PACIFIC JOURNAL OF MATHEMATICS
Vol. 82, No. 2, 1979

EXAMPLES OF LOCALLY COMPACT NON-COMPACT MINIMAL TOPOLOGICAL GROUPS

SUSANNE DIEROLF AND ULRICH SCHWANENGEL

In 1971, R. M. Stephenson, Jr., [4], showed that an abelian locally compact topological group must be compact if it is minimal (i.e., if it does not admit a strictly coarser Hausdorff group topology). He left open the question, whether there exist locally compact noncompact minimal topological groups.

In this note we give an example of a closed noncompact subgroup of GL (2; R) which is minimal. Moreover we prove that every discrete topological group is topologically isomorphic to a subgroup of a locally compact minimal topological group. Another example shows that a minimal topological group can contain a discrete, nonminimal normal subgroup.

Let $(X, \mathfrak{T})$ be a topological group. Then $\mathfrak{U}_e(X, \mathfrak{T})$ denotes the filter of all $\mathfrak{T}$-neighborhoods of the neutral element $e \in X$. If $G \subset X$ is a subgroup, let $\mathfrak{T}|G$ denote the relative topology induced by $\mathfrak{T}$ on G, and let $\mathfrak{T}/G$ denote the quotient topology on the left coset space X/G.

In the sequel we will need the following technical result.

LEMMA 1. *Let X be group and $G \subset X$ be a subgroup. Let $\mathfrak{S}, \mathfrak{T}$ be group topologies on X such that $\mathfrak{S} \subset \mathfrak{T}$, $\mathfrak{S}|G = \mathfrak{T}|G$, and $\mathfrak{S}/G = \mathfrak{T}/G$. Then $\mathfrak{S} = \mathfrak{T}$.*

Proof. Let $U \in \mathfrak{U}_e(X, \mathfrak{T})$. Then there is $V \in \mathfrak{U}_e(X, \mathfrak{S})$ such that $(V^{-1}V) \cap G \subset U$. Because of $U \cap V \in \mathfrak{U}_e(X, \mathfrak{T})$, there exists $W \in \mathfrak{U}_e(X, \mathfrak{S})$, $W \subset V$, such that $W \subset (U \cap V)G$.

Let $w \in W$; then there are $x \in U \cap V$ and $y \in G$ satisfying $w = xy$ whence $y = x^{-1}w \in ((U \cap V)^{-1}W) \cap G \subset (V^{-1}V) \cap G \subset U$. Thus $w = xy \in (U \cap V)U \subset U^2$. This proves $W \subset U^2$, hence $U^2 \in \mathfrak{U}_e(X, \mathfrak{S})$.

Given a group X, let Aut X denote the group of all automorphisms $f: X \to X$.

Let G, H be groups and let $\sigma: H \to \operatorname{Aut} G$ be a homomorphism; by $G \times_\sigma H$ we denote the corresponding semi-direct product, i.e., the set $G \times H$ provided with the group structure $(x, y)\cdot(x', y') := (x\cdot\sigma(y)(x'), y\cdot y')$ $(x, x' \in G,\ y, y' \in H)$ (cf. [1; Ch. III, §2, Prop. 27]).

In this situation we will often identify G with the normal subgroup $G \times \{e\} \subset G \times_\sigma H$ as well as H with the subgroup $\{e\} \times H$.

Let $\mathfrak{S}$ and $\mathfrak{T}$ be group topologies on G and H, respectively; then the product topology $\mathfrak{S} \times \mathfrak{T}$ is a group topology on $G \times_\sigma H$ if and only if the map

$$(G, \mathfrak{S}) \times (H, \mathfrak{T}) \longrightarrow (G, \mathfrak{S}) , \quad (x, y) \longmapsto \sigma(y)(x) ,$$

is continuous (cf. [1; Ch. III, §2, Prop. 28]). If $\mathfrak{S} \times \mathfrak{T}$ is a group topology on $G \times_\sigma H$, we will call the topological group $(G \times_\sigma H, \mathfrak{S} \times \mathfrak{T})$ the topological semi-direct product of the topological groups $(G, \mathfrak{S})$ and $(H, \mathfrak{T})$, and use the notation $(G, \mathfrak{S}) \times_\sigma^{\text{top}} (H, \mathfrak{T})$.

$$(*) \quad \begin{cases} \text{Moreover, if } \mathfrak{Z} \text{ is any group topology on } G \times_\sigma H, \text{ then the} \\ \text{map } (G, \mathfrak{Z}|G) \times (H, \mathfrak{Z}|H) \to (G, \mathfrak{Z}|G), (x, y) \mapsto \sigma(y)(x), \text{ is con-} \\ \text{tinuous (see the passage after Prop. 28 in [1; Ch. III, §2]).} \end{cases}$$

In the following examples topological semi-direct products and $(*)$ will be the main tools.

DEFINITION. A Hausdorff topological group $(X, \mathfrak{T})$ is called minimal, if there does not exist a Hausdorff group topology $\mathfrak{S}$ on X which is strictly coarser than $\mathfrak{T}$.

REMARK. Let $(X, \mathfrak{T})$ be a locally compact topological group, which admits a separating family $(p_i)_{i \in I}$ of continuous irreducible unitary finite-dimensional representations. If $(X, \mathfrak{T})$ is minimal, then $(X, \mathfrak{T})$ is compact. In fact, let $\mathfrak{S}$ denote the initial topology on X with respect to the representations p_i $(i \in I)$. Then $\mathfrak{S} \subset \mathfrak{T}$; moreover $(X, \mathfrak{S})$ is a Hausdorff precompact topological group. $(X, \mathfrak{T})$ being minimal, one obtains $\mathfrak{T} = \mathfrak{S}$, whence $(X, \mathfrak{T})$ is a precompact and complete topological group hence compact (cf. also [4]). It is clear that the above statement remains true, if we only assume $(X, \mathfrak{T})$ to be complete in its two-sided uniformity instead of being locally compact.

EXAMPLE 1. Let X be the group of all matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ in the general linear group $\mathrm{GL}(2; \boldsymbol{R})$ such that $a \in \boldsymbol{R}_+ := \{c \in \boldsymbol{R}: c > 0\}$ and $b \in \boldsymbol{R}$. Then X provided with its usual locally compact, noncompact group topology $\mathfrak{T}$, induced by $\boldsymbol{R}^4$, is a minimal topological group.

Proof. It is well-known that $(X, \mathfrak{T})$ may be identified with the topological semi-direct product $\boldsymbol{R} \times_\sigma^{\text{top}} \boldsymbol{R}_+$ with respect to the homomorphism $\sigma: \boldsymbol{R}_+ \to \operatorname{Aut} \boldsymbol{R}$, $\sigma(y)(x) := xy$ $(y \in \boldsymbol{R}_+, x \in \boldsymbol{R})$, where the groups $\boldsymbol{R} = (\boldsymbol{R}, +)$ and $\boldsymbol{R}_+ = (\boldsymbol{R}_+, \cdot)$ are given their standard topologies. In fact, the map

$$(X, \mathfrak{T}) \longrightarrow \boldsymbol{R} \times_{\sigma}^{\mathrm{top}} \boldsymbol{R}_{+}\ , \quad \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \longmapsto (b, a)\ ,$$

is a topological isomorphism. $G: = \boldsymbol{R} \times \{1\}$ is a normal subgroup of X, which we will write additively, $H: = \{0\} \times \boldsymbol{R}_{+}$ is a nonnormal subgroup of X, which we will write multiplicatively.

Let $\mathfrak{S}$ be a Hausdorff group topology on X such that $\mathfrak{S} \subset \mathfrak{T}$. We have to prove that $\mathfrak{S} = \mathfrak{T}$.

(a) We first show that $\mathfrak{S}|G = \mathfrak{T}|G$.

Because of $(*)$ and because of $\mathfrak{T} \supset \mathfrak{S}$, we obtain that the map $w: (G, \mathfrak{S}|G) \times \boldsymbol{R}_{+} \to (G, \mathfrak{S}|G)$, $((x, 1), y) \mapsto (xy, 1)$, is continuous. $\mathfrak{S}$ being Hausdorff, there exists $U \in \mathfrak{U}_0(G, \mathfrak{S}|G)$ such that $U - U \neq G$. Choose $V \in \mathfrak{U}_0(G, \mathfrak{S}|G)$ and $\varepsilon > 0$ satisfying $w(V \times [1 - \varepsilon, 1 + \varepsilon]) \subset U$. Hence $\bigcup_{(x,1) \in V} [-\varepsilon x, \varepsilon x] \times \{1\} = \bigcup_{(x,1) \in V} (x \cdot [1 - \varepsilon, 1 + \varepsilon] - x) \times \{1\} \subset w(V \times [1 - \varepsilon, 1 + \varepsilon]) - w(V \times \{1\}) \subset U - U \neq G$; consequently there is $M > 0$ such that $V \subset [-M, M] \times \{1\}$. Thus V is contained in a compact subset of $(G, \mathfrak{T}|G)$, hence $\mathfrak{T}|V = \mathfrak{S}|V$, which implies $\mathfrak{T}|G = \mathfrak{S}|G$.

(b) Next we show that $\mathfrak{S}/G = \mathfrak{T}/G$.

Because of (a), $(G, \mathfrak{S}|G)$ is a complete subgroup of the Hausdorff topological group $(X, \mathfrak{S})$, whence G is closed in $(X, \mathfrak{S})$. Consequently $\mathfrak{S}/G$ is a Hausdorff group topology on the factor group X/G. Let $q: X \to X/G$ denote the quotient map.

Because of (a) there exists $U \in \mathfrak{U}_e(X, \mathfrak{S})$ such that $U \cap G = [-1, 1] \times \{1\}$; choose $V \in \mathfrak{U}_e(X, \mathfrak{S})$ such that $V = V^{-1}$ and $V^3 \subset U$. There exists $\varepsilon \in]0, 1[$ satisfying $[-\varepsilon, \varepsilon] \times \{1\} \subset V$. If $(x, y) \in V$, then $(\varepsilon y, 1) = (x, y) \cdot (\varepsilon, 1) \cdot (x, y)^{-1} \in V^3 \cap G \subset U \cap G$, whence $y \leqq 1/\varepsilon$. V being symmetric, we obtain that $q(V) \subset q(R \times [\varepsilon, 1/\varepsilon]) = q(\{0\} \times [\varepsilon, 1/\varepsilon])$. Thus $q(V)$ is contained in a compact subset of $(X/G, \mathfrak{T}/G)$, hence $(\mathfrak{S}/G)|q(V) = (\mathfrak{T}/G)|q(V)$, which implies $\mathfrak{S}/G = \mathfrak{T}/G$.

Now, from (a) and (b) we obtain $\mathfrak{S} = \mathfrak{T}$ by Lemma 1.

Example 1 shows that a minimal locally compact topological group may have nonminimal normal subgroups and nonminimal factor groups, since clearly $\boldsymbol{R}$ and $\boldsymbol{R}_{+}$ are nonminimal topological groups.

REMARK. We mention without proof that for all $n \in \boldsymbol{N}$, the groups $\mathrm{GL}(n; \boldsymbol{K})$ $(\boldsymbol{K} \in \{\boldsymbol{R}, \boldsymbol{C}\})$ are not minimal.

EXAMPLE 2. Let K be a compact topological group, and let H be a discrete topological group. Let $G: = K^H$ be endowed with the product topology.

$$\sigma: H \longrightarrow \operatorname{Aut} G\ , \quad \sigma(k)((x_h)_{h \in H}): = (x_{hk})_{h \in H} \ (k \in H,\ (x_h)_{h \in H} \in G)$$

is a homomorphism. Moreover, the map

$$w: G \times H \longrightarrow G\ , \quad ((x_h)_{h\in H}, k) \longmapsto (x_{hk})_{h\in H}\ ,$$

is continuous, as can easily be verified.

Thus the topological semi-direct product $(X, \mathfrak{T}) := G \times_\sigma^{\mathrm{top}} H$ is a well-defined locally compact topological group.

From now on we assume that $K \neq \{e\}$. We prove that $(X, \mathfrak{T})$ is a minimal topological group.

Let $\mathfrak{S}$ be a Hausdorff group topology on X such that $\mathfrak{S} \subset \mathfrak{T}$. We have to show that $\mathfrak{S} = \mathfrak{T}$.

(a) $(G, \mathfrak{T}|G)$ being compact, we have $\mathfrak{S}|G = \mathfrak{T}|G$.

(b) We show that $\mathfrak{S}|H = \mathfrak{T}|H$.

K being nontrivial and Hausdorff, there exists $U \in \mathfrak{U}_e(K)$ such that $U \neq K$. Because of $(*)$ and (a), the map

$$w: (G, \mathfrak{T}|G) \times (H, \mathfrak{S}|H) \longrightarrow (G, \mathfrak{T}|G),\ ((x_h)_{h\in H}, k) \longmapsto (x_{hk})_{h\in H}\ ,$$

is continuous. Thus there exist $V \in \mathfrak{U}_e(H, \mathfrak{S}|H)$ and a finite subset $E \subset H$ such that the following implication holds:

$$\left.\begin{array}{c}(x_h)_{h\in H} \in G,\ x_h = e \ \text{ for all } \ h \in E \\ k \in V\end{array}\right\} \Longrightarrow x_k \in U\ .$$

Now we easily deduce that $k \in E$ for all $k \in V$. Thus V is finite, whence $-\mathfrak{S}|H$ being Hausdorff $-$ $\mathfrak{S}|H$ equals the discrete topology on H.

(c) Next we obtain that $\mathfrak{S}/H = \mathfrak{T}/H$.

In fact, $\mathfrak{S}|H$ being discrete, H is a closed subgroup of $(X, \mathfrak{S})$. Consequently, $\mathfrak{S}/H$ is Hausdorff and coarser than the compact topology $\mathfrak{T}/H$ (mind that $(G, \mathfrak{T}|G)$ and $(X/H, \mathfrak{T}/H)$ are homeomorphic).

Now, from (b) and (c) we obtain $\mathfrak{S} = \mathfrak{T}$ by Lemma 1.

Specializing for instance $K := \boldsymbol{Z}/2\boldsymbol{Z}$ in the above Example 2, we obtain:

PROPOSITION 1. *For every group H there exists a locally compact minimal topological group $(X, \mathfrak{T})$ containing H as a subgroup such that $\mathfrak{T}|H$ equals the discrete topology. Moreover, X can be chosen such that* $\operatorname{card} X \leqq 2^{\operatorname{card} H}$.

The following example shows that minimal topological groups can even contain discrete nonminimal normal subgroups.

EXAMPLE 3. Let $p \in \boldsymbol{Z}$ be a prime number, $K := \boldsymbol{Z}/p\boldsymbol{Z}$, and let X be an infinite-dimensional vector space over K. Let X be provided with its discrete topology. X being algebraically isomorphic to $K^{(I)} := \{(x_i)_{i\in I} \in K^I : \{i \in I : x_i \neq 0\} \text{ is finite}\}$ for some index set I, it is clear that X is not a minimal topological group (consider the relative product topology on $K^{(I)}$). Moreover, every subgroup of $(X, +)$ is a linear subspace of the K-vector space X, and for every finite subset $E \subset X$ the subgroup $\langle E \rangle$ generated by E is finite.

The group $Z(X)$ of all bijections $f: X \to X$ provided with the topology of pointwise convergence, is a topological group which is complete in its two-sided uniformity according to [2; §3, Ex. 19]. Cf. also [3]. — Clearly, $\operatorname{Aut} X$ is a closed subgroup of $Z(X)$. Thus $\operatorname{Aut} X$ provided with the relative topology induced by $Z(X)$ is a Hausdorff topological group, which is complete in its two-sided uniformity. Moreover, $\operatorname{Aut} X$ is metrizable and separable if X is countable.

We mention that $\operatorname{Aut} X$ does not have a group completion. In fact, let $(e_i)_{i \in I}$ be a basis of the K-vector space X, and let $Z(I)$ denote the group of all bijections $\tau: I \to I$ provided with the topology of pointwise convergence. On account of [2; §3, Ex. 19], $Z(I)$ does not have a group completion. The map

$$j: Z(I) \longrightarrow \operatorname{Aut} X,\ j(\tau)\Big(\sum_{i \in I} \alpha_i e_i\Big) := \sum_{i \in I} \alpha_i e_{\tau(i)}\ (\tau \in Z(I),\ (\alpha_i)_{i \in I} \in K^{(I)}),$$

being a topological isomorphism of $Z(I)$ onto a subgroup of $\operatorname{Aut} X$, also $\operatorname{Aut} X$ does not have a group completion (hence $\operatorname{Aut} X$ is not locally compact).

Obviously, the map

$$w: X \times \operatorname{Aut} X \longrightarrow X,\ (x, f) \longmapsto f(x),$$

is continuous; thus the topological semi-direct product $(Y, \mathfrak{T}) := X \times_{\mathrm{id}}^{\mathrm{top}} \operatorname{Aut} X$ is a well-defined Hausdorff topological group without a group completion, which is metrizable and separable if X is countable. We show that $(Y, \mathfrak{T})$ is a minimal topological group.

Let $\mathfrak{S} \subset \mathfrak{T}$ be a Hausdorff group topology on Y.

(a) We first show that $\mathfrak{S} | X = \mathfrak{T} | X$.

There exists $U \in \mathfrak{U}_0(X, \mathfrak{S} | X)$ such that $X \backslash U$ is infinite. Because of (*) and because of $\mathfrak{T} \supset \mathfrak{S}$, the map

$$w: (X, \mathfrak{S} | X) \times (\operatorname{Aut} X, \mathfrak{T} | \operatorname{Aut} X) \longrightarrow (X, \mathfrak{S} | X),\ (x, f) \longmapsto f(x),$$

is continuous. Thus there exist $V \in \mathfrak{U}_0(X, \mathfrak{S} | X)$ and a finite subset $E \subset X$ such that the following implication holds:

$$\left.\begin{array}{r} x \in V \\ f \in \operatorname{Aut} X,\ f(y) = y \text{ for all } y \in E \end{array}\right\} \Longrightarrow f(x) \in U .$$

$\langle E \rangle$ being finite, there exists $y \in X \backslash (\langle E \rangle \cup U)$. — For every $z \in X \backslash \langle E \rangle$ we can construct $f \in \operatorname{Aut} X$ such that $f(z) = y$ and $f(x) = x$ for all

$x \in \langle E \rangle$; because of $f(z) = y \notin U$ we obtain that $z \notin V$. This implies $V \subset \langle E \rangle$, whence V is finite. Consequently, $\mathfrak{S} | X$ equals the discrete topology.

(b) Next we show that $\mathfrak{S} | \mathrm{Aut}\, X = \mathfrak{T} | \mathrm{Aut}\, X$.

$$w\colon (X, \mathfrak{S}|X) \times (\mathrm{Aut}\, X, \mathfrak{S}|\mathrm{Aut}\, X) \longrightarrow (X, \mathfrak{S}|X)$$

being continuous, we obtain—using (a)—that for every $x \in X$ the set $\{f \in \mathrm{Aut}\, X\colon f(x) = x\}$ belongs to $\mathfrak{U}_{id}(\mathrm{Aut}\, X, \mathfrak{S}|\mathrm{Aut}\, X)$, whence clearly $\mathfrak{U}_{id}(\mathrm{Aut}\, X, \mathfrak{T}|\mathrm{Aut}\, X) \subset \mathfrak{U}_{id}(\mathrm{Aut}\, X, \mathfrak{S}|\mathrm{Aut}\, X)$. This proves $\mathfrak{T}|\mathrm{Aut}\, X \subset \mathfrak{S}|\mathrm{Aut}\, X$.

(c) Finally we show that $\mathfrak{S}/\mathrm{Aut}\, X = \mathfrak{T}/\mathrm{Aut}\, X$ (which together with (b) implies $\mathfrak{S} = \mathfrak{T}$ by Lemma 1).

Because of (b) and the fact that $(\mathrm{Aut}\, X, \mathfrak{T}|\mathrm{Aut}\, X)$ is complete in its two-sided uniformity, $\mathrm{Aut}\, X$ is a closed subgroup of the Hausdorff topological group $(Y, \mathfrak{S})$, as can easily be verified. Consequently, $(Y/\mathrm{Aut}\, X, \mathfrak{S}/\mathrm{Aut}\, X)$ is Hausdorff.

Let $q\colon Y \to Y/\mathrm{Aut}\, X$ denote the quotient map. There exists $U \in \mathfrak{U}_e(Y, \mathfrak{S})$ such that $(Y/\mathrm{Aut}\, X) \backslash q(U)$ is infinite. Let $V \in \mathfrak{U}_e(Y, \mathfrak{S})$ such that $V^2 \subset U$. Then there is a finite subset $E \subset X$ such that $(0, f) \in V$ for all $f \in \mathrm{Aut}\, X$ satisfying $f(x) = x$ $(x \in E)$. Fix $y \in X \backslash \langle E \rangle$ such that $q(y, id) \notin q(U)$.

Let $z \in X \backslash \langle E \rangle$ and let $g \in \mathrm{Aut}\, X$. Then there exists $f \in \mathrm{Aut}\, X$ such that $f(z) = y$ and $f(x) = x$ for all $x \in \langle E \rangle$. Thus $(0, f) \in V$. Because of $q((0, f)\cdot(z, g)) = q(y, fg) = q(y, id) \notin q(U)$ we obtain $(z, g) \notin V$. — Consequently $V \subset \langle E \rangle \times \mathrm{Aut}\, X$, whence $q(V) \subset q(\langle E \rangle \times \{id\})$ is finite. $(Y/\mathrm{Aut}\, X, \mathfrak{S}/\mathrm{Aut}\, X)$ being a Hausdorff topological homogeneous space, we obtain that $\mathfrak{S}/\mathrm{Aut}\, X$ is discrete, whence $\mathfrak{S}/\mathrm{Aut}\, X = \mathfrak{T}/\mathrm{Aut}\, X$.

Added in proof. From the construction of Example 2 it is clear that H is topologically isomorphic to the factor group X/G. Thus we obtain the following.

PROPOSITION 1′. *For every group H there exists a locally compact minimal topological group $(X, \mathfrak{T})$ containing a normal subgroup N such that $X/N = H$ and such that $\mathfrak{T}/N$ equals the discrete topology on H.*

REFERENCES

1. N. Bourbaki, *Topologie Générale*, Ch. III-IV, Hermann Paris, 1960.
2. ———, *Topologie Générale*, Ch. X, Hermann Paris, 1960.
3. S. Dierolf and U. Schwanengel, *Un exemple d'un groupe topologique q-minimal mais non précompact*, Bull. Sci. Math., 2e série, **101** (1977), 265–269.
4. R. M. Stephenson, Jr., *Minimal topological groups*, Math. Ann., **192** (1971), 193–195.

Received August 23, 1978.

MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
THERESIENSTR. 39
D-8000 MÜNCHEN 2, WEST GERMANY
AND
LERCHENAUER STR. 9
D-8000 MÜNCHEN 40
WEST GERMANY