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BOREL BOXES

DOROTHY MAHARAM AND A. H. STONE

This note gives a simple necessary and sufficient condition for a box, in a product of uncountably many metrizable spaces, to be a Borel set.

Throughout, $X = \prod\{X_a: a \in A\}$ where A is an uncountable index set and each X_a is a nonempty topological space; X is given the usual product topology. The ath co-ordinate of a point x of X is denoted by $x_a (a \in A)$. A sub-product $\prod\{X_a: a \in B\}$, where $B \subset A$, is denoted by X_B. A set of the form $Y = \prod\{Y_a: a \in A\}$, where $Y_a \subset X_a$ for each $a \in A$, will be called a "box" in X. The object of this note is to point out that, when each X_a is metrizable, there is a surprisingly simple characterization of the boxes that are Borel subsets of X. We require the following three lemmas, of which at least the first two are well known. (See for instance [2].) The third is a special case of our main result.

LEMMA 1. *Suppose each X_a is a separable metrizable space. Then each Baire set E in X is of the form $E_0 \times X_{A-B}$, where B is a countable subset of A and E_0 is a Borel subset of the countable product X_B. Conversely, each set of this form is a Baire subset of X.*

LEMMA 2. *Again suppose each X_a is separable and metrizable. Then, given a Borel set D in X, there exist Baire sets V, W in X such that $V \subset D \subset W$ and $W - V$ is meagre (in X).*

LEMMA 3. *Suppose each X_a is compact and metrizable, and p_a is a nonisolated point of $X_a (a \in A)$. Define*

$$D = \{x \in X: \textit{for all } a \in A,\, x_a \neq p_a\} = \prod\{X_a - \{p_a\}: a \in A\}\,.$$

Then D is not Borel in X.

Proofs. When E is a cozero set, the assertion of Lemma 1 holds by [3], and the general case follows. The converse is elementary.

When D is open, the assertion of Lemma 2 holds; it suffices to take $V =$ union of a maximal collection (necessarily countable) of pairwise disjoint nonempty elementary open sets contained in D (as in [3]), and $W = \bar{V}$. Since the family of sets D for which Lemma 2 holds is a Borel field, it includes all Borel sets.

In Lemma 3, suppose D is Borel, and apply Lemma 2 to obtain Baire sets V, W with $V \subset D \subset W$ and $W - V$ meagre. By Lemma 1, $V = V_0 \times X_{A-B}$ where B is a countable subset of A. If $V \neq \emptyset$, pick $q \in V$ and define $x \in X$ by:

$$x_a = q_a \quad \text{if} \quad a \in B; \; x_a = p_a \quad \text{if} \quad a \in A - B .$$

Since A is uncountable, $x \notin D$. But $x \in V_0 \times X_{A-B} = V \subset D$, a contradiction. This shows $V = \emptyset$, and therefore W is meagre.

On the other hand, $D \subset W = W_0 \times X_{A-C}$ for some countable $C \subset A$. We have $\prod \{X_a - \{p_a\}: a \in C\} \subset W_0$, so that $X_C - W_0 \subset \bigcup \{\{p_a\} \times X_{C-\{a\}}: a \in C\}$, a countable union of closed subsets of X_C that are nowhere dense in X_C (because p_a is not isolated in X_a). Thus $X_C - W_0$ is meagre in X_C, whence $X - W$ is meagre in X. Since W also is meagre, X is meagre, contradicting Baire's theorem.

THEOREM. *Let $Y = \prod_a Y_a$ be a nonempty box in $X = \prod_a X_a$, and suppose each X_a is a first countable Hausdorff space. Then Y is a Borel set in X if, and only if,*

(i) *for all $a \in A$, Y_a is Borel in X_a,*

(ii) *for all but at most a countable set of a's, Y_a is closed in X_a.*

Proof. Put $B = \{a \in A: Y_a \text{ is not closed in } X_a\}$. Assuming (i) and (ii), we have that B is countable, and $Y = E \times \prod \{Y_a: a \in B\}$ where $E = \prod \{Y_a: a \in A - B\}$ is closed in X_{A-B} and each Y_a is Borel in X_a. As a countable product of Borel sets, Y is Borel.

Conversely, suppose Y is Borel in X. It is easy to see that (i) holds, and that the sub-product $Y_B = \prod \{Y_a: a \in B\}$ is Borel in X_B. Suppose (ii) is false, so that B is uncountable. For each $b \in B$, pick $p_b \in \bar{Y}_b - Y_b$ and pick a sequence of points $q_{b1}, q_{b2}, \cdots$, of Y_b, distinct from p_b and from each other, converging to p_b. Put $Z_b = \{q_{bn}: n = 1, 2, \cdots\}$, $T_b = Z_b \cup \{p_b\}$. Thus T_b is a compact metric space (homeomorphic to the subspace $\{0, 1, 1/2, \cdots, 1/n, \cdots\}$ of the real line), and p_b is a nonisolated point of it. Write $Z_B = \prod \{Z_b: b \in B\}$, $T_B = \prod \{T_b: b \in B\}$; then $Z_B = Y_B \cap T_B$, where T_B is closed and Y_B is Borel in X_B. Thus Z_B is Borel in X_B. But $Z_B = \{t \in T_B: \text{for all } b \in B, t_b \neq p_b\}$, so that Lemma 1 is contradicted. Thus (ii) holds, and the proof is complete.

COROLLARY 1. *If Y is a box in $X = \prod_a X_a$, where each X_a is a separable metric space, then Y is Borel in X if, and only if, Y is a Baire set relative to its closure $\bar{Y}$ in X.*

(From the theorem and Lemma 1.)

COROLLARY 2. *Suppose, for each* $a \in A$, X_a *and* X'_a *are non-empty separable metric spaces and* f_a *is a Borel isomorphism of* X_a *onto* X'_a. *Then* $\prod_a f_a$ *is a Borel isomorphism of* X *onto* $X'(=\prod_a X'_a)$ *if, and only if,* f_a *is a homeomorphism for all but (at most) countably many* a's.

The "only if" in Corollary 2 is a straightforward deduction from the theorem. The "if" is an easy consequence of the theorem and the following lemma.

LEMMA 4. *If (for* $i = 1, 2$) g_i *is a Borel measurable map from* Z_i *to* Z'_i, *and if at least one of the spaces* Z'_1, Z'_2 *has a countable base, then* $g_1 \times g_2$ *is Borel measurable.*

Proof. Let $V_1, V_2, \cdots$, be a countable base for (say) Z'_2. It is easy to see that an open set G in $Z'_1 \times Z'_2$ can always be written in the form $G = \bigcup\{(U_n \times V_n): n = 1, 2, \cdots\}$, where $U_1, U_2, \cdots$, are suitable open sets in Z'_1. From this it is clear that $(g_1 \times g_2)^{-1}(G)$ is a Borel set in $Z_1 \times Z_2$.

It would be interesting to know whether the separability hypothesis can be dropped in Corollary 2. It is not needed for the "only if" part of the corollary, and it is at least relatively consistent that separability is unnecessary for the "if" part. This is because the appropriate variant of Lemma 4 (with Z_2, Z'_2 both metrizable, though not necessarily separable) holds in a model of set theory given by Fleissner [1].

REFERENCES

1. W. G. Fleissner, *All Borel additive decompositions can be σ-discretely decomposable,* to appear.

2. Dorothy Maharam, *Realizing automorphisms of category algebras of product spaces,* General Topology and Appl., **10** (1979), 161-174.

3. K. A. Ross and A. H. Stone, *Products of separable spaces,* Amer. Math. Monthly, **71** (1964), 398-403.

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UNIVERSITY OF ROCHESTER
ROCHESTER, NY 14627