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GENERAL PEXIDER EQUATIONS (PART I): EXISTENCE OF INJECTIVE SOLUTIONS

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Given open connected Ω, $\tilde{\Omega}\subseteq \boldsymbol{R}^n$ and given $T\colon \Omega\to \boldsymbol{R}$ continuous, $F\colon \tilde{\Omega}\to \boldsymbol{R}$ strictly monotonic, in each variable separately. The equation is $h\circ T=F\circ\pi$ for the unknowns $h\colon T(\Omega)\to \boldsymbol{R}$, $\pi\colon \Omega\to\tilde{\Omega}$ with $\pi=(f_1,\cdots,f_n)$ a product mapping - e.g., $h\{T(x,y)\}=F\{f(x),g(y)\}$. If T is one-one in each variable, then any continuous solution π must be injective or constant on Ω; conversely, if an injective solution π exists then T must be one-one in each variable separately.

1. Introduction. Given a subset $\Omega\subseteqq \boldsymbol{R}^n$ for $n\geqq 2$, let Ω_i denote its projection on the ith coordinate axis. By a *product mapping* $\pi\colon \Omega\to\tilde{\Omega}\subset \boldsymbol{R}^n$ is understood the restriction to Ω of a map $(f_1,\cdots,f_n)\colon X_1^n\Omega_i\to \boldsymbol{R}^n$ defined by n functions $f_i\colon \Omega_i\to\tilde{\Omega}_i\subseteqq \boldsymbol{R}$. For given $T\colon \Omega\to \boldsymbol{R}$ and $F\colon\tilde{\Omega}\to \boldsymbol{R}$, equations of the form

$$h\{T(x_1,\cdots,x_n)\}=F\{f_1(x_1),\cdots,f_n(x_n)\} \tag{1}$$

for the unknowns $h\colon T(\Omega)\to \boldsymbol{R}$ and $\pi\colon\Omega\to\tilde{\Omega}$ are generalizations of Pexider equations[1]. For the most part the literature concerns the case in which T and F are specified, usually the sum and/or product of the arguments. In [3] C. T. Ng recently gave a uniqueness theorem for continuous solutions π, assuming T continuous but with $F(u_1,\cdots,u_n)=u_1+\cdots+u_n$; a generalization to certain topological spaces appears in Ng [4] and [2]. A simple case of (1) was used by J. Lester and the author [5] to characterize Lorentz transformations in $\boldsymbol{R}^n$.

2. Formulation of results. Given Ω, $\tilde{\Omega}\subseteqq \boldsymbol{R}^n$ for $n\geqq 2$ and given $T\colon\Omega\to \boldsymbol{R}$, $F\colon\tilde{\Omega}\to \boldsymbol{R}$. Henceforth assume:

(A-1) *T continuous in each variable separately,*
(A-2) *F one-to-one in each variable separately,*
(A-3) *Ω open and connected.*

THEOREM 1. *With* (A-1, 2, 3) *assume $T\circ h=F\circ\pi$ satisfied on Ω, where $h\colon T(\Omega)\to \boldsymbol{R}$ and where $\pi\colon\Omega\to\tilde{\Omega}$ is an injective product mapping. Then T must be strictly monotonic in each variable separately on Ω.*

The existence of an injective solution π then places a severe

[1] For literature see [1]; J. V. Pexider studied $h(x\dot{+}y)=f(x)\dot{+}g(y)$.

condition on T; the following theorems indicate that if continuous solutions π are to exist, injectivity or at least some local one-to-one property of π is to be expected. A function will be called *locally nonconstant* if it is not constant on any open set.

THEOREM 2. *If in addition to* (A-1, 2, 3), *T is locally nonconstant in each variable separately then for any continuous product map $\pi: \Omega \to \tilde{\Omega}$ and corresponding $h: T(\Omega) \to \boldsymbol{R}$ satisfying $h \circ T = F \circ \pi$ on Ω, either π is also locally nonconstant in each variable separately or π is constant on Ω.*

The following theorem is a partial converse to Theorem 1.

THEOREM 3. *If in addition to* (A-1, 2, 3), *both T and F are strictly monotonic in each variable separately, then for any continuous product map, $\pi: \Omega \to \tilde{\Omega}$ and corresponding $h: T(\Omega) \to \boldsymbol{R}$ satisfying $h \circ T = F \circ \pi$ on Ω, either π is injective or π is constant on Ω.*

3. Proof of Theorem 1. By symmetry it suffices to consider T in its first variable for all choices of the remaining variables, denoted by $X = (x_2, \cdots, x_n)$. If (a, X) and (b, X) are elements of Ω with $a \neq b$, then by (A-2), $T(a, X) = T(b, X)$ implies $\pi(a, X) = \pi(b, X)$ for product functions π; π would not be injective. Hence each X determines a line λ parallel to the x_1 axis and $T(\cdot; X)$ is one-to-one on $\lambda \wedge \Omega$. Hence T is one-to-one and continuous in each variable separately. Since Ω was not assumed convex, the domain of $T(\cdot; X)$ for given X need not be connected (in R) and it remains to prove that T is in fact strictly monotonic in each variable for all choices of the remaining variables (either increasing for all, or decreasing for all). For each point $(x_1, \cdots, x_n) \in \Omega$, some open ball around this point is contained in Ω and define $V: \Omega \to \boldsymbol{R}^n$ by $V(x_1, \cdots, x_n) = (\pm 1, \cdots, \pm 1)$ according as T is strictly increasing $(+1)$ or decreasing (-1) in each variable within that open ball. Since V is constant on some neighborhood of each point in Ω, V is continuous on Ω and all of the $2n$ sets $V^{-1}(\pm 1, \cdots, \pm 1)$ are closed and disjoint. Since Ω is connected, all but one of these sets must be empty.

4. Proof of Theorem 2. Consider the two dimensional case $h\{T(x, y)\} = F\{f(x), g(y)\}$, valid on some open connected $\Omega \subset \boldsymbol{R}^2$; Ω_x, Ω_y denote the projections of Ω on the x and y axes, f and g are continuous on Ω_x and Ω_y respectively. Let $N_\varepsilon(x) :=]x - \varepsilon, x + \varepsilon[$, the open interval.

LEMMA 2. *For (x_0, y_0) in Ω, if f is constant on some $N_\varepsilon(x_0)$*

then g is also constant on some $N_\delta(y_0)$ and conversely.

Proof. Choose $\varepsilon > 0$ sufficiently small so that $N_\varepsilon(x_0) \times N_\varepsilon(y_0) \subset \Omega$ with $f(x) = k$ constant on $N_\varepsilon(x_0)$. Since $T(\cdot, y_0)$ is locally nonconstant and continuous, $T(N_\varepsilon(x_0), y_0)$ contains an open interval I; since $h\{T(x, y_0)\} = F\{k, g(y_0)\}$ is also constant, h must be constant on I. With x_1 chosen in $N_\varepsilon(x_0)$ such that $T(x_1, y_0)$ is in I, so also is $T(x_1, y)$ in I for all y in some $N_\delta(y_0)$; hence $h\{T(x_1, y)\} = F\{k, g(y)\}$ is constant, that is, $g(y)$ is constant by (A-2) for y in $N_\delta(y_0)$. Similarly for the converse.

LEMMA 3. *If f is constant on some closed interval $[a, b] \subset \Omega_x$, $a < b$, then for some $\delta > 0$, f is also constant on $]a - \delta, b + \delta[\subset \Omega_x$. Similarly for g relative to intervals in Ω_y.*

Proof. With $b \in \Omega_x$ so also $(b, y_0) \in \Omega$ for some y_0 and since Ω is open, $[b - \varepsilon, b + \varepsilon] \times [y_0 - \varepsilon, y_0 + \varepsilon] \subset \Omega$ for some $\varepsilon > 0$. Choose $x_0 \in]b - \varepsilon, b[= N(x_0)$, a neighbourhood of x_0 on which f is constant; by Lemma 2, g is constant on some $N(y_0)$. But again $(b, y_0) \in \Omega$ with g constant on $N(y_0)$ implies f constant on some $N(b)$, thus extending $[a, b]$ to $[a, b + \delta[$. Similarly for the end point a and for g relative to Ω_y.

If f is constant on some open interval, so also on the closure in Ω_x of the maximal extension of the interval on which f is constant; this maximal extension must also be open in Ω_x by Lemma 3. Since Ω_x is connected, f must be constant on Ω_x itself. In view of Lemma 2, g will be constant on Ω_y. A similar argument applied to any two of the arguments of π in $\boldsymbol{R}^n$ proves the theorem.

5. **Proof of Theorem 3**[2]**.** With T strictly monotonic in each variable separately, T is locally nonconstant in each variable also; the results of §4 are therefore applicable and it remains only to prove that if π is not injective on Ω, then some f_i is constant on some open set in Ω_i. Consider again the $\boldsymbol{R}^2$ case using f, g as before. If $f(a) = f(b)$ for some $a < b$, then for some $a < c < b$, $f(c)$ extremizes f (choosing max. or min. as required) on $[a, b]$ and in every $N_\varepsilon(c)$ two points x_0, x_2 can be found satisfying $f(x_0) = f(x_2)$. With $c \in \Omega_x$ so also $(c, y_0) \in \Omega$ for some $y_0 \in \Omega_y$, and for sufficiently small $\varepsilon > 0$ so also $N_\varepsilon(c) \times \{y_0\} \subset \Omega$. Hence $f(x_0) = f(x_2)$ with $[x_0, x_2] \times \{y_0\} \subset \Omega$; for this x_0, x_2 choose x_1 in the open interval $]x_0, x_2[$ such that $f(x_1)$ extremizes f on $[x_0, x_2]$. Assume $f(x_1) \geqq f(x)$ for all

[2] A similar argument may be found in [**3**].

$x_0 \leqq x \leqq x_2$ and note that $x_0 < x_1 < x_2$. Since T is strictly monotonic in each variable assume $T(x_0, y_0) < T(x_1, y_0) < T(x_2, y_0)$ and define $\Gamma_1, \Gamma_2, \Gamma_3 \subset \Omega_y$ as follows: $\Gamma_1 = \{y \mid T(x_0, y_0) < T(x_1, y) < T(x_2, y_0)\}$, $\Gamma_2 = \{y \mid T(x_0, y) < T(x_1, y_0)\}$, and $\Gamma_3 = \{y \mid T(x_1, y_0) < T(x_2, y)\}$. By continuity each Γ_i is open and $y_0 \in \Gamma_1 \wedge \Gamma_2 \wedge \Gamma_3$ thus defining a neighborhood $N(y_0)$ of y_0. For every $y \in N(y_0)$ follows $T(x_0, y_0) < T(x_1, y) < T(x_2, y_0)$ and $T(x_0, y) < T(x_1, y_0) < T(x_2, y)$; therefore there exist points $\alpha, \beta \in]x_0, x_1[$ satisfying $T(\alpha, y_0) = T(x_1, y)$ and $T(\beta, y) = T(x_1, y_0)$. The equation $h \circ T = F \circ \pi$ then implies $F\{f(\alpha), g(y_0)\} = F\{f(x_1), g(y)\}$ and $F\{f(\beta), g(y)\} = F\{f(x_1), g(y_0)\}$. But $f(x_1) \geqq f(\alpha)$ and $\geqq f(\beta)$ and since F is now strictly monotonic in each variable, $g(y) = g(y_0)$ follows. Hence g is constant on $N(y_0)$, and by § 4, g is constant on Ω_y and f is constant on Ω_x. When applied to any two arguments of the original equation in $\boldsymbol{R}^n$, $n \geqq 2$, the theorem follows.

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