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A COMMUTATIVE BANACH ALGEBRA OF FUNCTIONS OF GENERALIZED VARIATION

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It is known that the space of functions, anchored at a, and having bounded variation form a commutative Banach algebra under the total variation norm. We show that functions of bounded kth variation also form a Banach algebra under a norm defined in terms of the total kth variation.

1. Introduction. Let $BV_1[a, b]$ denote the space of functions of bounded variation on the closed interval $[a, b]$, and denote the total variation of f on that interval by $V_1(f)$ or $V_1(f; a, b)$. If

$$BV_1^*[a, b] = \{f;\ V_1(f) < \infty,\ f(a) = 0\}\ ,$$

then it is a well known result that $BV_1^*[a, b]$ is a Banach space under the norm $\|\cdot\|_1$, where $\|f\|_1 = V_1(f)$. What appears to be less well known is that, using pointwise operations, $BV_1^*[a, b]$ is a commutative Banach algera with a unit under $\|\cdot\|_1$ — see for example [1] and Exercise 17.35 of [2].

In [4] it was shown that $BV_k[a, b]$ is a Banach space under the norm, $\|\cdot\|_k$, where

$$\|f\|_k = \sum_{s=0}^{k-1} |f^{(s)}(a)| + V_k(f; a, b)\ , \tag{1}$$

and where the definition of $V_k(f; a, b) \equiv V_k(f)$ can be found in [3]. The subspace

$$BV_k^*[a, b] = \{f;\ f \in BV_k[a, b],\ f(a) = f'(a) = \cdots = f^{(k-1)}(a) = 0\}$$

is clearly also a Banach space under the norm $\|\cdot\|_k^*$, where

$$\|f\|_k^* = \alpha_k V_k(f)\ , \tag{2}$$

and $\alpha_k = 2^{k-1}(b-a)^{k-1}(k-1)!$.

If we define the product of two functions in $BV_k^*[a, b]$ by pointwise multiplication, then we show, in addition, that $BV_k^*[a, b]$ is a commutative Banach algebra under the norm given in (2). It is obvious that $BV_k^*[a, b]$ is commutative, so our main programme now is to show that if f and g belong to $BV_k^*[a, b]$, then so does fg, and that

$$V_k(fg) \leqq 2^{k-1}(k-1)!\,(b-a)^{k-1} V_k(f) V_k(g)\ , \qquad k \geqq 1\ .$$

We accept the case $k = 1$ as being known, so restrict our discussion to $k \geqq 2$. Because the same procedure does not appear to be applicable to the cases $k = 2$ and $k \geqq 3$, we present different treatments for these cases.

In order to achieve the stated results it was found convenient to work with two definitions of bounded kth variation, one defined with quite arbitrary subdivisions $a = x_0, x_1, \cdots, x_n = b$ of $[a, b]$, and the other using subdivisions in which all sub-intervals are of equal length. If we call the two classes of functions so obtained $BV_k[a, b]$ and $\overline{BV}_k[a, b]$ respectively, then we show that provided we restrict our functions to being continuous, then these classes are identical. More specifically, if we denote $C[a, b]$, $BV_k[a, b]$, and $\overline{BV}_k[a, b]$ by C, BV_k and $\overline{BV}_k$ respectively, then we show that

$$C \cap BV_k = \overline{BV}_k \, .$$

2. Notation and preliminaries.

DEFINITION 1. We shall say that a set of points $x_0, x_1, \cdots, x_n$ is a π-subdivision of $[a, b]$ when $a \leqq x_0 < x_1 < \cdots < x_n = b$.

DEFINITION 2. If $h > 0$, then we will denote by π_h a subdivision $x_0, x_1, \cdots, x_n$ of $[a, b]$ such that $a = x_0 < x_1 < \cdots < x_n \leqq b$, where $x_i - x_{i-1} = h$, $i = 1, 2, \cdots, n$, and $0 \leqq b - x_n \leqq h$.

Before introducing the two definitions of bounded kth variation we need the definition and some properties of kth divided differences, and for this purpose we refer the reader to [3]. In addition, we make use of the difference operator Δ_h^k defined by

$$\Delta_h^1 f(x) = f(x + h) - f(x) \, ,$$

and

$$\Delta_h^k f(x) = \Delta_h^1[\Delta_h^{k-1} f(x)] \, .$$

DEFINITION 3. The total kth variation of a function f on $[a, b]$ is defined by

$$V_k(f; a, b) = \sup_{\pi} \sum_{i=0}^{n-k} (x_{i+k} - x_i) \, | Q_k(f; x_i, \cdots, x_{i+k}) | \, .$$

If $V_k(f; a, b) < \infty$, we say that f is of bounded kth variation on $[a, b]$, and write $f \in BV_k[a, b]$.

DEFINITION 4. If f is continuous on $[a, b]$, then we define the total kth variation of f on $[a, b]$ (restricted form) by

$$\bar{V}_k(f; a, b) = \sup_{\pi_h} \sum_{i=0}^{n-k} \left| \frac{\Delta_h^k f(x_i)}{h^{k-1}} \right| .$$

If $\bar{V}_k(f; a, b) < \infty$, we say that f is of restricted bounded kth variation on $[a, b]$, and write $f \in \overline{BV}_k[a, b]$.

As before, we will usually write $V_k(f)$ and $\bar{V}_k(f)$ for $V_k(f; a, b)$ and $\bar{V}_k(f; a, b)$ respectively.

We now show that $C \cap BV_k = \overline{BV}_k$, and point out at this stage that the restriction to continuous functions is not nearly as severe as it first may appear, because functions belonging to $BV_k[a, b]$, when $k \geqq 2$, are automatically continuous — see Theorem 4 of [3].

LEMMA 1. *Let $I_1, I_2, \cdots, I_n$ be a set of n adjoining closed intervals on the real line having lengths $p_1/q_1, p_2/q_2, \cdots, p_n/q_n$ respectively, where $p_1, p_2, \cdots, p_n, q_1, q_2, \cdots, q_n$ are positive integers. Then it is possible to subdivide the intervals $I_1, I_2, \cdots, I_n$ into sub-intervals of equal length.*

The proof is easy and will be omitted.

LEMMA 2. *If $k \geqq 1$, then $C \cap BV_k \subset \overline{BV}_k$, using abbreviated notation.*

Proof. This is easy and will not be included.

LEMMA 3. *If $k \geqq 1$, then $C \cap BV_k \supset \overline{BV}_k$.*

Proof. Let us suppose that f is continuous, belongs to $\overline{BV}_k[a, b]$, but $f \notin BV_k[a, b]$. Then for an arbitrarily large number K, and an arbitrarily small positive number ε, there exists a subdivision $\pi_1(x_0, x_1, \cdots, x_n)$ of $[a, b]$ such that

$$S_{\pi_1} \equiv \sum_{i=0}^{n-k} (x_{i+k} - x_i) | Q_k(f; x_i, \cdots, x_{i+k}) | > K + \varepsilon .$$

If not all the lengths $(x_{i+1} - x_i)$, $i = 0, 1, \cdots, n-1$ are rational, then because f is continuous we can obtain a subdivision $\pi_2(y_0, y_1, \cdots, y_n)$ of $[a, b]$ in which all the lengths $(y_{i+1} - y_i)$, $i=0, 1, \cdots, n-1$ are rational, and such that $|S_{\pi_1} - S_{\pi_2}| < \varepsilon$, S_{π_2} being the approximating sum of $V_k(f; a, b)$ corresponding to the π_2 subdivision. Consequently,

$$S_{\pi_2} \geqq S_{\pi_1} - |S_{\pi_1} - S_{\pi_2}| > K .$$

In the π_2 subdivision, all sub-intervals have rational length, so we can apply Lemma 1 to obtain a π_h subdivision of $[a, b]$ in which each sub-interval has length h. If S_{π_h} is the corresponding approximating sum of $\bar{V}_k(f; a, b)$, then it follows from Theorem 3 of [3] that

$$\frac{1}{(k-1)!} S_{\pi_h} \geqq S_{\pi_2} > K ,$$

since for any π_h subdivision, and each $i = 0, 1, \cdots, n-k$,

$$\frac{\Delta_h^k f(x_i)}{h^{k-1}} = (k-1)!\,(x_{i+k} - x_i) Q_k(f; x_i, \cdots, x_{i+k}) .$$

Thus $S_{\pi_h} > (k-1)!\,K$, and this is a contradiction to the assumption that $f \in \overline{BV}_k[a, b]$. Hence $f \in \overline{BV}_k[a, b]$, and so $\overline{BV}_k \subset C \cap BV_k$.

THEOREM 1. *If* $k \geqq 1$, *then* $C \cap BV_k = \overline{BV}_k$; *and if* f *is a continuous function on* $[a, b]$, *then*

$$\bar{V}_k(f; a, b) = (k-1)!\; V_k(f; a, b) , \qquad k \geqq 1 . \tag{3}$$

Proof. The first part follows from Lemmas 2 and 3. For the second part we first observe that

$$\bar{V}_k(f; a, b) \leqq (k-1)!\; V_k(f; a, b) . \tag{4}$$

Let $\varepsilon > 0$ be arbitrary. Then there exists a π_1 subdivision of $[a, b]$ and the corresponding approximating sum S_{π_1} to $V_k(f; a, b)$ such that

$$S_{\pi_1} > V_k(f; a, b) - \frac{\varepsilon}{2(k-1)!} .$$

If not all the sub-intervals of π_1 have rational lengths, then we can proceed as in Lemma 3 to obtain a π_h subdivision of $[a, b]$ in which all sub-intervals are of equal length h. Then, if S_{π_h} is the corresponding approximating sum to $\bar{V}_k(f; a, b)$, we can show that

$$\begin{aligned}\frac{1}{(k-1)!} S_{\pi_h} &\geqq S_{\pi_1} - \frac{\varepsilon}{2(k-1)!} \\ &> V_k(f; a, b) - \frac{\varepsilon}{(k-1)!} .\end{aligned}$$

Consequently,

$$\begin{aligned}\bar{V}_k(f; a, b) &\geqq S_{\pi_h} \\ &> (k-1)!\; V_k(f; a, b) - \varepsilon ,\end{aligned}$$

from which it follows that $\bar{V}_k(f; a, b) \geqq (k-1)!\; V_k(f; a, b)$. This inequality together with (4) gives (3).

LEMMA 4. *If f and g are any two real valued functions defined on $[a, b]$, $h > 0$ and $a \leqq x < x + kh \leqq b$, then*

$$\begin{aligned} \Delta_h^k[f(x)g(x)] &= f(x+kh)\Delta_h^k g(x) + \binom{k}{1}\Delta_h^1 f(x+(k-1)h)\Delta_h^{k-1}g(x) + \cdots \\ &\quad + \binom{k}{s}\Delta_h^s f(x+(k-s)h)\Delta_h^{k-s}g(x) + \cdots + \Delta_h^k f(x)\Delta_h^0 g(x) \qquad (5) \\ &= \sum_{s=0}^{k}\binom{k}{s}\Delta_h^s f(x+(k-s)h)\Delta_h^{k-s}g(x), \text{ where } \Delta_h^0 g(x) = g(x) . \end{aligned}$$

Proof. The proof by induction is straightforward and will not be included.

LEMMA 5. *If f and g belong to $BV_k[a, b]$, $k \geqq 1$, then $fg \in BV_k[a, b]$.*

Proof. The result for $k = 1$ is well known, so we assume that $k \geqq 2$, in which case f and g are continuous in $[a, b]$. Consequently, in view of Theorem 1, there will be no loss of generality in working with equal sub-intervals of $[a, b]$. Using (5) we have, suppressing the "h" in "Δ_h^k",

$$\begin{aligned} \frac{\Delta^k[f(x)g(x)]}{h^{k-1}} &= f(x+kh)\frac{\Delta^k g(x)}{h^{k-1}} + \cdots + \binom{k}{s}\frac{\Delta^s f(x+(k-s)h)}{h^s}\,\frac{\Delta^{k-s}g(x)}{h^{k-s-1}} \qquad (6) \\ &\quad + \cdots + \frac{\Delta^{k-1}f(x+h)}{h^{k-1}}\Delta^1 g(x) + \frac{\Delta^k f(x)}{h^{k-1}}g(x) . \end{aligned}$$

It follows from Theorem 4 of [3] that

$$\frac{\Delta^s f(x+(k-s)h)}{h^s}, \; s = 0, 1, \cdots, k-1$$

is uniformly bounded. Hence we can conclude from (6) that $fg \in \overline{BV}_k[a, b]$ by summing over any π_h subdivision of $[a, b]$, and noting that f and g belong to $\overline{BV}_k[a, b] \subset \overline{BV}_{k-1}[a, b] \subset \cdots \subset \overline{BV}_1[a, b]$ — see Theorem 10 of [3]. Since fg is continuous it follows from Theorem 1 that $fg \in BV_k[a, b]$.

3. Main results.

We now make an application of Theorem 1 to obtain a relationship between $V_{k-1}(f)$ and $V_k(f)$ when $f \in BV_k^*[a, b]$.

THEOREM 2. *If $f \in BV_k^*[a, b]$, $k \geqq 2$, then*

$$V_{k-1}(f) \leqq (k-1)(b-a)V_k(f) , \qquad (7)$$

or

$$\bar{V}_{k-1}(f) \leqq (b-a)\bar{V}_k(f) .$$

Proof. It follows from Theorem 10 of [3] that $f \in BV_{k-1}^*[a, b]$, so $V_{k-1}(f) < \infty$. We now establish the inequality. Since $f \in BV_k^*[a, b]$, $f^{(k-1)}(a) = 0$. Hence for any $\varepsilon > 0$, we can choose a π_h subdivision of $[a, b]$ such that

$$\left|\frac{\Delta_h^{k-1} f(a)}{h^{k-1}}\right| < \frac{\varepsilon}{(b-a)} . \tag{8}$$

There is no loss of generality in choosing such a subdivision in view of Theorem 3 of [3] which tells us that the approximating sums for total kth variation are not decreased by the addition of extra points of subdivision. Accordingly, let $a = x_0, x_1, \cdots, x_n \leqq b$ be a π_h subdivision of $[a, b]$ with property (8). Then, suppressing the "h" in "Δ_h^{k-1}" and "Δ_h^k", we obtain

$$\begin{aligned}
\sum_{i=0}^{n-k+1} |\Delta^{k-1} f(x_i)| &= \sum_{i=0}^{n-k+1} \left|\sum_{s=1}^{i} [\Delta^{k-1} f(x_s) - \Delta^{k-1} f(x_{s-1})] + \Delta^{k-1} f(x_0)\right| \\
&= \sum_{i=0}^{n-k+1} \left|\sum_{s=1}^{i} \Delta^k f(x_{s-1}) + \Delta^{k-1} f(x_0)\right| \\
&\leqq \sum_{i=0}^{n-k+1} \sum_{s=1}^{i} |\Delta^k f(x_{s-1})| + \sum_{i=0}^{n-k+1} |\Delta^{k-1} f(x_0)| \\
&\leqq n \sum_{s=1}^{n-k} |\Delta^k f(x_{s-1})| + n|\Delta^{k-1} f(x_0)| \\
&\leqq (b-a) \sum_{s=1}^{n-k} \left|\frac{\Delta^k f(x_{s-1})}{h}\right| + (b-a)\left|\frac{\Delta^{k-1} f(x_0)}{h}\right| .
\end{aligned}$$

Therefore, dividing both sides by h^{k-2}, we obtain

$$\begin{aligned}
\sum_{i=0}^{n-k+1} \left|\frac{\Delta^{k-1} f(x_i)}{h^{k-2}}\right| &\leqq (b-a) \sum_{s=1}^{n-k} \left|\frac{\Delta^k f(x_{s-1})}{h^{k-1}}\right| + (b-a)\left|\frac{\Delta^{k-1} f(x_0)}{h^{k-1}}\right| \\
&\leqq (b-a)\bar{V}_k(f) + \varepsilon ,
\end{aligned}$$

from which it follows that

$$\bar{V}_{k-1}(f) \leqq (b-a)\bar{V}_k(f) . \tag{9}$$

Consequently, using (2) we obtain

$$V_{k-1}(f) \leqq (k-1)(b-a)V_k(f) ,$$

as required.

COROLLARY. *Let p be an integer such that $1 \leqq p < k$. If $f \in BV_k^*[a, b]$, then $f \in BV_p^*[a, b]$, and*

$$V_p(f) \leqq p(p+1)\cdots(k-1)(b-a)^{k-p} V_k(f) , \tag{10}$$

or

$$\bar{V}_p(f) \leqq (b-a)^{k-p} \bar{V}_k(f) .$$

Proof. The proof follows from repeated applications of (7), and Theorem 10 of [3].

We now proceed to obtain a relationship between $V_k(fg)$, $V_k(f)$ and $V_k(g)$ when f and g belong to $BV_k^*[a, b]$. It appears convenient to treat the cases $k = 2$, and $k \geqq 3$ separately, so we begin by considering $k = 2$.

THEOREM 3. *If f and g belong to $BV_2^*[a, b]$, then $fg \in BV_2^*[a, b]$, and*

$$\begin{aligned} V_2(fg) &\leqq V_2(f)V_1(g) + V_1(f)V_2(g) \\ &\leqq 2(b-a)V_2(f)V_2(g) . \end{aligned} \tag{11}$$

Proof. There is no loss of generality in considering π_h subdivisions of $[a, b]$. Let $a = x_0, x_1, \cdots, x_n$ be such a subdivision. Then, noting that $f(a) = 0 = g(a)$ when $f, g \in BV_2^*[a, b]$, and writing $f(x_{s+1}) - f(x_s) = \Delta f(x_s)$, we obtain for $i \geqq 1$,

$$\begin{aligned} \Delta^2 f(x_i)g(x_i) &= \Delta[\Delta f(x_i)g(x_i)] \\ &= \Delta[f(x_{i+1})\Delta g(x_i) + (\Delta f(x_i))g(x_i)] \\ &= \Delta\left[\left(\sum_{s=0}^{i} \Delta f(x_s)\right)\Delta g(x_i) + \Delta f(x_i)\sum_{s=0}^{i-1}\Delta g(x_s)\right] \\ &= \sum_{s=0}^{i} \Delta(\Delta f(x_s)\Delta g(x_i)) + \sum_{s=0}^{i-1} \Delta(\Delta f(x_i)\Delta g(x_s)) \\ &= \sum_{s=0}^{i} [\Delta f(x_{s+1})\Delta^2 g(x_i) + \Delta^2 f(x_s)\Delta g(x_i)] \\ &\quad + \sum_{s=0}^{i-1} [\Delta f(x_{i+1})\Delta^2 g(x_s) + \Delta^2 f(x_i)\Delta g(x_s)] . \end{aligned} \tag{12}$$

Therefore, noting that the last summation in (12) is zero when $i = 0$, we have

$$\begin{aligned} \sum_{i=0}^{n-2} |\Delta^2 f(x_i)g(x_i)| &\leqq \sum_{i=0}^{n-2} [|\Delta f(x_1)| + \cdots + |\Delta f(x_{i+1})|]|\Delta^2 g(x_i)| \\ &\quad + \sum_{i=0}^{n-2} [|\Delta^2 f(x_0)| + \cdots + |\Delta^2 f(x_i)|]\Delta g(x_i)| \\ &\quad + \sum_{i=0}^{n-2} |\Delta f(x_{i+1})|[|\Delta^2 g(x_0)| + \cdots + |\Delta^2 g(x_{i-1})|] \\ &\quad + \sum_{i=1}^{n-2} |\Delta^2 f(x_i)|[\Delta g(x_0)| + \cdots + |\Delta g(x_{i-1})|] , \end{aligned}$$

which after some re-arrangement is equal to

$$\left(\sum_{i=1}^{n-1}|\Delta f(x_i)|\right)\left(\sum_{i=0}^{n-2}|\Delta^2 g(x_i)|\right) + \left(\sum_{i=0}^{n-2}|\Delta^2 f(x_i)|\right)\left(\sum_{i=0}^{n-2}|\Delta g(x_i)|\right).$$

Therefore, dividing by h, and using Definition 4, we observe that $fg \in BV_2^*[a, b]$, and obtain

$$\bar{V}_2(fg) \leqq \bar{V}_1(f)\bar{V}_2(g) + \bar{V}_2(f)\bar{V}_1(g), \tag{13}$$

or

$$V_2(fg) \leqq V_1(f)V_2(g) + V_2(f)V_1(g),$$

using Theorem 1.

To complete the proof we employ (7) with $k = 2$.

We are now in a position to consider the general case $k \geqq 3$ for which we adopt a different procedure. When $k \geqq 3$ we make use of the fact that $f^{(k-2)} \in BV_2^*[a, b]$, and consequently exists throughout $[a, b]$, and is in fact absolutely continuous in that interval.

THEOREM 4. *Let f and g belong to $BV_k^*[a, b]$ when $k \geqq 3$. Then $fg \in BV_k^*[a, b]$, and*

$$\bar{V}_k(fg) \leqq 2^{k-1}(b-a)^{k-1}\bar{V}_k(f)\bar{V}_k(g), \tag{14}$$

or

$$V_k(fg) \leqq 2^{k-1}(b-a)^{k-1}(k-1)!\ V_k(f)V_k(g). \tag{15}$$

Proof. We first observe from Lemma 5 that $fg \in BV_k^*[a, b]$. It follows from Theorems 2 and 8 of [5] that

$$\begin{aligned}
\bar{V}_k(fg) &= \bar{V}_2((fg)^{(k-2)}) \\
&= \bar{V}_2\left(\sum_{s=0}^{k-2}\binom{k-2}{s}f^{(k-s-2)}g^{(s)}\right) \\
&\leqq \sum_{s=0}^{k-2}\binom{k-2}{s}\bar{V}_2(f^{(k-s-2)}g^{(s)}) \\
&\leqq 2(b-a)\sum_{s=0}^{k-2}\binom{k-2}{s}\bar{V}_2(f^{(k-s-2)})\bar{V}_2(g^{(s)}), \text{ using (11)} \\
&= 2(b-a)\sum_{s=0}^{k-2}\binom{k-2}{s}\bar{V}_{k-s}(f)\bar{V}_{s+2}(g) \\
&\leqq 2(b-a)\sum_{s=0}^{k-2}\binom{k-2}{s}(b-a)^s\bar{V}_k(f)\cdot(b-a)^{k-s-2}\bar{V}_k(g), \\
&\qquad\qquad \text{using (10)}
\end{aligned}$$

$$= 2(b-a)^{k-1}\bar{V}_k(f)\bar{V}_k(g)\sum_{s=0}^{k-2}\binom{k-2}{s}$$
$$= 2^{k-1}(b-a)^{k-1}\bar{V}_k(f)\bar{V}_k(g)\,, \quad \text{as required for (14).}$$

To obtain (15) we employ (3).

Combining Theorems 3 and 4 gives

THEOREM 5. *If f and g belong to $BV_k^*[a, b]$, $k \geqq 1$, then $fg \in BV_k^*[a, b]$, and*

$$V_k(fg) \leqq \alpha_k V_k(f) V_k(g)\,,$$

where $\alpha_k = 2^{k-1}(k-1)!\,(b-1)^{k-1}$.

Our final theorem is now apparent.

THEOREM 6. *If k is a positive integer, then $BV_k^*[a, b]$ is a commutative Banach algebra under the norm $||\cdot||_k^*$, where*

$$||f||_k^* = \alpha_k V_k(f)\,,$$

and $\alpha_k = 2^{k-1}(k-1)!\,(b-a)^{k-1}$.

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