ can be defined regarding $P(r)$ as a density on $\Omega(\lambda)$ in the same way as $\alpha(P)$ at $\lambda = 0$. It will be shown that $\alpha_\lambda(P)$ is a decreasing function of λ on $[0, 1)$ and that

$$\alpha(P) = \lim_{\lambda \rightarrow 0} \alpha_\lambda(P).$$

Also, the interrelation of generators of $\mathcal{S}(\lambda)$ and $\mathcal{S}(0)$ will be studied.

As for the Martin compactification with respect to the elliptic operator, we can find the work of S. Ito [4] and L. Hunt—R. Wheeden [3], among others. Neither the regularity of an elliptic operator nor the continuity of the solutions are assumed at the ideal boundary in the recent research on the Picard principle for which we refer to M. Nakai [8], [10], and [11], among others.

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1. Let $P(r)$ be any rotation free density on $\lambda < |z| \leq 1$ ($\lambda \geq 0$) and consider the elliptic differential equation

$$L_n e(z) \equiv L_{P,n} e(z) = \Delta e(z) - (P(|z|) + n^2/|z|^2)e(z) = 0 \quad (n = 0, 1, \dots)$$

on $\lambda < |z| \leq \rho$ for any ρ fixed in $(\lambda, 1]$. We will denote by $f(r; \rho)$ a function defined on $\lambda < |z| \leq \rho$ and by $f(r)$ the function $f(r; 1)$.

Let $e_n(r; \rho)$ ($n \geq 0$) be the rotation free solution of $L_n e(z) = 0$ on $\lambda < |z| \leq \rho$ ($|z| = r$) with $\lim_{r \rightarrow \lambda} e_n(r; \rho) = 0$ and $e_n(\rho; \rho) = 1$ for $\lambda > 0$, and the unique bounded rotation free solution of $L_n e(z) = 0$ with $e_n(\rho; \rho) = 1$ for $\lambda = 0$. The existence of such a solution can be deduced by the solvability of the Dirichlet problem, Harnack principle, and the maximum principle (cf [8]). However we note that the applications of the maximum principle for the operator L_n can be made in quite different ways according as $\lambda > 0$ and $\lambda = 0$.

Consider another rotation free density $P_1(r)$ such that

$$P(r) \leq P_1(r)$$

on $\lambda < |z| \leq 1$ and denote by $\bar{e}_n(r; \rho)$ the corresponding solution to the density $P_1(r)$, i.e., the solution of $L_n e(z) = L_{P_1, n} e(z) = 0$ with the same properties as $e_n(r; \rho)$. Then,

$$(1) \quad e_n(r; \rho) = e_n(r)/e_n(\rho), \quad \bar{e}_n(r; \rho) = \bar{e}_n(r)/\bar{e}_n(\rho)$$

on $\lambda < |z| \leq \rho$. The maximum principle (cf. e.g., [7]) or equivalently the comparison principle [9] yields that

$$(2) \quad e_0(r; \rho) \geq e_1(r; \rho) \geq \dots \geq e_n(r; \rho) \geq \dots \geq 0,$$

and

$$(3) \quad \bar{e}_0(r; \rho) \geq \bar{e}_1(r; \rho) \geq \dots \geq \bar{e}_n(r; \rho) \geq \dots \geq 0, \\ e_n(r; \rho) \geq \bar{e}_n(r; \rho) \quad (n \geq 0)$$

on $\lambda < |z| \leq \rho$. Setting

$$v_n(r; \rho) = e_n(r; \rho)/e_0(r; \rho) \quad (n \geq 1),$$

we obtain from (1) and (2) that $v_n(r; \rho) = (e_n(r)/e_0(r))/(e_n(\rho)/e_0(\rho)) \leq 1$ for any r and ρ with $\lambda < r \leq \rho \leq 1$. Hence $v_n(r)$ is an increasing function of r on $(\lambda, 1]$ and the same is true for $v_n(r; \rho)$ on $(\lambda, \rho]$, since $v_n(r; \rho) = v_n(r)/v_n(\rho)$ on $(\lambda, \rho]$. Thus we have

$$(4) \quad \frac{d}{dr} v_n(r; \rho) \geq 0 \quad (n \geq 1)$$

on $(\lambda, \rho]$ for any ρ in $(\lambda, 1]$.

2. Consider the function $u_n(r)$ ($n \geq 0$) on $(\lambda, 1]$ defined by

$$u_n(r) = \bar{e}_n(r)/e_n(r).$$

Then, $u_n(r)$ ($r = |z|$) satisfies the elliptic equation

$$M_n u(z) = \Delta u(z) + 2\nabla \log e_n(|z|) \cdot \nabla u(z) - (P_1(|z|) - P(|z|))u(z) = 0 \\ (n \geq 0)$$

on $\lambda < |z| < 1$, where ∇u is the gradient vector field $(\partial u/\partial x, \partial u/\partial y)$ of u . Since $(\bar{e}_n(r)/\bar{e}_n(\rho)) \leq (e_n(r)/e_n(\rho)) \leq 1$ on $(r, \rho]$ by (1) and (3), $u_n(r) \leq u_n(\rho)$ on $(\lambda, 1]$ for any r and ρ in $(\lambda, 1]$ with $r < \rho$ which implies that

$$\frac{d}{dr} u_n(r) \geq 0 \quad (n \geq 0)$$

on $(\lambda, 1)$. Consequently, $\lim_{r \rightarrow 1} u_n(r)$ exists and $1 > \lim_{r \rightarrow 1} u_n(r) \geq 0$, because the constant 1 is not a solution of $M_n u(z) = 0$.

We define the function $h_n(r; \rho)$ by

$$h_n(r; \rho) = \log(1/e_n(r; \rho)).$$

Then, a direct computation yields that

$$\begin{aligned} M_n h_n(r) &= -[\nabla \log e_n(r) \cdot \nabla \log e_n(r) + P(r) + (n/r)^2 + (P_1(r) \\ &\quad - P(r))h_n(r)] \leq 0 \quad (r = |z|) \end{aligned}$$

on $\lambda < |z| < 1$. Hence the operator M_n is parabolic for an end $\lambda < |z| < 1$ (cf. [5] and [14]).

To assert that $u_n(r) \geq u_0(r)$ ($n \geq 1$) on $(\lambda, 1]$, observe that

$$M_n u_0(r) = 2\nabla \log v_n(r) \cdot \nabla u_0(r) \geq 0 \quad (r = |z|)$$

on $\lambda < |z| < 1$. Then, $M_n(u_n(r) - u_0(r) + \varepsilon h_n(r)) \leq 0$ on $\lambda < |z| < 1$ for every $\varepsilon > 0$. Since $\lim_{r \rightarrow \lambda}(u_n(r) - u_0(r) + \varepsilon h_n(r)) \geq 0$ and $u_n(r) - u_0(r) + \varepsilon h_n(r) = 0$ at $r = 1$, the maximum principle implies that $u_n(r) - u_0(r) + \varepsilon h_n(r) \geq 0$ on $\lambda < |z| < 1$ for every $\varepsilon > 0$. Letting $\varepsilon \rightarrow 0$, we deduce that

$$(5) \quad u_n(r) \geq u_0(r) \quad (n \geq 1)$$

on $(\lambda, 1]$.

3. We now define the quantity

$$\alpha_n(P) = \lim_{r \rightarrow \lambda} v_n(r)$$

for any nonnegative rotation free density $P(r)$ on $\lambda < |z| \leq 1$ ($\lambda \geq 0$) which will be referred to as the n th *singular index* of P at $r = \lambda$. The quantity $\alpha_1(P)$ is abbreviated as $\alpha(P)$ and simply called the *singular index* of P at $r = \lambda$ ([18]). In particular $\alpha(0)$ with $P = 0$ will be referred to as the *harmonic index* at $r = \lambda$.

First, in view of (5), we have $\bar{e}_n(r)/\bar{e}_0(r) \geq e_n(r)/e_0(r)$ on $(\lambda, 1]$ and hence $\alpha_n(P_1) \geq \alpha_n(P)$. It is needless to say that $\alpha_n(P_1)$ ($n \geq 1$) indicate the n th singular index of P_1 at $r = \lambda$. For $P = 0$ and $\lambda > 0$, observe that $e_0(r) = \log(\lambda/r)/\log \lambda$ and $e_1(r) = (r - \lambda^2 r^{-1})(1 - \lambda^2)^{-1}$. Then, by the L'Hospital theorem, we deduce that $\alpha(0) = (-2\lambda \log \lambda)(1 - \lambda^2)^{-1}$. In the case of $\lambda = 0$, beside $P = 0$ there are many densities $P(r) \geq 0$ which satisfies $\alpha(P) = 0$ ([8], also see (34) in [6]). We summarize the result in:

THEOREM 1. *Let $P(r)$ and $P_1(r)$ be any rotation free densities which satisfy the condition $P(r) \leq P_1(r)$ on $\lambda < |z| \leq 1$ ($\lambda \geq 0$). Then,*

$$0 \leq \alpha_n(P) \leq \alpha_n(P_1) < 1 \quad (n \geq 1).$$

In particular, the harmonic index $\alpha(0)$ at $r = \lambda$ which is equal to $(-2\lambda \log \lambda)(1 - \lambda^2)^{-1}$ for $\lambda > 0$ minimizes singular indices among nonnegative rotation free densities on $\lambda < |z| \leq 1$.

The above theorem intuitively indicates that in the case $\lambda > 0$ we can make $\alpha(P)$ sufficiently close to 1, selecting a sufficiently large density P . To see our inspection is certainly true, we first observe that

$$L_0((r/\rho)^m - e_0(r; \rho)) = (r^{m-2}/\rho^m)(m^2 - P(r)r^2)$$

on $\lambda < |z| \leq \rho$ with $r = |z|$. Then, for an $m > 0$ satisfying

$$(6) \quad m^2 - P(r)r^2 \leq 0$$

on $\lambda < |z| \leq 1$, $(r/\rho)^m \geq e_0(r; \rho)$ is valid on $(\lambda, \rho]$ by the maximum principle and hence we deduce that $d/dr(e_0(r)/r^m) \geq 0$ on $(\lambda, 1]$ for such an $m > 0$. Thus we can easily see that

$$L_1(e_1(r) - r^c e_0(r)) = -2cr^{c-1}e_0(r) \frac{d}{dr} \log(e_0(r)/r^m) \leq 0$$

on $(\lambda, 1)$ for any c fixed in $(0, 1]$ and satisfying (6) with $m = (1 - c^2)/2c$. A fortiori, $e_1(r)/e_0(r) \geq r^c$ on $(\lambda, 1]$. Thus we deduce:

PROPOSITION 1. *Let $P(r)$ be any rotation free density on $\lambda < |z| \leq 1$ ($\lambda > 0$) which satisfies the condition (6) with $m = (1 - c^2)/2c$, c being fixed in $(0, 1]$. Then,*

$$\lambda^c \leq \alpha(P) < 1$$

is valid.

4. We observe that the function $v_n(r; \rho)$ ($r = |z|$) in no. 1 is a bounded solution of the elliptic equation

$$I_n v(z) = \Delta v(z) + 2\mathcal{F} \log e_0(|z|; \rho) \cdot \mathcal{F} v(z) - (n/r)^2 v(z) = 0 \quad (n \geq 1)$$

on $\lambda < |z| \leq \rho$. A direct computation yields that

$$I_n h_0(r; \rho) = -[P(r) + \mathcal{F} \log e_0(r; \rho) \cdot \mathcal{F} \log e_0(r; \rho) + (n/r)^2 h_0(r; \rho)] < 0$$

on $\lambda < |z| < \rho$. Hence each operator I_n possesses an Evans super-solution $h_0(r; \rho)$ on $\lambda < |z| < \rho$ (cf. e.g., [5], [14]).

To compare $v_n(r; \rho)$ for $n = 1, 2, \dots$, we observe that, for $m > 0$,

$$I_n[v_1(r; \rho)^m] = m(m-1)v_1(r; \rho)^{m-2} \left(\frac{d}{dr} v_1(r; \rho) \right)^2 - ((n^2 - m)/r^2)v_1(r; \rho)^m \quad (r = |z|)$$

is valid on $\lambda < |z| < \rho$. Taking $m = n^2$, $I_n[v_1(r: \rho)^{n^2}] \geq 0$ on $\lambda < |z| < \rho$ for $n > 1$. Also, setting $m = n$,

$$I_n[v_1(r: \rho)^n] = n(n-1)v_1(r: \rho)^{n-2} \left[\left(\frac{d}{dr} v_1(r: \rho) \right)^2 - (v_1(r: \rho)/r)^2 \right]$$

on $\lambda < |z| < \rho$. Considering $L_1(e_1(r: \rho) - (r/\rho)e_0(r: \rho) + \varepsilon \log(r/\rho))$ on $\lambda < |z| < \rho$ with every $\varepsilon > 0$ for $\lambda = 0$ and $\varepsilon = 0$ for $\lambda > 0$. Then, $v_1(r)/r \geq v_1(\rho)/\rho$ is valid for any r and ρ such that $\lambda < r \leq \rho \leq 1$. This inequation shows that $v_1(r)/r$ is decreasing on $(\lambda, 1]$, and the same is true for $v_1(r: \rho)/r$ on $(\lambda, \rho]$. Hence, $d/dr \log(v_1(r: \rho)/r) \leq 0$ on $(\lambda, \rho]$, and a fortiori $d/dr v_1(r: \rho) - v_1(r: \rho)/r \leq 0$ on (λ, ρ) . Consequently, $I_n[v_1(r: \rho)^n] \leq 0$ on $\lambda < |z| < \rho$. From these estimates it can be deduced that $I_n[v_n(r: \rho) - v_1(r: \rho)^{n^2} + \varepsilon h_0(r: \rho)] \leq 0$ ($n \geq 2$) and $I_n[v_1(r: \rho)^n - v_n(r: \rho) + \varepsilon h_0(r: \rho)] \leq 0$ ($n \geq 1$) on $\lambda < |z| < \rho$ for every $\varepsilon > 0$. Therefore, the maximum principle yields that

$$(7) \quad v_1(r: \rho)^{n^2} \leq v_n(r: \rho) \leq v_1(r: \rho)^n \quad (n \geq 1)$$

on $\lambda < |z| \leq \rho$ for any ρ in $(\lambda, 1]$. Taking $\rho = 1$ in the above inequations, we can deduce:

LEMMA 1. For any rotation free density $P(r) \geq 0$ on $\lambda < |z| \leq 1$ ($\lambda \geq 0$),

$$\alpha(P)^{n^2} \leq \alpha_n(P) \leq \alpha(P)^n \quad (n \geq 1).$$

5. Consider the Green's function $G(z, \zeta)$ of $\Delta u = Pu$ on $\lambda < |z| < 1$ ($\lambda \geq 0$) with pole at ζ . For any z and $\zeta = re^{i\sigma}$ in $\lambda < |z| < 1$ with $|z| \neq r$, expand $G(z, \zeta)$ into Fourier series:

$$G(z, re^{i\sigma}) = c_0(z: r)/2 + \sum_{n=1}^{\infty} (c_n(z: r) \cos n\sigma + s_n(z: r) \sin n\sigma),$$

where

$$\begin{aligned} c_n(z: r) &= \pi^{-1} \int_0^{2\pi} G(z, re^{i\sigma}) \cos n\sigma d\sigma \quad (n \geq 0), \quad s_n(z: r) \\ &= \pi^{-1} \int_0^{2\pi} G(z, re^{i\sigma}) \sin n\sigma d\sigma \quad (n \geq 1). \end{aligned}$$

Since $G(z, \zeta)$ is dominated by the harmonic Green's function $\log(|1 - \bar{\zeta}z|/|z - \zeta|)$ of the unit disk, the method of M. Nakai [8] can be modified to the hyperbolic case. But for the sake of completeness, we draw the outline of the method in [8]. The $c_n(z: r)$ ($n \geq 0$) and $s_n(z: r)$ ($n \geq 1$) are bounded continuous functions of r on $(\lambda, 1]$, and satisfy $L_n c_n(z: r) = 0$ ($n \geq 0$) and $L_n s_n(z: r) = 0$ ($n \geq 1$) for $r \neq |z|$. Therefore

$$c_n(z: r)/c_n(z: \rho) = e_n(r: \rho), \quad s_n(z: r)/s_n(z: \rho) = e_n(r: \rho)$$

for every r and ρ with $\lambda < r \leq \rho < |z|$, and

$$(8) \quad |c_n(z: \rho)|, |s_n(z: \rho)| \leq c_0(z: \rho) \leq 2 \min(\log \rho^{-1}, \log |z|^{-1}).$$

Since $\sup_{r \in (\lambda, \tau]} v_n(r: \rho) = \rho_\tau < 1$ for any τ such that $\tau < \rho$, by the Weierstrass double convergence theorem, the limit

$$L(z, \theta: \rho) = \lim_{r \rightarrow \lambda, \sigma \rightarrow \theta} G(z, re^{i\sigma})/e_0(r: \rho)$$

exists for any ρ in $(\lambda, 1]$ with $\rho < |z|$ which is positive function of z on $\rho < |z| < 1$, where

$$(9) \quad L(z, \theta: \rho) = c_0(z: \rho)/2 + \sum_{n=1}^{\infty} (c_n(z: \rho) \cos n\theta + s_n(z: \rho) \sin n\theta) \alpha_n(P) v_n(\rho)^{-1}$$

for any z, ρ such that $\lambda < \rho < |z| < 1$, and any θ in T . Denoting $G(z, \zeta)/G(z_0, \zeta)$ by $K(z, \zeta)$ for some fixed z_0 in $\lambda < |z| < 1$, we obtain from the relation $K(z, re^{i\sigma}) = (G(z, re^{i\sigma})/e_0(r: \rho))/(G(z_0, re^{i\sigma})/e_0(r: \rho))$ that

$$(10) \quad \lim_{r \rightarrow \lambda, \sigma \rightarrow \theta} K(z, re^{i\sigma}) = L(z, \theta: \rho)/L(z_0, \theta: \rho)$$

for every z in $\lambda < |z| \leq 1$ and any θ in $T \equiv (-\infty, \infty)/\text{mod } 2\pi$, where ρ is any number in $(\lambda, |z|) \cap (\lambda, |z_0|)$. Since the left hand side of the above equation is independent of ρ , we can define the function

$$(11) \quad k(z, \theta) = L(z, \theta: \rho)/L(z_0, \theta: \rho)$$

on $\lambda < |z| \leq 1$ for any θ in T and any z_0 in $\lambda < |z| < 1$.

In passing we note that the following result which had shown for $\lambda = 0$ in [8] is also valid for $\lambda > 0$, since the proof in [8] is not affected by the parabolicity of the origin.

LEMMA 2. *The system $c_n(z: r)$ ($n \geq 0$) and $s_n(z: r)$ ($n \geq 1$) is linearly independent as functions of z in the following sense: Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be arbitrary absolutely convergent real series. Suppose that*

$$\sum_{n=0}^{\infty} a_n c_n(z: r) + \sum_{n=1}^{\infty} b_n s_n(z: r) = 0$$

on a nonempty open subset of $\lambda < |z| < 1$ ($\lambda > 0$) for any fixed r in $(\lambda, 1)$. Then, $a_0 = a_n = b_n = 0$ ($n \geq 1$).

6. We denote $\lambda < |z| \leq 1$ by $\Omega(\lambda)$ and consider the Martin compactification $\Omega(\lambda)_F^*$ of an relatively closed annulus $\Omega(\lambda)$ ($\lambda > 0$)

with respect to $\Delta u = Pu$ toward the ideal boundary $|z| = \lambda$. As usual, the Martin topology can be given by the completion of $\Omega(\lambda)$ with respect to the metric d defined by

$$(12) \quad d(\zeta_1, \zeta_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|K(z_n, \zeta_1) - K(z_n, \zeta_2)|}{1 + |K(z_n, \zeta_1) - K(z_n, \zeta_2)|}$$

for any ζ_1 and ζ_2 in $\Omega(\lambda)$, where $\{z_n\}_{n=1}^{\infty}$ is any dense subset of $\lambda < |z| < 1$. A point ζ^* in $\Gamma_P = \Omega(\lambda)_P^* - \Omega(\lambda)$ is called *minimal* if $K(z, \zeta^*)$ is minimal in the half module $\mathcal{P}(\lambda)$ of the positive solutions u of $\Delta u = Pu$ on $\Omega(\lambda)$ with $u = 0$ on $|z| = 1$. Since $\alpha(P) > 0$, we can deduce by the same arguments as in [8] that each $k(z, \theta)$ is minimal, and that $k(z, \theta_1)$ and $k(z, \theta_2)$ are nonproportional for each pair $\theta_1 \neq \theta_2$ in T by virtue of Lemma 2. Then, the relation between $k(z, \theta)$ and $K(z, \zeta^*)$ are given by

$$k(z, \theta) = K(z, \arg \pi(\zeta^*))$$

for any ζ^* in Γ_P with $\theta = \arg \pi(\zeta^*)$, where π is the homeomorphism of $\Omega(\lambda)_P^*$ onto the closed annulus $\lambda \leq |z| \leq 1$. Denoting by (r, θ) the polar coordinate of a point ζ in $\lambda \leq |z| \leq 1$, the mapping $\tau \cdot \pi$ defines a homeomorphism of $\Omega(\lambda)_P^*$ onto $\alpha(P) \leq |z| \leq 1$, where τ is given by $\tau(r, \theta) = (1 - \lambda)^{-1}[(1 - \alpha(P))r + \alpha(P) - \lambda]e^{i\theta}$. Thus we have the counterpart of Theorem 5 in M. Nakai [8].

THEOREM 2. *The Martin compactification $\Omega(\lambda)_P^*$ of $\Omega(\lambda)$ ($\lambda > 0$) with respect to any rotation free density $P(r) \geq 0$ on $\Omega(\lambda)$ is homeomorphic to the closed annulus $\alpha(P) \leq |z| \leq 1$ in such a way that every boundary point is minimal.*

As an immediate consequence of Theorems 1 and 2, we have

$$\dim \mathcal{P}(\lambda) = c \quad (\lambda > 0),$$

c being the cardinal number of continuum. Thus, we conclude:

COROLLARY. *The Picard principle does not hold on a hyperbolic end $\Omega(\lambda)$ ($\lambda > 0$) for any nonnegative rotation free density on $\Omega(\lambda)$.*

In contrast to $\lambda > 0$, $\dim \mathcal{P}(0) = 1$ or c for any nonnegative rotation free density P on $0 < |z| \leq 1$ ([8]). Thus the comparison of the elliptic dimension is meaningful for $\lambda = 0$. As a byproduct of Theorem 1, and Theorem 5 in [8], we have the following proposition which was shown as a consequence of the b -test by M. Nakai [9].

PROPOSITION 2. *Let P and P_1 be any nonnegative rotation free densities such that $P(r) \leq P_1(r)$ on $0 < |z| = r \leq 1$. Then,*

$$\dim \mathcal{P}(0) \leq \dim \mathcal{P}_1(0),$$

where $\dim \mathcal{P}_1(0)$ is the elliptic dimension of P_1 at $|z| = 0$.

7. Consider a nonnegative rotation free density P on $0 < |z| \leq 1$ which will be fixed in the sequel. Then beside the n th singular index $\alpha_n(P)$ ($n \geq 1$) of P at $r = 0$, we can define the n th singular index $\alpha_{n\lambda}(P)$ of P at $r = \lambda$ for any λ in $(0, 1)$, regarding P as a density on $\lambda < |z| \leq 1$. We are concerned with an interrelation of the $\alpha_n(P)$ at $r = 0$ and the $\alpha_{n\lambda}(P)$ at $r = \lambda$. To avoid the confusion in notations, we will denote any function $f_n(r; \rho)$ by $f_{n\lambda}(r; \rho)$ for $\lambda > 0$ and $f_n(r; \rho)$ for $\lambda = 0$ which are defined for a density P on $0 < |z| \leq 1$. Likewise, $I_{n\lambda}$ stands for the operator I_n in no. 4 considered on $\lambda < |z| < \rho$.

We consider the function $v_{n\lambda}(r; \rho)$ and $v_{n\eta}(r; \rho)$ for any λ, η , and ρ with $0 < \lambda \leq \eta < \rho < 1$. Since $I_{n\eta}v(z) = \Delta v(z) + 2\nabla \log e_{0\eta}(|z|; \rho) \cdot \nabla v(z) - (n/|z|)^2 v(z) = 0$ on (η, ρ) , we have by a direct computation

$$\begin{aligned} I_{n\eta}h_{0\eta}(r; \rho) &= -[P(r) + \nabla \log e_{0\eta}(r; \rho) \cdot \nabla \log e_{0\eta}(r; \rho) \\ &\quad + (n/r)^2 h_{0\eta}(r; \rho)] \leq 0 \end{aligned}$$

and

$$I_{n\eta}v_{n\lambda}(r; \rho) = 2\nabla \log (e_{0\eta}(r; \rho)/e_{0\lambda}(r; \rho)) \cdot \nabla v_{n\lambda}(r; \rho)$$

on $\eta < |z| < \rho$. Then, since $1 \geq e_{0\eta}(r; \rho)/e_{0\lambda}(r; \rho) = (e_{0\eta}(r)/e_{0\lambda}(r))/(e_{0\eta}(\rho)/e_{0\lambda}(\rho))$ on $[\eta, \rho]$, the function $e_{0\eta}(r)/e_{0\lambda}(r)$ is increasing on $(\eta, 1)$ and the same is true for $e_{0\eta}(r; \rho)/e_{0\lambda}(r; \rho)$ on (η, ρ) . Consequently $I_{n\eta}v_{n\lambda}(r; \rho) \leq 0$ on $\eta < |z| < \rho$ and hence $I_{n\eta}(\varepsilon h_{0\eta}(r; \rho) + v_{n\eta}(r; \rho) - v_{n\lambda}(r; \rho)) \leq 0$ on $\eta < |z| < \rho$ for every $\varepsilon > 0$. Therefore it follows from the maximum principle that

$$v_{n\lambda}(r; \rho) \leq v_{n\eta}(r; \rho) \quad (n \geq 1)$$

on (η, ρ) for any λ, η , and ρ such that $0 < \lambda \leq \eta < \rho < 1$. Since $v_{n\eta}(r; \rho) = v_{n\eta}(r)/v_{n\eta}(\rho)$, $v_{n\lambda}(\eta)v_{n\lambda}(\rho)^{-1} \leq \alpha_{n\eta}(P)v_{n\eta}(\rho)^{-1}$ and by the monotone increasingness of $v_{n\lambda}(r)$, we have

$$(13) \quad \alpha_{n\lambda}(P)v_{n\lambda}(\rho)^{-1} \leq \alpha_{n\eta}(P)v_{n\eta}(\rho)^{-1} \quad (n \geq 1)$$

for any λ, η , and ρ with $0 < \lambda \leq \eta < \rho \leq 1$.

Let $\bar{v}_{n\lambda}(r; \rho)$ be the continuous function on $[0, 1]$ defined by $\bar{v}_{n\lambda}(r; \rho) = v_{n\lambda}(r; \rho)$ on $(\lambda, \rho]$ and $\bar{v}_{n\lambda}(r; \rho) = \alpha_{n\lambda}(P)v_{n\lambda}(\rho)^{-1}$ on $[0, \lambda]$ for any λ in $(0, 1)$. Since $\{\bar{v}_{n\lambda}(r; \rho)\}$ forms a decreasing net as $\lambda \rightarrow 0$, the limit $\bar{v}_n(r; \rho) = \lim_{\lambda \rightarrow 0} \bar{v}_{n\lambda}(r; \rho)$ exists. On the other hand, the

convergence of $e_{n\lambda}(r: \rho) \rightarrow e_n(r: \rho)$ as $\lambda \rightarrow 0$ is uniform on $[\kappa, \rho]$ for any κ in (λ, ρ) by virtue of the Harnack principle. Thus $\{\bar{v}_{n\lambda}(r: \rho)\}$ converges uniformly to $v_n(r: \rho)$ on $[\kappa, \rho]$ and $\bar{v}_n(r: \rho) = v_n(r: \rho)$ on $[\kappa, \rho]$ for any κ such that $0 < \lambda < \kappa < \rho$. By the unique solvability of the Cauchy problem for I_n on $0 < |z| < \rho$ (cf. no. 4) we have $\bar{v}_n(r: \rho) = v_n(r: \rho)$ on $[0, \rho]$, setting $v_n(0, \rho) = \alpha_n(P)v_n(\rho)^{-1}$. Then $\lim_{\lambda \rightarrow 0} \bar{v}_{n\lambda}(r: \rho) = v_n(r: \rho)$, where $\{v_{n\lambda}(r: \rho)\}$ converges monotone decreasingly to the function $v_n(r: \rho)$ on $[0, \rho]$, and $\lim_{\lambda \rightarrow 0} \bar{v}_{n\lambda}(r: \rho) = \alpha_n(P)v_n(\rho)^{-1}$ exists. Hence, by the Weierstrass double convergence theorem, we deduce that, for any λ in $(0, \rho)$,

$$(14) \quad \begin{aligned} \alpha_n(P)v_n(\rho)^{-1} &\leq \alpha_{n\lambda}(P)v_{n\lambda}(\rho)^{-1}, \\ \lim_{\lambda \rightarrow 0} \alpha_{n\lambda}(P)v_{n\lambda}(\rho)^{-1} &= \alpha_n(P)v_n(\rho)^{-1} \quad (n \geq 1). \end{aligned}$$

Taking $\rho = 1$ in the above discussion, we summarize the result in:

THEOREM 3. *Let P be any nonnegative rotation free density on $0 < |z| \leq 1$. Then, for any λ and η with $0 < \lambda < \eta < 1$,*

$$\alpha_{n\lambda}(P) \leq \alpha_{n\eta}(P), \quad \lim_{\lambda \rightarrow 0} \alpha_{n\lambda}(P) = \alpha_n(P) \quad (n \geq 1).$$

8. Consider a net $\{\lambda < |z| < 1\}_{(\lambda > 0)}$ which may be regarded as a regular exhaustion of a parabolic end $0 < |z| < 1$. The Green's functions of $\Delta u = Pu$ on $\lambda < |z| \leq 1$ and $0 < |z| \leq 1$ with pole at ζ are denoted by $G_\lambda(z, \zeta)$ and $G(z, \zeta)$, respectively. We maintain that for each θ fixed in T a net of minimal functions $\{k_\lambda(z, \theta)\}$ converges to a minimal function $k(z, \theta)$ uniformly on each compact subset of $0 < |z| \leq 1$. We first note that in view of (8) the following inequalities are valid on $\eta \leq |z| \leq 1$ for any η with $\rho < \eta$: For $n = 1, 2, \dots$,

$$\begin{aligned} c_n(z, \theta, \lambda) &\equiv |c_{n\lambda}(z: \rho) \cos n\theta \alpha_{n\lambda}(P)v_{n\lambda}(\rho)^{-1} - c_n(z: \rho) \cos n\theta \alpha_n(P)v_n(\rho)^{-1}|, \\ s_n(z, \theta, \lambda) &\equiv |s_{n\lambda}(z: \rho) \sin n\theta \alpha_{n\lambda}(P)v_{n\lambda}(\rho)^{-1} - s_n(z: \rho) \sin n\theta \alpha_n(P)v_n(\rho)^{-1}| \\ &\leq (2 \log \rho^{-1})(\alpha_{n\lambda}(P)v_{n\lambda}(\rho)^{-1} - \alpha_n(P)v_n(\rho)^{-1}) \\ &\quad + \left\{ \frac{1}{\pi} \int_0^{2\pi} |G_\lambda(z, \rho e^{i\theta}) - G(z, \rho e^{i\theta})| d\theta \right\} \alpha_n(P)v_n(\rho)^{-1}. \end{aligned}$$

We have from (7) that $\alpha_{n\lambda}(P)v_{n\lambda}(\rho)^{-1} \leq (\alpha_{1\lambda}(P)v_{1\lambda}(\rho)^{-1})^n$ and $\alpha_n(P)v_n(\rho)^{-1} \leq (\alpha(P)v_1(\rho)^{-1})^n$ ($n \geq 1$). Then by (8) and (9), $|L_\lambda(z, \theta: \rho)| \leq 4(\log \eta^{-1})(1 - \alpha_{1\lambda}(P)v_{1\lambda}(\rho)^{-1})^{-1}$ and $|L(z, \theta: \rho)| \leq 4(\log \eta^{-1})(1 - \alpha(P)v_1(\rho)^{-1})^{-1}$ on $\eta \leq |z| \leq 1$ which shows that both functions $L_\lambda(z, \theta: \rho)$ and $L(z, \theta: \rho)$ converge uniformly and absolutely on $\eta \leq |z| \leq 1$. Thus the function $|L_\lambda(z, \theta: \rho) - L(z, \theta: \rho)|$ is dominated by

$$|c_{0\lambda}(z; \rho) - c_0(z; \rho)|/2 + \sum_{n=1}^{\infty} (c_n(z, \theta, \lambda) + s_n(z, \theta, \lambda))$$

on $\eta \leq |z| \leq 1$. Consequently, we have from the above estimates that

$$\begin{aligned} & |L_\lambda(z, \theta; \rho) - L(z, \theta; \rho)| \\ & < \left\{ \frac{2}{\pi} \int_0^{2\pi} |G_\lambda(z, \rho e^{i\theta}) - G(z, \rho e^{i\theta})| d\theta \right\} (1 - \alpha(P)v_1(\rho)^{-1})^{-1} \\ & + (4 \log \rho^{-1}) \sum_{n=1}^{\infty} (\alpha_{n\lambda}(P)v_{n\lambda}(\rho)^{-1} - \alpha_n(P)v_n(\rho)^{-1}) \end{aligned}$$

on $\eta \leq |z| \leq 1$. Setting $\beta_n(\lambda) = \alpha_{n\lambda}(P)v_{n\lambda}(\rho)^{-1} - \alpha_n(P)v_n(\rho)^{-1}$, (13) and (14) yield that $\beta_n(\lambda)$ is a nonnegative function of λ on $[0, 1)$ which tends monotone decreasingly to 0 as $\lambda \rightarrow 0$. Also, $\sum_{n=1}^{\infty} \beta_n(\lambda)$ exists for each λ in $[0, 1)$, since $\sum_{n=1}^{\infty} \beta_n(\lambda) \leq 2\alpha_{n\lambda}(P)v_{n\lambda}(\rho)^{-1}/(1 - \alpha_{n\lambda}(P)v_{n\lambda}(\rho)^{-1})$. Thus we have $\lim_{\lambda \rightarrow 0} \sum_{n=1}^{\infty} \beta_n(\lambda) = 0$. On the other hand, $\int_0^{2\pi} |G_\lambda(z, \rho e^{i\theta}) - G(z, \rho e^{i\theta})| d\theta$ tends to 0 uniformly on $\eta \leq |z| \leq 1$ as $\lambda \rightarrow 0$. Since η is arbitrary in $(\rho, 1)$, we conclude that as $\lambda \rightarrow 0$ a net $\{L_\lambda(z, \theta; \rho)\}$ converges to the function $L(z, \theta; \rho)$ on each compact subset of $\rho < |z| \leq 1$ for each θ fixed in T .

The functions $L_\lambda(z, \theta; \rho)$ and $L(z, \theta; \rho)$ are positive and finite on $\eta \leq |z| \leq 1$ for any η in $(\rho, 1)$ (cf. no. 5). Taking z_0 in $\eta \leq |z| \leq 1$, the inequation

$$\begin{aligned} |k_\lambda(z, \theta) - k(z, \theta)| & \leq L_\lambda(z_0, \theta; \rho)^{-1} |L_\lambda(z, \theta; \rho) - L(z, \theta; \rho)| \\ & + L_\lambda(z_0, \theta; \rho)L(z_0, \theta; \rho)^{-1}M |L(z_0, \theta; \rho) - L_\lambda(z_0, \theta; \rho)| \end{aligned}$$

is valid on $\eta \leq |z| \leq 1$, where $M = \max_{\eta \leq |z| \leq 1} |L(z, \theta; \rho)|$. As $\lambda \rightarrow 0$, $\{k_\lambda(z, \theta)\}$ converges to $k(z, \theta)$ uniformly on $\eta \leq |z| \leq 1$. Since both functions $k_\lambda(z, \theta)$ and $k(z, \theta)$ can be defined independently of the choice of ρ as far as $\rho < \min(|z_0|, |z|)$ (cf. no. 5) and η is arbitrary in $(\rho, 1)$, a net $\{k_\lambda(z, \theta)\}$ converges to the function $k(z, \theta)$ uniformly on each compact subset of $0 < |z| \leq 1$. Thus we have:

THEOREM 4. *Let P be any nonnegative rotation free density on $0 < |z| \leq 1$. Then, for each θ fixed in T , a net $\{k_\lambda(z, \theta)\}_{(\lambda > 0)}$ converges to $k(z, \theta)$ uniformly on each compact subset of $0 < |z| \leq 1$ as $\lambda \rightarrow 0$, where $k_\lambda(z, \theta)$ and $k(z, \theta)$ are minimal functions with respect to $\Delta u = Pu$ on $\lambda < |z| \leq 1$ and $0 < |z| \leq 1$, respectively.*

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