PACIFIC JOURNAL OF MATHEMATICS
Vol. 82, No. 1, 1979

EMBEDDING OPEN 3-MANIFOLDS IN COMPACT 3-MANIFOLDS

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We consider open 3-manifolds that are monotone unions of compact 3-manifolds each bounded by a torus. We give necessary and sufficient conditions for embedding such an open 3-manifold in a compact 3-manifold. We also show that if the open 3-manifold embeds in a compact 3-manifold, then it embeds in a compact 3-manifold as the complement of the intersection of a decreasing sequence of solid tori.

1. Introduction and preliminary definitions. Kister and McMillan [5] showed that a particular contractible open 3-manifold does not embed in S^3. Haken [4] then showed that this same open 3-manifold does not embed in any compact 3-manifold. Haken's major tool was his finiteness theorem stating that there is an upper bound on the number of incompressible nonparallel surfaces in a compact 3-manifold. See [4] and § VI of [12]. This theorem is also the major tool in this paper.

The spaces and maps that we will consider are intended to be in the piecewise linear category. An *n-manifold* is a separable metric space such that each point has a neighborhood homeomorphic to an n-cell. A *submanifold* of an n-manifold is a subspace that is also an n-manifold. A k-manifold F is *properly embedded* in an n-manifold N if and only if F is a closed subset of N and $F \cap \partial N = \partial F$. A *surface* is a connected 2-manifold; a *planar surface* is a surface that can be embedded in a disk. A *punctured disk* is a planar surface obtained by removing from the interior of a disk the interiors of a finite collection of disjoint subdisks. Similarly a *punctured 3-cell* is a compact 3-manifold obtained by removing from the interior of a 3-cell the interiors of a finite collection of disjoint 3-cells.

Let I denote the unit interval $[0, 1]$; let I_i denote the interval $[-i, i]$.

Two properly embedded surfaces F and F' are *parallel* in a 3-manifold N if and only if there is an embedding of $(F \times I, \partial F \times I)$ into $(N, \partial N)$ such that F is the image of $F \times \{0\}$ and F' is the image of $F \times \{1\}$. A collection of properly embedded surfaces is *parallel* if and only if any two disjoint surfaces in the collection are parallel. The corresponding definitions for parallel simple closed curves in a 2-manifold are similar.

A 2-manifold F properly embedded in a 3-manifold N is *compressible* if and only if either F is a 2-sphere that bounds a 3-cell in

N, or there is a *compressing disk* for F, that is, a disk D in N such that $D \cap \partial F = \partial D$ and ∂D is a nontrivial curve on F. Otherwise F is *incompressible*.

An open 3-manifold N is *simply connected at infinity* if and only if each compact subset of N is contained in a compact submanifold B of N such that $N - B$ is simply connected.

In this paper M will denote an open 3-manifold such that $M = \bigcup_{i=0}^{\infty} M_i$ where each M_i is a compact, connected 3-manifold with a single, torus boundary component, and $M_i \subsetneqq \mathrm{Int}\,(M_{i+1})$. For $i > 0$ let $K_i = M_i - \mathrm{Int}\,(M_{i-1})$.

2. Conditions for embedding M in a compact 3-manifold.

Theorems 2.3 and 2.4 provide a necessary and sufficient condition for M to embed in a compact 3-manifold. The following two preliminary lemmas will be used in the proof of Theorem 2.3.

LEMMA 2.1. *Suppose E and F are tori and E is embedded in $\mathrm{Int}\,(F \times I)$ so that it separates the components of $\partial(F \times I)$. Then E is parallel to $\partial(F \times I)$.*

Proof. Notice that E must be incompressible in $F \times I$. Otherwise we could use a compressing disk for E to obtain a 2-sphere in $F \times I$ that separates the components of $\partial(F \times I)$. This contradicts the fact that $F \times I$ is irreducible. The conclusion now follows by Haken's lemma on page 91 of [4].

LEMMA 2.2. *Suppose M is embedded in a compact 3-manifold N. Suppose there is an integer j such that for $i > j$, ∂M_{i-1} and ∂M_i are nonparallel and incompressible in K_i. Then for some $n \geqq j$ there is a compressing disk D for ∂M_n in $N - \mathrm{Int}\,(M_n)$. Furthermore, if any such disk D and an open regular neighborhood of D in $N - \mathrm{Int}\,(M_n)$ satisfy the following three properties for $n < i \leqq k$, then D and the open regular neighborhood can be modified in $N - \mathrm{Int}\,(M_k)$ so that these properties hold for $n < i \leqq k + 1$.*

(1) *The components of $D \cap K_i$ are punctured disks that are parallel and properly embedded in K_i and each has exactly one boundary component on ∂M_{i-1}.*

(2) *The components of $D \cap (N - \mathrm{Int}\,(M_i))$ are compressing disks for ∂M_i in $N - \mathrm{Int}\,(M_i)$.*

(3) *The open regular neighborhood of D in $N - \mathrm{Int}\,(M_n)$ is homeomorphic to $D \times \boldsymbol{R}$ under a homeomorphism that maps the intersection with ∂M_i onto $(D \cap \partial M_i) \times \boldsymbol{R}$ and the intersection with K_i onto $(D \cap K_i) \times \boldsymbol{R}$.*

Proof. Consider the infinite collection of surfaces ∂M_i for $i \geqq j$ in the compact 3-manifold $N - \mathrm{Int}\,(M_j)$. If any two of these surfaces were parallel, Lemma 2.1 would imply that all of the intermediate surfaces would be parallel. This is contrary to the hypothesis that no two adjacent surfaces are parallel. Haken's finiteness theorem implies that there is a compressing disk D for one of the surfaces ∂M_n. If D were contained in $M - \mathrm{Int}\,(M_j)$, it would intersect at most a finite number of the ∂M_i. We could put it in general position with respect to these surfaces so that the intersection would be a finite collection of simple closed curves. Then we could modify D to eliminate the curves of intersection that were trivial on the surfaces. Of the remaining curves, one that was innermost on D would bound a subdisk of D that is contained in some K_i. This is contrary to the hypothesis that ∂M_{i-1} and ∂M_i are incompressible in K_i. Thus D is properly embedded in $N - \mathrm{Int}\,(M_n)$.

Suppose there is an integer $k \geqq n$ such that D and an open regular neighborhood of D in $N - \mathrm{Int}\,(M_n)$ satisfy the three properties for $n < i \leqq k$. We can modify D in $N - \mathrm{Int}\,(M_k)$ to eliminate any simple closed curves of intersection between D and ∂M_{k+1} that are trivial on ∂M_{k+1}. The intersection $D \cap \partial M_{k+1}$ must still be nonempty since ∂M_k is incompressible in K_{k+1} and the curves of $D \cap \partial M_k$ are nontrivial on ∂M_k. Of the curves of $D \cap \partial M_{k+1}$, choose one that is innermost on D and consider the subdisk E of D that it bounds. Since ∂M_{k+1} is incompressible in K_{k+1}, E must be properly embedded in $N - \mathrm{Int}\,(M_{k+1})$. Also choose one of the components of $D \cap K_{k+1}$. By property (2), this component is a punctured disk disk with exactly one boundary component on ∂M_k. Because of the above modifications of D, the remaining boundary curves of this punctured form a collection of parallel simple closed curves that are nontrivial on ∂M_{k+1}. For each of the curves of $D \cap \partial M_k$, modify D in K_{k+1} by attaching copies of this punctured disk that are parallel in K_{k+1}. Then to each of the resulting curves of intersection with ∂M_{k+1}, attach parallel copies of E in $N - \mathrm{Int}\,(M_{k+1})$.

Finally choose an open regular neighborhood of $D \cap (N - \mathrm{Int}\,(M_k))$ in $N - \mathrm{Int}\,(M_k)$ such that its intersection with ∂M_{k+1} is an open regular neighborhood of $D \cap \partial M_{k+1}$ in ∂M_{k+1}, its intersection with K_{k+1} is an open regular neighborhood of $D \cap K_{k+1}$ in K_{k+1}, and its intersection with ∂M_k agrees with the intersection of ∂M_k with the original open regular neighborhood of D. Then the original open regular neighborbood in M_k and the new open regular neighborhood in $N - \mathrm{Int}\,(M_k)$ can be combined to form the open regular neighborhood satisfying property (3).

THEOREM 2.3. *Suppose M can be embedded in a compact* 3-

manifold. Then either M is simply connected at infinity and contains no infinite collection of disjoint fake 3-cells, or there is an integer n such that for each $i > n$ there is a punctured disk P_i properly embedded in K_i with the following properties.

(1) *$P_i \cap \partial M_{i-1}$ is a simple closed curve that is nontrivial on ∂M_{i-1}, and $P_i \cap \partial M_i$ is a nonempty collection of simple closed curves that are parallel and nontrivial on ∂M_i.*

(2) *When a solid torus is attached to K_i along ∂M_i so that the components of $P_i \cap \partial M_i$ are meridional curves of the solid torus, the resulting 3-manifold is a solid torus.*

(3) *For $i > n + 1$, the simple closed curve $P_i \cap \partial M_{i-1}$ is parallel to the curves of $P_{i-1} \cap \partial M_{i-1}$ on ∂M_{i-1}.*

Proof. Suppose M is embedded in a compact 3-manifold N. If an infinite number of the surfaces ∂M_i are parallel in $N - \mathrm{Int}\,(M_0)$, Lemma 2.1 implies that we can choose n so that all the surfaces ∂M_i for $i \geqq n$ are parallel. Then we can choose vertical annuli P_i in $K_i \cong \partial M_i \times I$ for $i > n$ that satisfy the three properties.

If this is not the case, then for any i there is a $k \geqq i$ such that ∂M_i is parallel to ∂M_k but not parallel to ∂M_{k+j} for $j \geqq 1$. Lemma 2.1 implies that $\partial M_i, \partial M_{i+1}, \cdots, \partial M_k$ are parallel. Thus, a punctured disk properly embedded in $M_k - \mathrm{Int}\,(M_i)$ that satisfies the three properties can be adjusted so that the components of its intersection with $M_{i+1} - \mathrm{Int}\,(M_i), \cdots, M_k - \mathrm{Int}\,(M_{k-1})$ satisfy the three properties. Therefore by choosing a subsequence of the surfaces ∂M_i and re-indexing, we can assume no two of these surfaces are parallel.

If there is an infinite number of values of i such that ∂M_i is compressible in $K_i \cup K_{i+1}$, we can find either an infinite sequence of even values of i or an infinite sequence of odd values of i with this property. In either case we can modify each of the corresponding submanifolds M_i by attaching a 2-handle in K_{i+1} to M_i or cutting M_i along a 1-handle in K_i. Notice that if M_i is modified, then M_{i-1} and M_{i+1} are unchanged. It follows that M can be written as the union of a nested sequence of compact, connected submanifolds each having a single, 2-sphere boundary component. Of the complementary domains of N determined by the union of these 2-spheres, Kneser's theorem [6] implies that all but a finite number must be homeomorphic to $S^2 \times I$. It thus follows that M is simply connected at infinity and contains no infinite collection of disjoint fake 3-cells.

Now we need to consider only the case that there is an integer j such that none of the tori ∂M_i for $i \geqq j$ is parallel in $N - \mathrm{Int}\,(M_0)$ to any other, and such that for each $i > j$, ∂M_{i-1} and ∂M_i are incompressible in K_i. By Lemma 2.2 there is an integer $n \geqq j$ and a compressing disk D for ∂M_n in $N - \mathrm{Int}\,(M_n)$. For any $k > n$ we can inductively modify D and the open regular neighborhood of D so that the three properties of Lemma 2.2 are satisfied for $n < i \leqq k$. Let P_i be a component of $D \cap K_i$ and let D_i be the component of $D \cap (N - \mathrm{Int}(M_i))$ that contains P_{i+1}. Notice that as long as k is chosen to be greater than i, the punctured disks P_i do not depend on the value of k. Notice also that the P_i satisfy properties (1) and (3) of the conclusion of this theorem.

Let us identify the open regular neighborhood of D in $N - \mathrm{Int}(M_n)$ with $D \times \boldsymbol{R}$ by a homeomorphism as indicated in property (3) of Lemma 2.2. Let

$$K_i^* = (M_i \cup D_{i-1} \times I_i) - \mathrm{Int}(M_{i-1} \cup D_{i-1} \times I_{i-1}) .$$

By adding the orientation preserving 1-handle $D_{i-1} \times I_{i-1}$ to K_i^*, we obtain $(M_i \cup D_{i-1} \times I_i) - \mathrm{Int}(M_{i-1})$. By attaching 3-cells to the 2-sphere boundary components of $(M_i \cup D_{i-1} \times I_i) - \mathrm{Int}(M_{i-1})$, we obtain the 3-manifold described in property (2) of the conclusion of this theorem. Hence this 3-manifold is a solid torus if and only if K_i^* is a punctured 3-cell.

In any case K_i^* is bounded by 2-spheres. So any K_i^* that is not a punctured 3-cell contributes at least one nontrivial factor to a connected sum decomposition of the compact 3-manifold N. By Kneser's theorem [6] there can only be a finite number of such values of i. Therefore we can choose an integer n so that property (2) is also satisfied for all $i > n$.

THEOREM 2.4. *Suppose there is an integer n such that for $i > n$ there is a punctured disk P_i properly embedded in K_i satisfying the three properties of the conclusion of Theorem* 2.3. *Then M can be embedded in a compact* 3*-manifold N so that $N - \mathrm{Int}(M_i)$ is a solid torus for all $i \geqq n$.*

REMARK. If M is simply connected at infinity and contains no infinite collection of disjoint fake 3-cells, then, by the techniques of Edwards [3], the one point compactification of M is a compact 3-manifold. Together with this fact, Theorem 2.4 thus provides the converse to Theorem 2.3.

Proof of Theorem 2.4. We want to use the punctured disks P_i to construct a planar surface P that has a single boundary component and is properly embedded in $M - \mathrm{Int}(M_n)$. The planar surface P can be constructed inductively so that it satisfies the following defining properties. First $P \cap K_{n+1} = P_{n+1}$. Second, for $i > n + 1$, $P \cap K_i$ consists of copies of P_i that are properly embedded and parallel in

K_i and are such that the boundary curves on ∂M_{i-1} coincide with the boundary curves of $P \cap M_{i-1}$ on ∂M_{i-1}.

Next we want to embed M in an open 3-manifold M^* so that P is contained in a disk D in M^*. We can choose a regular neighborhood of P in $M - \mathrm{Int}\,(M_n)$ that intersects each ∂M_i in a regular neighborhood of $P \cap \partial M_i$ in ∂M_i. The topological interior of this regular neighborhood in $M - \mathrm{Int}\,(M_n)$ is an open regular neighborhood of P in $M - \mathrm{Int}\,(M_n)$. We can identify this open regular neighborhood with $P \times \boldsymbol{R}$ by a homeomorphism that identifies the intersection of the open neighborhood and ∂M_i with $(P \cap \partial M_i) \times \boldsymbol{R}$.

We can embed P as a subset of a disk D so that ∂D is the image of ∂P. Then for each $i \geqq n$, $P \cap \partial M_i$ determines a collection of simple closed curves on D each bounding a subdisk of D that contains a component of $P \cap (M - \mathrm{Int}\,(M_i))$. Let D_i be the union of these subdisks. Let M_i^* be the compact 3-manifold obtained by attaching to a copy of M_i a collection of 2-handles $D_i \times I_i$ along ∂M_i. Choose the attaching maps so that the product structure of $\partial D_i \times I_i$ agrees with the product structure of $(P \cap \partial M_i) \times I_i$. Notice that for $i > n$, there is an embedding of M_{i-1}^* into M_i^* determined by the embedding of M_{i-1} in M_i and the standard embedding of $D_i \times I_{i-1}$ into $D_i \times I_i$. Let $K_i^* = M_i^* - \mathrm{Int}\,(M_{i-1}^*)$. Now let M^* be the open 3-manifold obtained from M_n^* by inductively attaching the K_i^* so that

$$M^* = M_n^* \cup \Big(\bigcup_{i=n+1}^{\infty} K_i^* \Big) = \bigcup_{i=n}^{\infty} M_i^* \,.$$

Since P can be embedded in D and each M_i can be embedded in the corresponding M_i^*, this will give the desired embeddings of M and P into M^*.

By removing the interiors of a collection of disjoint 3-cells from the solid torus described in property (2) of Theorem 2.3, we can obtain $M_i^* - \mathrm{Int}\,(M_{i-1})$. Furthermore, by cutting $M_i^* - \mathrm{Int}\,(M_{i-1})$ along the 1-handles $D_{i-1} \times I_{i-1}$ we can obtain

$$K_i^* = M_i^* - \mathrm{Int}\,(M_{i-1}^*) = M_i^* - \mathrm{Int}\,(M_{i-1} \cup D_{i-1} \times I_{i-1}) \,.$$

Thus, K_i^* is the union of a collection of punctured 3-cells. Notice that each component of K_i^* intersects M_{i-1}^* in a single 2-sphere.

Let N be the compact 3-manifold obtained by attaching a solid torus to a copy of M_n along ∂M_n so that ∂D is a meridional curve of the solid torus. Then we can extend the embedding of M_n into N to an embedding of $M_n^* = M_n \cup D \times I_n$ into N. Finally, we can inductively embed the punctured 3-cells of $K_{n+1}^*, K_{n+2}^*, \cdots$ in the remaining 3-cells of the solid torus. This gives the desired embedding of M^*, and hence also of M, into N. It follows from the construction that $N - \operatorname{Int}(M_i)$ is a solid torus for all $i \geqq n$.

COROLLARY 2.5. *Suppose M can be embedded in a compact 3-manifold. Then there is an integer n such that $M - \operatorname{Int}(M_n)$ can be embedded in a solid torus as the complement of the intersection of a decreasing sequence of solid tori. In particular, M is orientable at infinity.*

Proof. By Theorem 2.3, M is either simply connected at infinity and contains no infinite collection of disjoint fake 3-cells, or M contains the punctured disks satisfying the three properties. In the first case, the conclusion follows by the remark made after the statement of Theorem 2.4. Otherwise we can apply Theorem 2.4.

REMARK. The referee has pointed out that this corollary may be proved by using the main result of a paper [7] by Knoblauch. If M is embedded in a compact, orientable 3-manifold, Knoblauch's result implies that $M - \operatorname{Int}(M_j)$ can be embedded in S^3 for some j. As in the proof of Theorem 2.3, we can assume that for $i \geqq j$ none of the tori ∂M_i is parallel in $M - \operatorname{Int}(M_j)$ to any other, and that for each $i > j$, ∂M_{i-1} and ∂M_i are incompressible in K_i; otherwise the result of the corollary follows easily. Now with $M - \operatorname{Int}(M_j)$ embedded in S^3 we can use Haken's theorem to find an integer n so that if $i > n$, ∂M_i bounds a solid torus containing $M - \operatorname{Int}(M_i)$.

On the other hand, suppose that M is embedded in a compact, nonorientable 3-manifold N. For some n we must have that $M - \operatorname{Int}(M_n)$ is orientable. Otherwise for some integer j we could find an orientation reversing loop α in $\operatorname{Int}(M_j)$ and an orientation reversing loop β in $M - M_j$ so that α and β represent the same element of the finite group $H_1(N; \boldsymbol{Z}_2)$. A Mayer-Vietoris sequence argument would give a loop γ in ∂M_j that represents this same element of $H_1(N; \boldsymbol{Z}_2)$. But ∂M_j is orientable and 2-sided, so γ would be orientation preserving. Now since $M - \operatorname{Int}(M_n)$ is orientable, it embeds in the orientable double cover of N, and we can apply the argument of the previous paragraph.

3. The existence of the punctured disks.

In this section we will consider conditions on $K_i = M_i - \operatorname{Int}(M_{i-1})$ that are necessary for the existence of a properly embedded punctured disk satisfying properties (1) and (2) of Theorem 2.3.

The following notation and conventions will be used throughout this section. All homology groups will be with coefficients in $\boldsymbol{Z}$. If c is a collection of oriented simple closed curves in an n-manifold, let $[c]$ denote the sum of the elements of the first homology group of the n-manifold represented by the curves of c. Let us assume there is a homeomorphism $h: K_i \rightarrow L$ where L is the complement of the interiors of two disjoint solid tori T and T' in a homology 3-sphere S. In Corollary 2.5 we have shown that if M embeds in a compact 3-manifold, then this assumption holds for all but a finite number of values of i. Let us also assume that $h(\partial M_i) = \partial T$ and $h(\partial M_{i-1}) = \partial T'$. If c is a union of simple closed curves in S and d is a simple closed curve disjoint from c, then the *linking number* of c with d is the absolute value of the multiple of a generator determined by $[c]$ in $H_1(S - d) \cong \boldsymbol{Z}$. Let q denote the linking number of the centerline of T with the centerline of T'. It can be shown that q is independent of the choice of h, L, and S. Choose transverse oriented simple closed curves l and m on ∂T so that l bounds a surface in $S - \operatorname{Int}(T)$ and m bounds a disk in T. A simple closed curve on ∂T that represents $\pm\alpha[l] \pm \beta[m] \neq 0$ in $H_1(\partial T)$ for $\alpha \geqq 0$ and $\beta \geqq 0$ will be called an *(α, β)-curve* on ∂T. It can be shown that α and β are relatively prime. On $\partial T'$ choose l' and m' similarly. The corresponding definition of an (α, β)-curve on $\partial T'$ is also similar.

THEOREM 3.1. *Let P be a punctured disk properly embedded in L with the following properties.*

(1) *$P \cap \partial T'$ is an (α, β)-curve on $\partial T'$, and $P \cap \partial T$ is a nonempty collection of simple closed curves that are parallel and nontrivial on ∂T.*

(2) *When a solid torus T^* is attached to L along ∂T so that the components of $P \cap \partial T$ are meridional curves of T^*, $L \cup T^*$ is a solid torus.*

Give the components of ∂P the orientations induced from an orientation chosen for P. If $q = 0$, then

(a) *$\alpha = 1$ and $\beta = 0$,*

(b) *$[P \cap \partial T] = 0$ in $H_1(\partial T)$, and*

(c) *each component of $P \cap \partial T$ is a $(\gamma, 1)$-curve on ∂T for some integer γ.*

If $q > 0$, then

(d) *β is divisible by q^2,*

(e) *$[P \cap \partial T] = \pm(\beta/q)[l] \pm \alpha q[m]$ in $H_1(\partial T)$,*

(f) *each component of $P \cap \partial T$ is a $(\beta/q^2, \alpha)$-curve on ∂T, and*

(g) *if $q > 1$, then $\alpha \neq 0$.*

REMARK. Properties (1) and (2) of this theorem are restatements of the corresponding properties of Theorem 2.3.

Proof of Theorem 3.1. Let us first consider the case that $q = 0$. Then $P \cap \partial T$ does not link the centerline of T', so neither does $P \cap \partial T'$. This implies that $\beta = 0$, which in turn implies that $\alpha = 1$. Since we can extend P to a meridional disk of the solid torus $L \cup T^*$, it follows that $T' \cup L \cup T^*$ is homeomorphic to the 3-sphere S^3. In particular $H_1(T' \cup L \cup T^*) \cong (0)$. Since $T' \cup L$ is the complement of $\mathrm{Int}(T)$ in the homology 3-sphere S, $H_1(T' \cup L) \cong \boldsymbol{Z}$ and is generated by $[m]$. Thus a meridional disk of T^* must have been attached to a $(\gamma, 1)$-curve on ∂T for some integer γ. Since any such curve links the centerline of T, but $P \cap \partial T'$ does not, we must have $[P \cap \partial T] = 0$ in $H_1(\partial T)$.

Let us now consider the case that $q > 0$. As in the first case, $T' \cup L \cup T^*$ is homeomorphic to $S^2 \times S^1$ if $\alpha = 0$, S^3 if $\alpha = 1$, or to the lens space $L_{\alpha,\beta}$ if $\alpha > 1$. In particular $H_1(T' \cup L \cup T^*) \cong \boldsymbol{Z}/\alpha\boldsymbol{Z}$.

Also recall that $H_1(T' \cup L) \cong \boldsymbol{Z}$ and is generated by $[m]$. Thus we can also compute $H_1(T' \cup L \cup T^*)$ by considering the way a meridional disk of T^* is attached to $T' \cup L$. Now the oriented simple closed curve $P \cap \partial T'$ has linking number αq with the centerline of T and β with the centerline of T'. Since the oriented curves $P \cap \partial T$ are homologous to $P \cap \partial T'$, $P \cap \partial T$ also has linking number αq with the centerline of T and β with the centerline of T'. Since each factor of $[l]$ in an element of $H_1(\partial T)$ corresponds to linking the centerline of T' a total of q times, it follows that $[P \cap \partial T] = \pm(\beta/q)[l] \pm \alpha q[m]$ in $H_1(\partial T)$. Thus each component of $P \cap \partial T$ is a $(\beta/dq, \alpha q/d)$-curve where d is the greatest common divisor β/q and αq. Therefore $H_1(T' \cup L \cup T^*) \cong \boldsymbol{Z}/(\alpha q/d)\boldsymbol{Z}$, and so $\alpha q/d = \alpha$. Now if $\alpha \neq 0$, then $q = d$. If $\alpha = 0$, then $\beta = 1$, so we must have $q = 1$. In either case each component of $P \cap \partial T$ is a $(\beta/q^2, \alpha)$-curve.

4. Applications and examples.

In the previous section we investigated the existence of a punctured disk properly embedded in K_i satisfying properties (1) and (2) of Theorem 2.3. In this section we will give conditions that also involve the matching of the boundary curves of the punctured disk at one stage with the boundary curves of the punctured disk at the previous stage, that is condition (3) of Theorem 2.3. We will also consider several examples as typical applications of these results.

Throughout this section we will assume there in an integer n such that for each $i > n$ there is a homeomorphism $h_i: K_i \to L_i$ where L_i is the complement of the interiors of two disjoint solid tori T_i and T'_i in the 3-sphere S^3. As in §3 we assume $h_i(\partial M_i) = \partial T_i$ and $h_i(\partial M_{i-1}) = \partial T'_i$. We also define the linking number q_i and choose the longitudinal and meridional curves l_i, m_i, l'_i, m'_i similarly. For each $i > n$ let

$$g_i = (h_i | \partial M_i) \circ (h_{i+1}^{-1} | \partial T'_{i+1}) \colon \partial T'_{i+1} \longrightarrow \partial T_i \,.$$

Then g_i describes the matching between L_{i+1} and L_i.

A solid torus T is *unknotted* in a closed 3-manifold if and only if there is a compressing disk D for ∂T such that ∂D is transverse to a meridional curve of T. The *wrapping number* of a solid torus T' in the interior of a solid torus T is the minimum number of components in the intersection between T' and a meridional disk of T.

THEOREM 4.1. *Suppose each* $q_i = 0$. *Let* Q_i *be the 3-manifold obtained when a solid torus is attached to* $T'_i \cup L_i$ *so that* $g_i(l'_{i+1})$ *is a meridional curve of the solid torus. Then* M *embeds in a compact 3-manifold if and only if either* M *is simply connected at infinity and contains no infinite collection of disjoint fake 3-cells, or there is an integer* n *such that the following three properties hold for* $i > n$.

(1) T'_i *is unknotted in* Q_i.

(2) Q_i *is homeomorphic to the 3-sphere.*

(3) $g_i(l'_{i+1})$ *is a* $(\gamma_i, 1)$*-curve on* ∂T_i *for some integer* γ_i.

Proof. Suppose M embeds in a compact 3-manifold and M is not simply connected at infinity. Let P_i denote the image in L_i of the punctured disks given by Theorem 2.3. Thus we can apply Theorem 3.1. Part (a) implies that $P_{i+1} \cap \partial T'_{i+1}$ is homologous to l'_{i+1} on $\partial T'_{i+1}$. Part (c) implies that $P_i \cap \partial T_i$ is a $(\gamma_i, 1)$-curve on ∂T_i. Therefore property (3) of Theorem 2.3 gives property (3) of this theorem. Property (2) of Theorem 2.3 implies that $Q_i - \operatorname{Int}(T'_i)$ is a solid torus. Now P_i can be extended to a meridional disk D_i of this solid torus. Part (a) of Theorem 3.1 again implies that ∂D_i is a (1, 0)-curve of $\partial T'_i$. Therefore Q_i is homeomorphic to the 3-sphere and T'_i is unknotted in Q_i.

Conversely, suppose the three properties of this theorem hold. Then a (1, 0)-curve on $\partial T'_i$ bounds a disk D_i in $Q_i - \operatorname{Int}(T'_i)$. The disk D_i can be chosen so that each component of $D_i \cap \partial T_i$ is a meridional curve on the boundary of the solid torus $Q_i - \operatorname{Int}(T'_i \cup L_i)$. Now if $D_i \cap \partial T_i$ is empty for an infinite number of values of i, then M is the union of some M_k and a sequence of punctured 3-cells, each of which intersects the union of M_k and the previous punctured 3-cells in a single 2-sphere boundary component. Thus M is simply connected at infinity, contains no infinite collection of disjoint fake 3-cells, and can be embedded in a compact 3-manifold. Otherwise we can apply Theorem 2.4 to complete the proof of Theorem 4.1.

The examples in this section will be constructed as follows. Let

L_0 be the complement of the interior of an unknotted solid torus T_0 in S^3; for $i \geqq 1$, let L_i be the complement of the interiors of two disjoint solid tori T_i and T'_i in S^3. Let $g_i: \partial T'_{i+1} \to \partial T_i$ be homeomorphisms. Then we can define the open 3-manifold M as the identification space of the disjoint union of the L_i where x is identified with $g_i(x)$ for $x \in \partial T'_{i+1}$. Let K_i be the image under the identification map of L_i in M.

EXAMPLE 4.2. Let L_i be as indicated in Figure 1. We can construct the contractible open 3-manifold of Whitehead [10] by letting g_i be a homeomorphism that takes l'_{i+1} to m_i and m'_{i+1} to l_i. Theorem 4.1 can be used to show that this open 3-manifold embeds in S^3.

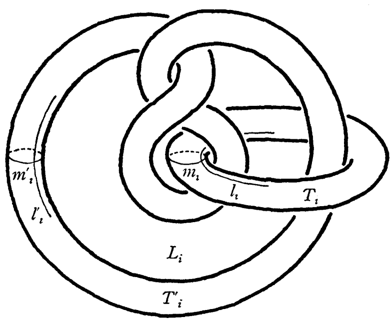

FIGURE 1

EXAMPLE 4.3. Again let L_i be as indicated in Figure 1. We want to know what homeomorphisms g_i will give an open 3-manifold that can be embedded in a compact 3-manifold. By Theorem 4.1 a necessary condition is that there is an integer n such that for $i > n$, $g_i(l'_{i+1})$ is a $(\gamma_i, 1)$-curve on ∂T_i for some integer γ_i. Since T_i is unknotted in S^3 there is an embedding $h: T'_i \cup L_i \to S^3$ of the solid torus $T'_i \cup L_i$ onto $T'_i \cup L_i$ in S^3 so that $hg_i(l'_{i+1})$ is a $(0, 1)$-curve of ∂T_i Thus, the compact 3-manifold Q_i of Theorem 4.1 is homeomorphic to S^3. But if $\gamma_i \neq 0$, then $h(T'_i)$ is a regular neighborhood of a nontrivial twist knot. See [11] or [1] for the relevant facts about twist knots. Thus condition (1) of Theorem 4.1 is not met unless $\gamma_i = 0$.

REMARK. Suppose for each $i \geqq 0$ that $g_i(l'_{i+1})$ is a $(\gamma_i, 1)$-curve on ∂T_i for any integer γ_i. If we choose $g_i(m'_{i+1}) = l_i$ for each $i \geqq 0$, then M will be monotone union of solid tori. Since each solid torus will be contractible in the succeeding one, M will also be contractible.

EXAMPLE 4.4. Let L_i be as indicated in Figure 2. Notice that there is a punctured disk in L_i bounded by a (1, 0)-curve on $\partial T'_i$ and two (1, 0)-curves on ∂T_i. This punctured disk can be visualized in Figure 2 as the union of an annulus with boundary components $a_1 \cup b_1$ and d_1, a disk with boundary $b_1 \cup c_1 \cup b_2 \cup c_2$, and an annulus with boundary components $a_2 \cup b_2$ and d_2.

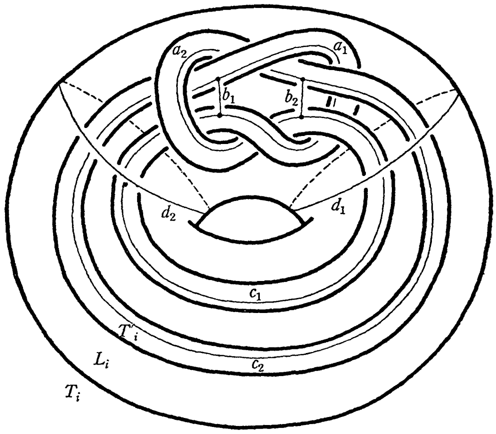

FIGURE 2

Nevertheless, if $g_i(l'_{i+1})$ is a (1, 0)-curve on ∂T_i, Q_i will be homeomorphic to $S^2 \times S^1$. In fact, in order for Q_i to be a 3-sphere, $g_i(l'_{i+1})$ must be a $(\gamma_i, 1)$-curve on ∂T_i. As in the previous example, we can reembed $T'_i \cup L_i$ so that the image of $g_i(l'_{i+1})$ is a meridional curve of the complementary solid torus. Figure 3 is a typical picture of the knotted torus T'_i in the 3-sphere Q_i.

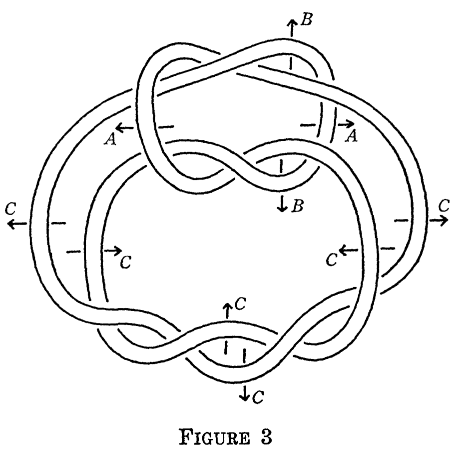

FIGURE 3

An epimorphism from the fundamental group of the complement of T'_i onto the trefoil knot group can be obtained as follows. Compute the Wirtinger presentation [2] for the group of the complement of T'_i. Then identify all generators that have the same label in Figure 3. The resulting group will be a presentation of the trefoil knot group:

$$\langle A, B, C \mid ACA^{-1}B^{-1}, BAB^{-1}C^{-1}, CBC^{-1}A^{-1}\rangle .$$

Therefore T'_i is a nontrivial knot, and so it does not satisfy condition (1) of Theorem 4.1.

Notice that ∂L_i is incompressible in L_i. Indeed $\partial T'_i$ is incompressible in $L_i \cup T_i$ since T'_i is knotted in S^3. Also a meridional disk of $T'_i \cup L_i$ that is disjoint from T'_i could be used together with the punctured disk in L_i to contradict the fact that T'_i is knotted. Thus M is not simply connected at infinity.

It follows that there are no sewing maps g_i for which the resulting open 3-manifold M will embed in a compact 3-manifold. In fact it follows from Theorem 2.3 that if there is an infinite number of values of i for which K_i is homeomorphic to such an L_i with ∂M_i corresponding to ∂T_i, then the open 3-manifold cannot be embedded in a compact 3-manifold.

THEOREM 4.5. *Suppose each $q_i = 1$. Suppose T_i is unknotted in S^3, T'_i is knotted in S^3, and T'_i has wrapping number one in the solid torus $S^3 - \mathrm{Int}(T_i)$. Then M can be embedded in a compact 3-manifold if and only if there is an integer n so that $g_i(m'_{i+1})$ is a $(1, 0)$-curve on ∂T_i for all $i > n$.*

Proof. Suppose M can be embedded in a compact 3-manifold. Using the fact that $\partial T'_i$ and ∂T_i are incompressible in L_i, it follows that M is not simply connected at infinity. Let P_i denote the images in L_i of the punctured disks given by Theorem 2.3.

Notice that $P_i \cap \partial T'_i$ is a $(0, 1)$-curve on $\partial T'_i$. For suppose it is an (α, β)-curve with $\alpha \neq 0$. Then there is a 3-cell in $S^3 - T_i$ that intersects L_i in a cube with a knotted hole C. We can modify P_i so that $P_i \cap C \cap \partial T'_i$ consists of exactly α arcs. We can further modify P_i so that each component of $P_i \cap \partial C$ is nontrivial on ∂C. Then each component of $P_i \cap C$ is a compressing disk for ∂C in C. This contradicts the hypothesis that T'_i is knotted. For more details see §5 of [4].

By part (f) of Theorem 3.1 each component of $P_i \cap \partial T_i$ is a $(1, 0)$-curve on ∂T_i. The result of the previous paragraph applied to P_{i+1} in L_{i+1} and property (3) of Theorem 2.3 imply that $g_i(m'_{i+1})$ is a $(1, 0)$-curve on ∂T_i.

Suppose conversely that there is an integer n so that $g_i(m'_{i+1})$ is a $(1, 0)$-curve on ∂T_i for all $i > n$. Since T'_i has wrapping number one in $S^3 - \operatorname{Int}(T_i)$, there is an annulus P_i in L_i bounded by the $(0, 1)$-curve m'_i on $\partial T'_i$ and the $(1, 0)$-curve l_i on ∂T_i.

Let Q_i be the 3-manifold obtained by attaching a solid torus to $T'_i \cup L_i$ along ∂T_i so that l_i is a meridional curve of the solid torus. Notice that Q_i is homeomorphic to $S^2 \times S^1$. To apply Theorem 2.4 it suffices now to show that $Q_i - \operatorname{Int}(T'_i)$ is a solid torus. We can extend P_i to a disk D_i in $Q_i - \operatorname{Int}(T'_i)$ by attaching a meridional disk of the solid torus $Q_i - \operatorname{Int}(T'_i \cup L_i)$ to $P_i \cap \partial T_i$. Now D_i can be extended to a nonseparating 2-sphere in Q_i by attaching a meridional disk of T'_i to ∂D_i. Since Q_i is a prime 3-manifold, cutting $Q_i - \operatorname{Int}(T'_i)$ along D_i yields a 3-cell. Thus $Q_i - \operatorname{Int}(T'_i)$ is a solid torus.

EXAMPLE 4.6. Let L_i be as indicated in Figure 4. This illustrates the situation described in Theorem 4.5.

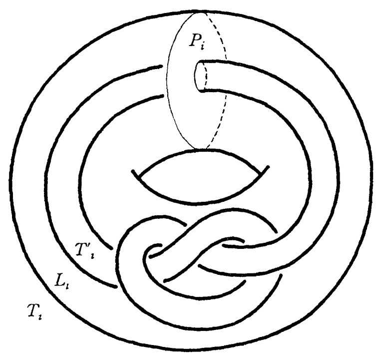

FIGURE 4

If $g_i(m'_{i+1})$ is a $(1, 0)$-curve on ∂T_i for all $i \geqq 0$ then the resulting open 3-manifold M is one of the open 3-manifolds given by Tucker in Example 3 of [9]. Even though M does not embed in any compact, irreducible 3-manifold, it does embed in $S^2 \times S^1$. To see this notice that there is a homeomorphism of L_i onto itself that interchanges the two boundary components. Indeed, cut L_i along the annulus P_i and let L_i^* denote the resulting cube with a knotted hole. There is an isotopy of L_i^* that interchanges $L_i^* \cap \partial T_i$ and $L_i^* \cap \partial T'_i$. This can be used to define the desired homeomorphism of L_i. It now follows that $M - \operatorname{Int}(M_0)$ can be embedded in a solid torus as the complement of the intersection of a decreasing sequence of solid tori each embedded in the previous one as T'_i is embedded in $S^3 - \operatorname{Int}(T_i)$ in Figure 4.

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Received November 3, 1977. The second author was partially supported by Western Michigan University Faculty Research Fund.

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