PACIFIC JOURNAL OF MATHEMATICS
Vol. 86, No. 1, 1980

ON A REMARKABLE CLASS OF POLYHEDRA IN COMPLEX HYPERBOLIC SPACE

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Dedicated to Gerhard Hochschild on the occasion of his 65th birthday

The Selberg, Piatetsky-Shapiro conjecture, now established by Margoulis, asserts that an irreducible lattice in a semi-simple group G is arithmetic if the real rank of G is greater than one. Arithmetic lattices are known to exist in the real-rank one group SO$(n, 1)$, the motion group of real hyperbolic n-space, for $n \leqq 5$. These examples due to Makarov for $n = 3$ and Vinberg for $n \leqq 5$ are defined by reflecting certain finite volume polyhedra in real hyperbolic n-space through their faces. The purpose of the present paper is to show that there are also nonarithmetic lattices in the real-rank one group PU(2, 1), the group of motions of complex hyperbolic 2-space which can be defined algebraically and leads to remarkable polyhedra. This serves to help determine the limits of the Selberg, Piatetsky-Shapiro conjecture. The analysis of these polyhedra also leads to the first known example of a compact negatively curved Riemannian space which is not diffeomorphic to a locally symmetric space.

This paper arose out of an attempt to determine the limits of validity of the Selberg, Piatetsky-Shapiro conjecture on the arithmeticity of lattice subgroups. In 1960 A. Selberg conjectured that apart from some exceptional G, an irreducible noncocompact lattice subgroup Γ of a semi-simple group G is arithmetic ("irreducible" in the sense that Γ is not commensurable with a direct product of its intersections with factors of G). Later Piatetsky-Shapiro conjectured: An irreducible lattice of a semi-simple group G is arithmetic if $\boldsymbol{R}$-rank $G > 1$.

The Selberg, Piatetsky-Shapiro conjecture was settled affirmatively by G. A. Margoulis in the striking paper that he submitted to the 1974 International Congress of Mathematicians in Vancouver.

The simple groups of $\boldsymbol{R}$-rank 1 are (up to a local isomorphism)

$$\mathrm{SO}(n, 1), \mathrm{SU}(n, 1), \mathrm{SP}(n, 1), F_4$$

which act as isometries on the hyperbolic space

$$\boldsymbol{R}h^n, \boldsymbol{C}h^n, \boldsymbol{H}h^n, \boldsymbol{O}h^2$$

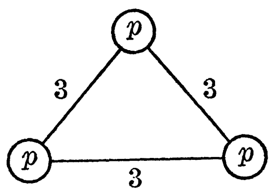
over the real numbers $\boldsymbol{R}$, the complex numbers $\boldsymbol{C}$, the quaternions (or Hamiltonians) $\boldsymbol{H}$, the octonians (or Cayley numbers) $\boldsymbol{O}$ respectively. Nonarithmetic lattices in SO(2, 1)$(=\mathrm{SL}_2(\boldsymbol{R})/\pm 1)$ have been known for a long time. Nonarithmetic lattices in $\mathrm{SO}(n, 1)$ for $n = 3$ where first found by V. S. Makarov (in 1965) and shortly thereafter E. B. Vinberg initiated an extensive investigation which turned up nonarithmetic lattices in $\mathrm{SO}(n, 1)$ for $n \leqq 5$. Both the Makarov and Vinberg examples are defined by reflecting a polyhedron F of finite volume in $\boldsymbol{R}h^n$ in its $(n-1)$-dimensional faces, which are pointwise fixed under the reflections. The $(n-1)$-dimensional faces thus must lie on geodesic subspaces of $\text{codimension}_{\boldsymbol{R}}$ 1. The group Γ generated by the reflections in the codimension-1 faces of F is a discrete group of isometries on $\boldsymbol{R}h^n$ and has F as a fundamental domain if and only if all the dihedral angles of F are of the form $\pi/\text{integer}$.

Straightforward generalization of this method for finding lattice subgroups of isometries on the other hyperbolic spaces is blocked by the fact that only in $\boldsymbol{R}$-hyperbolic spaces do there exist codimension 1 geodesic subspaces. Thus it is not a priori clear what to take as the bounding surfaces of a polyhedron F out of which we are to construct a group Γ of isometries with F as fundamental domain.

The principal result of this paper is the construction of a class of such polyhedra F in $\boldsymbol{C}h^2$. The guiding principle in the discovery of F is the exploitation of symmetry. The polyhedron F depends on two parameters, (p, t) where $p = 3, 4, 5$, and $|t| < 3(1/2 - 1/p)$. To each $F(p, t)$, there corresponds an infinite subgroup Γ of $U(2, 1)$ generated by three $\boldsymbol{C}$-reflections of order p. For some values of the parameter, the polyhedron F is stabilized by a subgroup of Γ of order 3. For only a finite number of values of the parameters is the group Γ discrete. Whenever Γ is discrete, it is a lattice subgroup.

The main theorems proved in this paper are:

THEOREM A (cf. § 17.3). *There exist in* $\mathrm{PU}(2, 1)$ *nonarithmetic lattices generated by* $\boldsymbol{C}$*-reflections of order* 3, 4, *or* 5. *Up to an isometry, any such lattice with Coxeter diagram*



and phase shift φ, $\varphi^3 = \exp \pi i t$, *is given by the seven values*

$$(p, t) = (3, 5/42), (3, 1/12), (3, 1/30)$$
$$(4, 3/20), (4, 1/12)$$
$$(5, 1/5), (5, 11/30)\ .$$

The noncocompact lattices $\Gamma(p, t)$ are arithmetic.

THEOREM (cf. § 19). *For every rational value of t, with $|t| < 1/2 - 1/p$, there is a complex analytic manifold $Y(p, t)$ and a canonical $\Gamma(p, t)$ map $\pi: Y(p, t) \to \boldsymbol{C}h^2$ such that $\Gamma(p, t)$ operates discontinuously and holomorphically on $Y(p, t)$.*

The quotient of Y by a torsion-free subgroup of Γ leads to the first known example of a compact negatively curved Riemannian space M which is not diffeomorphic to a locally symmetric space (cf. [8]). M is in fact an algebraic surface and is negatively curved with respect to a Kaehler metric.

The major effort in proving Theorem A once the groups $\Gamma(p, t)$ have been defined, is to decide when $\Gamma(p, t)$ is discrete. Apart from § 4 which gives a criterion for the arithmeticity of a lattice, most of § 3 to § 17 is aimed at the discreteness question. The choice of the complex analytic structure in Theorem B with respect to which Γ operates discontinuously and holomorphically depends on the results in § 6 (though not on § 6.5) and § 18. In particular, explicit information (cf. § 18.2) about the star of each vertex in the polyhedral space Y is exploited.

In anticipation of generalizing the construction given here to $n = 3$ and 4, the results in § 3 on spinal surfaces are presented for general n.

Part of the results contained in this paper were announced in [7].

A large part of this paper is devoted to computations. At an early stage of the investigation of discreteness, I profited greatly from computer exploration. Without the computer it would have been very difficult to recognize how much more complicated is the fundamental domain of the group of order 72 generated by two C-reflections with Coxeter diagram

$$\textcircled{3} \overset{4}{\text{———}} \textcircled{3}$$

than the fundamental domain for the group of order $24(p/6 - p)^2$ generated by two C-reflections with Coxeter diagram

$$\textcircled{p} \overset{3}{\text{————}} \textcircled{p}$$

Thus the discovery of the family $F(p, t)$ owes much to the vastly wider exploration which the computer permits. Once the target began to be discerned, it became possible to verify all the remarkable properties of the family $F(p, t)$ by a combination of geometric and algebraic methods. The account here is in the strictly logical (rather than psychological) sense totally independent of the computer.

I take pleasure in acknowledging my debt and gratitude to my associate and students at Yale who programmed the algorithms described in § 7, 8 and other related capabilities. In chronological order of their assistance, I was enormously helped by Dr. Sidnie Feit, Alex Feingold, James Cogdell, Daniel Bar-Yacov, and Edna Bar-Yacov.

2. Algebraic preliminaries.

2.1. *C-reflections.*

Let V be an n-dimensional vector space over $\boldsymbol{C}$. A $\boldsymbol{C}$-reflection in V is a linear map $R: V \to V$ of finite order with $n-1$ eigenvalues equal to 1. A $\boldsymbol{C}$-reflection can be expressed in the form $x \to x + \beta(x)e$ where β is a linear function and $1 + \beta(e)$ is a root of unity other than 1.

Let H be a nondegenerate hermitian form on V and denote $H(v, w)$ by $\langle v, w \rangle$ for all $v, w \in V$. Given $e \in V$ with $\langle e, e \rangle = 1$ and a positive integer p, we denote by $R_{e,p}$ the $\boldsymbol{C}$-reflection

$$x \longrightarrow x + (\xi - 1)\langle x, e \rangle e , \quad \xi = \exp 2\pi\sqrt{-1}/p .$$

$R_{e,p}$ fixes each point in $e^{\perp} = \{x \in V; \langle x, e \rangle = 0\}$ and has order p. Clearly $R_{e,p}$ preserves H.

Suppose $\{e_1, \cdots, e_n\}$ is a linearly independent set of unit vectors and let $R_i = R_{e_i,p_i}$. Set $\Gamma = \{\{R_1, R_2, \cdots, R_n\}\}$, the group generated by the set $\{R_1, \cdots, R_n\}$. Then $\Gamma \subset U(H)$, the unitary group of the hermitian form H.

The hermitian form H need not be positive definite. However if Γ is finite and irreducible, then H must be positive definite. For by Schur's lemma, an irreducible group stabilizes a unique hermitian form up to scalar factor; on the other hand, a finite group preserves a definite hermitian form. Since $H(e_i, e_i) = 1$, H is positive definite if Γ is irreducible. We will see in § 2.3 how to drop this last hypothesis.

2.2. *Finite groups generated by two C-reflections.*

Let $\{e_1, e_2\}$ be a base of the $\boldsymbol{C}$-vector space V and let H be a hermitian form on V. The condition that $\{R_{e_1,p_1}, R_{e_2,p_2}\}$ generate a finite group Γ may be determined as follows.

Consider the 1 dimensional projective space $\boldsymbol{CP}^1$ of one-dimensional subspaces of V; let $\pi: V - \{0\} \to \boldsymbol{CP}^1$ denote the canonical projection. Let R_i denote the action of R_{e_1,p_1} on CP^1, and denote $\pi(e_i^\perp)$ by $^*e_i^\perp (i = 1, 2)$.

CP^1 may be identified with the standard 2-sphere S^2, and Γ operates on S^2 by rotations. Thus Γ is finite if and only if the group $\{R_1, R_2\}$ generated by rotations of angle $2\pi/p_i$ at $^*e_i^\perp$ is finite. This is the case if and only if there is a γ in Γ such that the geodesic triangle with base $\gamma(^*e_1^\perp)^*e_2^\perp$ and base angles π/p_1, π/p_2 has as third angle $2\pi/q$ where

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{2}{q} \geqq 1 \text{ and } q \text{ is even if } p_1 \neq p_2 . \tag{2.2.1}$$

From spherical trigonometry the length of the base $^*e_1^\perp {}^*e_2^\perp$ in a geodesic triangle with angles π/p_1, π/p_2, $2\pi/q$ is given by

$$\cos d(^*e_1^\perp, {}^*e_2^\perp) = \frac{\cos \pi/p_1 \cdot \cos \pi/p_2 + \cos 2\pi/q}{\sin \pi/p_1 \cdot \sin \pi/p_2} .$$

On the other hand, the inner-product on V is related to the metric on S^2 by the formulae

$$|\langle e_1, e_2\rangle| = \cos \sigma,\ d(^*e_1^\perp, {}^*e_2^\perp) = 2\sigma .$$

Therefore, by the half-angle formula

$$|\langle e_1, e_2\rangle| = \left(\frac{\cos (\pi/p_1 - \pi/p_2) + \cos 2\pi/q}{2 \sin \pi/p_1 \cdot \sin \pi/p_2}\right)^{1/2} . \tag{2.2.2}$$

Let φ be any complex number with $|\varphi| = 1$ and let $H(\varphi)$ denote the hermitian form on C^2 given by

$$\langle e_i, e_i\rangle = 1, \quad (i = 1, 2)$$
$$\langle e_1, e_2\rangle = \alpha\varphi$$

where α is given by (2.2.2). Let $\Delta(\varphi)$ denote the determinant of the matrix $(\langle e_i, e_j\rangle)$. Then

$$\Delta(\varphi) = 1 - \alpha^2 = \sin^2\sigma > 0$$

if H is nondegenerate. We note that $\Delta(\varphi)$ is independent of φ. Let $\Gamma(\varphi)$ denote the group $\{\{R_{e_1,p_1}, R_{e_2,p_2}\}\}$ corresponding to H_φ. Replacing e_1 by $e_1\varphi$ does not change R_{e_1,p_1} and provides an isometry of $H(1)$ to $H(\varphi)$. Thus $\Gamma(\varphi)$ is independent of φ.

Set $R_i = R_{e_i,p_i} (i = 1, 2)$. Suppose that H satisfies (2.2.1) and (2.2.2). Then the group $\Gamma = \{\{R_1, R_2\}\}$ has the relations

$$R_1^{p_1} = 1,\ (R_2)^{p_2}$$
$$(R_1R_2)^{q/2} = (R_2R_1)^{q/2} .$$

If q is odd, the second relation signifies

$$R_1R_2\cdots R_1 = R_2R_1\cdots R_2$$

(cf. [4]). Moreover, the center of the group Γ is cyclic and generated by

$$\begin{cases}(R_1R_2)^q \text{ if } q \text{ is odd}\\ (R_1R_2)^{q/2} \text{ if } q \text{ is even}\end{cases} \tag{2.2.3}$$

(this corresponds to the fact that on S^2, R_1R_2 rotates about the vertex e_0 opposite ${}^*e_1^\perp {}^*e_2^\perp$ through an angle of $4\pi/q$).

Let s denote the reciprocal of the spherical excess of the geodesic triangle $e_0^* e_1^* e_2^\perp$ i.e., $s^{-1} = p_1^{-1} + p_2^{-1} + 2q^{-1} - 1$ then the order of the center is (cf. [4])

$$\begin{cases}\dfrac{2s}{q}\ , & q \text{ odd}\\[2mm] \dfrac{4s}{q}\ , & q \text{ even}\end{cases} \tag{2.2.4}$$

and the order of Γ is given by

$$\#\Gamma = \frac{8s^2}{q}\ . \tag{2.2.5}$$

2.3. *Coxeter diagrams, phase shifts.*

To each finite group generated by C-reflections, Coxeter has associated a diagram. In the case of a group generated by two reflections $\{R_{e_1,p_1}, R_{e_2,p_2}\}$ with

$$\langle e_1, e_2\rangle = \alpha\varphi$$

with α given by (2.2.2) and (2.2.1) satisfied, the Coxeter diagram consists of two nodes and a line

$$(p_1)\overset{q}{\text{―――――}}(p_2)$$

if $q > 2$; if $q = 2$, no line is drawn joining the nodes. This diagram determines $\{R_{e_1,p_1}, R_{e_2,p_2}\}$ uniquely up to an isomorphism.

To a group generated by n-reflections every two of which generate a finite group, one associates a diagram made up of n nodes, attaching p_i the order of the reflection R_i to the ith node; the nodes i, j are joined by a line labeled q_{ij} if $R_i = R_{e_i,p_i}$, $R_j = R_{e_j,p_j}$ and the q_{ij} (resp. $q_{ij}/2$) is the lowest power p of R_iR_j which is in the center of $\Gamma_{ij} = \{\{R_i, R_j\}\}$ if p is odd (resp. even). Conversely, to any diagram $\mathscr{D}$

$$\cdots (p_i)\overset{q_{ij}}{\text{―――――}}(p_j) \cdots$$

with n nodes, satisfying (2.2.1), one associates a family of groups

Γ generated by n-reflections as follows:

In $\boldsymbol{C}^n$, let $e_1, e_2, \cdots, e_n$ be the standard base. Define the hermitian form H:

$$\begin{aligned} \langle e_i, e_j\rangle &= -\alpha_{ij}\varphi_{ij} \qquad i \neq j \\ \langle e_i, e_i\rangle &= 1 \end{aligned} \tag{2.3.2}$$

where

$$\alpha_{ij} = \left(\frac{\cos(\pi/p_1 - \pi/p_2) + \cos 2\pi/q_{ij}}{2\sin\pi/p_1\cdot\sin\pi/p_2}\right)^{1/2}, \quad |\varphi_{ij}| = 1$$

and define $R_i = R_{e_i,p_i}(i = 1, \cdots, n)$. For any i, j, the group $\Gamma_{ij} = \{\{R_i, R_j\}\}$ is finite and independent of the choice of φ_{ij}, up to an isomorphism. More generally, if the graph of $\mathscr{D}$ is a tree, then one can replace e_i by $\varphi_i e_i$ $(i = 1, \cdots, n)\,|\varphi_i| = 1$, so as to get an isomorphism between any two groups Γ in the above family. For any loop $i_1 i_2\cdots i_k$ in the graph, the product $\varphi_{i_1 i_2}\cdot\varphi_{i_2 i_2}\cdots\varphi_{i_k i_1}$ is invariant under such changes, and indeed two groups Γ with the same diagram need not be isomorphic. The set $\{\varphi_{ij};\ i \neq j,\ i, j = 1, \cdots, n\}$ is called the *phase* shift of the hermitian form H. Two phase shifts define isomorphic groups Γ if the products of phase shifts over all closed loops are equal. A phase shift is called rational if it is a root of unity.

The data ($\mathscr{D}$, phase shifts) determines a unique hermitian form H and a unique group Γ generated by $\boldsymbol{C}$-reflections; we sometimes denote Γ by $\Gamma(\mathscr{D}, H)$.

(2.3.3) *If the Coxeter diagram of a group Γ is connected and the hermitian form H is nondegenerate, then the group operates irreducibly on the underlying vector space.*

The proof is essentially the same as for Coxeter groups of $\boldsymbol{R}$-reflections. (See Bourbaki, Groupes$\cdots$Lie, Ch. 5, § 4.7.)

Unlike the case of $\boldsymbol{R}$-reflection groups, groups with different Coxeter diagrams may be isomorphic.

As a consequence of (2.3.3), if the hermitian form H of the data, ($\mathscr{D}$, H) is nondegenerate and the group Γ is finite, then H is positive definite. For Γ is a direct sum of irreducible groups corresponding to the connected components on the diagram $\mathscr{D}$.

On the summand corresponding to each component, H is positive definite by the result in § 2.1. Hence H is positive definite.

2.4. *Finite groups generated by $\boldsymbol{C}$-reflections.*

Finite subgroups of PGL(n, $\boldsymbol{C}$) generated by $\boldsymbol{C}$-reflections were classified near the turn of the century (cf. G. Bagnera, I gruppi finiti di tranzformazioni lineari dello spazio che contegono omologie, Rend. Circ. Mat. Palermo, 19 (1905), pp. 1–56; C-reflections on projective space were named homologies in those days). Subsequently the classification was redone by G. L. Shephard and J. A. Todd in 1953 (see [10]) and then by Coxeter in 1966 (see [4]). Coxeter has listed the group diagrams.

Finite linear groups Γ in n-variables generated by C-reflections are characterized by the property:

> Γ possesses a set of n algebraically independent invariant forms $I_1, I_2, \cdots, I_n$ of degrees $m_1+1, m_2+1, \cdots, m_n+1$ such that
>
> (2.4.1) $$\prod_{i=1}^{n}(m_i+1) = \sharp\Gamma$$
>
> (see [10] or (Bourbaki, Groupes$\cdots$Lie, Ch. V § 5.5) for this and other properties).

(2.4.2) If Γ is a finite linear group generated by C-reflections $\{R_1, \cdots, R_n\}$, then any C-reflection $R \in \Gamma$ is conjugate to a power of some $R_i (i=1, \cdots, n)$: (See [3], Lemma 4.11 (iii).)

Given complex reflections R_{e_1,p_1} and R_{e_2,p_2} on a vector space V, they generate a group Γ' which stabilizes $W = Ce_1 + Ce_2$ and fixes each point of $e_1^\perp \cap e_2^\perp$. If

$$\det\begin{pmatrix}\langle e_1, e_1\rangle & \langle e_1, e_2\rangle\\ \langle e_2, e_1\rangle & \langle e_2, e_2\rangle\end{pmatrix} = 1 - |\langle e_1, e_2\rangle|^2 > 0$$

the restriction of the hermitian form H to W is positive definite. Hence Γ' is faithfully represented in the compact unitary group of the restriction of H to W. Thus Γ' is discrete if and only if it is finite.

One can use this observation to determine that some groups are not discrete. For this purpose, one needs a list of all possible *admissible* values of $|\langle e_1, e_2\rangle|$ if R_{e_1,p_1} and R_{e_2,p_2} are to lie in a compact group and be discrete. Such reflections lie in groups Γ generated by two C-reflections, such that $|\langle e_1, e_2\rangle|$ is given by the number α on the right hand side of (2.2.2). For a finite linear group Γ generated by C-reflections, any reflection is conjugate to a power of a generator (cf. (2.4.2)). Thus the admissible values can be read off from the list of $\{|\langle e_1, \gamma e_2\rangle|;\ \gamma \in \Gamma\}$ corresponding to Coxeter diagrams with two nodes. We list only the connected diagrams with $p_1 = p_2$ and values of $|\langle e_1, \gamma e_2\rangle| \neq 0, 1$.

(2.4.3)

$\mathscr{D}$	$\lvert\langle e_1, \gamma e_2\rangle\rvert$
② $\overset{q}{\text{———}}$ ②	$\cos k\pi/q$
③ $\overset{3}{\text{———}}$ ③	.577
④ $\overset{3}{\text{———}}$ ④	.707
⑤ $\overset{3}{\text{———}}$ ⑤	.851, .526
③ $\overset{4}{\text{———}}$ ③	.816, .577
③ $\overset{5}{\text{———}}$ ③	.934, .356, .951, .309

2.5. *Lattice subgroups.*

Let G be a locally compact group. A *lattice* Γ in G is a discrete subgroup such that G/Γ has finite Haar measure. Two subgroups Γ and Γ' of G are called *commensurable* if $\Gamma \cap \Gamma'$ is of finite index in both Γ and Γ'.

If G is a semi-simple matrix group having no compact normal subgroup of positive dimension and Γ is a lattice in G, then Γ is dense in the Zariski-topology of G; that is, any polynomial in the matrix entries of G which vanishes on Γ vanishes on G ([1], cf. [9]). In particular, if G acts irreducibly on the underlying vector space, so does Γ. This last remark holds equally well if G is *reductive* (i.e., G is a product of a semi-simple group and a commuting abelian subgroup), and G acts (absolutely) irreducibly on the underlying complex vector space.

If G is a Lie group, we denote by Ad G the representation of G on its Lie algebra G induced by $x \to gxg^{-1}$.

Suppose now that G is a reductive Lie group of matrices and Γ a lattice subgroup such that Ad Γ is Zariski-dense in Ad G. Let k denote the field generated over the field $\boldsymbol{Q}$ of rational numbers by $\mathrm{Tr}\,\mathrm{Ad}\,\Gamma = \{\mathrm{Tr}\,\mathrm{Ad}\,\gamma; \gamma \in \Gamma\}$. Let T denote the function $g \to \mathrm{Tr}\,\mathrm{Ad}\,g$ on G. Let $\mathscr{L}$ denote the $\boldsymbol{C}$-linear span of the functions

$$x \longrightarrow T(x\gamma), \qquad x \in G, \gamma \in \Gamma .$$

The group G acts on the space $\mathscr{F}$ of all polynomial functions on G via the translations:

$$f \longrightarrow g \cdot f, \qquad f \in \mathscr{F}, g \in G$$

where we define $(g \cdot f)(x) = f(xg)$. Clearly $\mathscr{L} \subset \mathscr{F}$ and $\Gamma \cdot \mathscr{L} \subset \mathscr{L}$. Since $T(x\gamma) = \mathrm{Tr}\,(\mathrm{Ad}\,x\gamma) = \mathrm{Tr}\,(\mathrm{Ad}\,x\,\mathrm{Ad}\,\gamma)$ and $\mathrm{Ad}\,\Gamma$ is Zariski-dense in G, it follows that $G \cdot \mathscr{L} \subset \mathscr{L}$. The set of functions $\{\gamma \cdot T; \gamma \in \Gamma\}$ span $\mathscr{L}$ and from among them we select a base $B = \{\gamma_1 \cdot T, \cdots, \gamma_n \cdot T\}$ (note: $n \leqq (\dim G)^2$ and is therefore finite).

In the representation ρ of G on $\mathscr{L}$, the matrix of $\rho(\gamma)$ with respect to the base B is k-valued.

Proof. (cf. [5]). For any $c \in G$, let $\hat{c}$ denote the evaluation map $\mathscr{F} \to \boldsymbol{C}$ given by $f \to f(c)$. $\hat{c}$ is linear. Since $\operatorname{Ad} \Gamma$ is Zariski dense in $\operatorname{Ad} G$, $\{\hat{\gamma}; \gamma \in \Gamma\}$ is a separating family of linear functions on $\mathscr{L}$. Hence we can select $\beta_1, \cdots, \beta_n$ in Γ so that $\{\widehat{\beta}_1, \cdots, \widehat{\beta}_n\}$ separates $\mathscr{L}$. For any $\gamma \in \Gamma$ and for fixed i,

$$\begin{aligned} \widehat{\beta}_k(\gamma \cdot \gamma_i T) &= \widehat{\beta}_k(\sum_j c_{ij} \gamma_j \cdot T) \\ T(\beta_k \gamma \gamma_i) &= \sum_{j=1} c_{ij} T(\beta_k \gamma_j) \qquad k = 1, \cdots, n \,. \end{aligned}$$

This gives a system of n equations for the n unknowns $c_{i1}, \cdots, c_{in}$, with all coefficients in k. Hence the solutions are in k. Since $\rho(\gamma) = (c_{ij})$, our assertion is proved.

The representation ρ provides a faithful representation of $\operatorname{Ad} G$. The Zariski-closure G^* of $\rho(G)$ in the full $n^2 \times n^2$ matrix algebra coincides with the Zariski-closure of $\rho(\Gamma)$. Let I denote the ideal of polynomial functions on $\operatorname{Hom}_{\boldsymbol{C}}(\mathscr{L}, \mathscr{L})$ which vanish on G^*. Since $\rho(\Gamma)$ is Zariski-dense in G^* and the matrix entries of $\rho(\Gamma)$ are in k, the ideal I has a base of polynomial functions with coefficients in k. Thus (strictly by definition),

(2.5.1) *The algebraic group G^* is defined over the field k.*

3. Geometric preliminaries.

3.1. *The ball B and its isometries.*

Let V be an $n+1$-dimensional vector space over $\boldsymbol{C}$ on which we fix a hermitian form H of signature $(n, 1)$ i.e., n plus signs and 1 minus sign. The unitary group of H operates on the projective space $\boldsymbol{P}_n$ of one-dimensional $\boldsymbol{C}$-subspaces of V; we denote the resulting group on $\boldsymbol{P}_n$ by $PU(H)$. Set $\langle p, q \rangle = H(p, q)$ and

$$\begin{aligned} V^- &= \{p \in V; \langle p, p\rangle < 0\} \\ V^0 &= \{p \in V; \langle p, p\rangle = 0\} \\ V^+ &= \{p \in V; \langle p, p\rangle > 0\} \,. \end{aligned}$$

Let π denote the canonical map of $V - \{\{0\}\}$ onto $\boldsymbol{P}_n$. Set

$$B = \pi(V^-), \ \operatorname{Aut} B = \text{the restriction of } PU(H) \text{ to } B \,.$$

On B one can define a Riemannian positive definite infinitesimal metric invariant under $\operatorname{Aut} B$ by the formula (cf. [6])

$$ds^2 = -\frac{1}{\langle p, p\rangle^2} \begin{vmatrix} \langle dp, dp\rangle & \langle dp, p\rangle \\ \langle p, dp\rangle & \langle p, p\rangle \end{vmatrix} \,. \tag{3.1.1}$$

Although there are many choices for p which map to the same $\pi(p)$, the formula above yields a well-defined metric. The *angle* between two tangent vectors to B at a point $\pi(p)$ is defined by the usual associated real-valued inner product formula

$$(x, y) = -\mathrm{Re}\,\frac{1}{\langle p, p\rangle^2}\begin{vmatrix}\langle x, y\rangle & \langle x, p\rangle \\ \langle p, y\rangle & \langle p, p\rangle\end{vmatrix},$$

thus

$$\cos \sphericalangle(x, y) = \frac{(x, y)}{[(x, x)(y, y)]^{1/2}} \tag{3.1.2}$$

In particular, if x and y are nonzero tangent vectors to B at a point $\pi(p)$ with $y = \alpha x$, $\alpha \in \boldsymbol{C}$, then $\sphericalangle(x, y) = \arg\alpha$.

The distance between two points $\pi(p)$ and $\pi(q)$ in B is given by the formula (cf. [6])

$$\cosh d(\pi(p), \pi(q)) = \frac{|\langle p, q\rangle|}{(\langle p, p\rangle\langle q, q\rangle)^{1/2}}. \tag{3.1.3}$$

For any vector $\boldsymbol{C}$-subspace W of V with $\dim_{\boldsymbol{C}} W = k + 1$ on which the signature of the restriction of H to W is $(k, 1)$, the intersection $W \cap V^-$ is nonempty. $\pi(W \cap V^-)$ is called a $\boldsymbol{C}$-k-plane in B; and W is called its *preimage in* V. In case $k = 1$, $\pi(W \cap V^-)$ is called a $\boldsymbol{C}$-*line* in B. A $\boldsymbol{C}$-k-plane is clearly a geodesic subspace of B for $k = 0, 1, \cdots, n$.

By the principal axis theorem, there is a base of $\boldsymbol{C}$-linear functions on V such that

$$H = -|x_0|^2 + |x_1|^2 + \cdots + |x_n|^2 .$$

Such a base β is called a *standardizing coordinate system*; it is not unique, any other differing from it by a unitary transformation of H. The associated nonhomogeneous coordinate system $\{x_1/x_0, \cdots, x_n/x_0\}$ on the complement of $x_0 = 0$ in $\boldsymbol{P}_n$ is called a *standard nonhomogeneous* system. In the nonhomogeneous coordinates $x_1/x_0, \cdots, x_n/x_0$ of β, B becomes the open unit ball in $\boldsymbol{C}^n$:

$$\left(\frac{x_1}{x_0}\right)^2 + \cdots + \left(\frac{x_n}{x_0}\right)^2 < 1 .$$

The point o of B defined by the n equations

$$0 = x_1 = x_2 = \cdots = x_n$$

is called the *center* of the standardizing and of the standard nonhomogeneous coordinates. Every point of B is the center of some standardizing β.

The stabilizer $(\mathrm{Aut}\, B)_0$ of the center o of β is the image in $\mathrm{Aut}\, B$ of the subgroup of $\mathrm{PU}(H)$ given in any standardizing coordinates by

$$x_0' = \alpha x_0$$

$$\begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix} = U \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

where ${}^t\bar{U} = U^{-1}$. The group $U(H)$ is transitive on $\pi(V^-)$ and the subgroup $(\mathrm{Aut}\, B)_o$ is transitive on the set of all $\boldsymbol{C}$-lines through o. It follows at once that $\mathrm{Aut}\, B$ is transitive on all pointed lines i.e., pairs (p, L) with p a point on the $\boldsymbol{C}$-line L of B.

The geodesic subspaces of B having constant negative curvature are of two distinct kinds. On the one hand by a direct computation one finds that all $\boldsymbol{C}$-lines have constant curvature -2. On the other hand, given in V any $\boldsymbol{C}$-linearly independent vectors $f_0, \cdots, f_k$ such that $\langle f_i, f_j\rangle \in \boldsymbol{R}(i, j = 0, \cdots, k)$, the subspace $\pi([\boldsymbol{R}f_0 + \cdots + \boldsymbol{R}f_k] \cap V^-)$ is called an $\boldsymbol{R}$-k-plane if it is nonempty; its sectional curvature is $-1/2$. If $k = 1$, we call the $\boldsymbol{R}$-1-plane an $\boldsymbol{R}$-line.

REMARK 1. The ratio of 4 between the curvatures of $\boldsymbol{C}$-lines and $\boldsymbol{R}$-planes arises from the existence of a restricted $\boldsymbol{R}$-root 2α. Any $\boldsymbol{C}$-line is isometric to real hyperbolic 2-space i.e., the Poincare disc.

REMARK 2. With respect to any nonhomogeneous coordinate system on B, a $\boldsymbol{C}$-k-plane is the intersection with B of a k-plane; or as we shall say for short, a $\boldsymbol{C}$-k-plane is linear with respect to any nonhomogeneous coordinate system. By contrast, an $\boldsymbol{R}$-k-plane $\pi([\boldsymbol{R}f_0 + \cdots + \boldsymbol{R}f_k] \cap V^-)$ is the intersection with B of a k-dimensional $\boldsymbol{R}$-linear subspace of $\boldsymbol{C}^n$ relative to nonhomogeneous coordinates $x_1/x_0, \cdots, x_n/x_0$ provided that $f_0, \cdots, f_k$ lies in the $\boldsymbol{R}$-linear span of the dual base to $x_0, \cdots, x_n$. Thus, with respect to a standard nonhomogeneous coordinate system, an $\boldsymbol{R}$-k-plane need not be *linear*; for example, a general $\boldsymbol{R}$-line is a circular arc meeting the boundary of A orthogonally, where A is the unique $\boldsymbol{C}$-line containing it.

REMARK 3. Any geodesic line P in B is an $\boldsymbol{R}$-line. For let p_0, p_1 be vectors in V^- such that $\pi(p_0)$ and $\pi(p_1)$ are distinct points of P. Set $M = \boldsymbol{C}p_0 + \boldsymbol{C}p_1$ and $L = \pi(M \cap V^-)$. Then L is the unique $\boldsymbol{C}$-line containing P. Set $e_0 = \langle p_0, p_0\rangle^{-1}p_0$ and $e_1 \in M \cap (\boldsymbol{C}e_0)^\perp$ where $\perp$ denotes the orthogonal subspace of V with respect to H. Then

$P = \pi((e_0 + \boldsymbol{R}e_1) \cap V^-)$ and is thus an $\boldsymbol{R}$-line. We note that $\langle p_0, p_1\rangle \neq 0$ since the restriction of the hermitian form H to M is of signature $(1, 1)$. Since $\pi(p_0) = \pi(\langle p_0, p_1\rangle^{-1}p_0)$, we find that $P = \pi(\langle p_0, p_1\rangle^{-1}p_0 + \boldsymbol{R}p_1)$ is the geodesic line through $\pi(p_0)$ and $\pi(p_1)$.

One can say that with respect to a standard nonhomogeneous coordinate z on a $\boldsymbol{C}$-line, an $\boldsymbol{R}$-line is linear with respect to $\operatorname{Re} z$ and $\operatorname{Im} z$ if and only if it passes through the origin.

REMARK 4. Inasmuch as $U(n)$ is transitive on all points of $\{z \in \boldsymbol{C}^n; |z| = 1\}$, it follows that $\operatorname{Aut} B$ is transitive on all pairs (p, P) with $p \in B$ and P an $\boldsymbol{R}$-line through p.

3.2. *Properties of an equidistant surface.*

Given two points $\pi(p_1)$ and $\pi(p_2)$ in B, the *equidistant surface* S of $\pi(p_1)$, $\pi(p_2)$ is by definition

$$S = \{\pi(x) \in B; d(\pi(x), \pi(p_1)) = d(\pi(x), \pi(p_2))\} .$$

The equation of S is easily obtained:

$$\cosh d(\pi(x), \pi(p_1)) = \cosh d(\pi(x), \pi(p_2))$$
$$\frac{|\langle x, p_1\rangle|}{(\langle x, x\rangle\langle p_1, p_1\rangle)^{1/2}} = \frac{|\langle x, p_2\rangle|}{(\langle x, x\rangle\langle p_2, p_2\rangle)^{1/2}} .$$

If moreover we normalize the points of V representing $\pi(p_1)$ and $\pi(p_2)$ so that $\langle p_1, p_1\rangle = \langle p_2, p_2\rangle$, the equation of S becomes simply

$$|\langle x, p_1\rangle| = |\langle x, p_2\rangle| .$$

The locus of this equation is easy to describe. Let M denote the 2 dimensional $\boldsymbol{C}$-linear subspace $\boldsymbol{C}p_1 + \boldsymbol{C}p_2$ of V. Let A denote the orthogonal complement to M in V with respect to H. We have $A \cap M = (0)$ since the restriction of H to M has signature $(1, 1)$ and thus H on A must be positive definite. $\pi(M \cap V^-)$ is a $\boldsymbol{C}$-line in B and $P = S \cap \pi(M \cap V^-)$ is the perpendicular bisector in the Poincare disc $\pi(M \cap V^-)$ of the points $\pi(p_1)$ and $\pi(p_2)$. Now for any x in V, if x satisfies $|\langle x, p_1\rangle| = |\langle x, p_2\rangle|$ so also do the points $x + A$. Hence S contains $\bigcup_{\pi(x) \in P} \pi(\boldsymbol{C}x + A) \cap B$.

Let $x_0, \cdots, x_n$ be standardizing coordinates with $x_2 = x_3 = \cdots = x_n = 0$ on M and $x_0 = x_1 = 0$ on A. With respect to nonhomogeneous coordinates $x_1/x_0, \cdots, x_n/x_0$ on $\boldsymbol{C}^n$ each of the planes $\pi(\boldsymbol{C}x + A)$ is parallel to $\pi(\boldsymbol{C}y + A)$ (i.e., they meet only in the line at infinity $x_0 = 0$). Thus we see that S contains the $\boldsymbol{R}$-$(2n-1)$-dimensional cylinder erected on the geodesic line P.

Consider now the orthogonal projection μ (with respect to the

Riemannian metric) of B onto the geodesic C-line $\pi(M \cap V^-)$. If $\pi(x) \in S$ it is easy to verify that in the Riemannian metric the C-$(n-1)$-plane $\pi(Cx + A) \cap B$ is orthogonal to the C-line $\pi(M) \cap B$. Hence $\mu(\pi(x)) \in P$. Thus $\pi(Cx + A) \cap B = \mu^{-1}\mu(\pi(x))$ for all $x \in S$,

$$S = \bigcup_{p \in P} \mu^{-1}(p)$$

and S is indeed the $(2n-1)$-dimensional cylinder elected on P with respect to both the Riemannian metric as well as nonhomogeneous standard coordinates on B. We call P the *spine* of S; for any $\pi(x) \in S$, we call each subset $\mu^{-1}\mu(\pi(x)) = \pi(Cx + A) \cap B$ the *slice* of S through $\pi(x)$ and it is denoted $A_{\pi(x)}$. An *S-function* is any function $c_0 + c_1 y_1 + \cdots + c_n y_n$ which is constant on all slices of S, where $y_1, \cdots y_n$ is a nonhomogeneous coordinate system on B.

The spine P of an equidistant surface S satisfies the two properties

(P1) P is a geodesic line contained in S.

(P2) Let L denote the unique C-line of B containing P, and let μ_L: $B \to L$ denote the orthogonal projection of B onto L. Then $S = \bigcup_{p \in P} \mu_L^{-1}(p)$.

The following simple lemma will be useful in showing that the spine of an equidistant surface is uniquely characterized by properties (P1) and (P2).

LEMMA 3.2.1. *Let C be a C-linear subspace of V which contains an open (in the Euclidean topology) subset 0 such that $\pi(V^- \cap 0)$ is a nonempty subset of the equidistant surface S. Let P be a subset of S satisfying properties* (P1) *and* (P2), *and let L be the C-line of B containing P. Then $\mu_L\pi(V^- \cap C)$ consists of a single point.*

Proof. Let M denote the preimage of L, i.e., the C-linear span of $\pi^{-1}(L)$. Then $\dim_C M = 2$. It is easy to verify that orthogonal projection μ_L of B onto L can be described as the map $\pi(v) \to \pi(\lambda(v))$, $v \in V^-$ where λ: $V \to M$ is the orthogonal projection of V onto M with respect to the hermitian form H. In particular, μ_L is a holomorphic map. On the other hand, the geodesic line P is an R-line and by hypothesis $\mu_L\pi(V^- \cap 0) \subset \mu_L(S) \subset P$. Consequently $\mu_L\pi$ is constant on a nonempty open subset of C and therefore is constant on $V^- \cap C$.

Let P be a geodesic line in an equidistant surface S satisfying property (P2) and let μ_L be as in (P2). For any $p \in P$, we call $\mu_L^{-1}(p)$ the *slice* at p with respect to P.

LEMMA 3.2.2. *Let P and P' be lines in the equidistant surface*

S satisfying (P1) *and* (P2). *Then $P = P'$.*

Proof. Let p and q be distinct points of P and let A_p, A_q be the slices at p, q respectively with respect to P. Let C_p denote the C-linear span of $\pi^{-1}(A_p)$. Let $\mu_{L'}$ denote the orthogonal projection of B onto the unique C-line L' containing P'. By Lemma 3.2.1, one sees that $\mu_{L'}(A_p) = \mu_{L'}\pi(C_p \cap V^-) = p'$, a single point of P'. Consequently $p' = A_p \cap P'$ and $A_p = A_{p'}$, the slice at p' with respect to P'. Similarly, $q' = A_q \cap P'$ and $A_q = A_{a'}$, the slice at q with respect to P and the slice at q' with respect to P'.

Consider now the quadrilateral p, q, q', p'. Since each slice meets P and P' at right angles, the sum of the four angles of the quadrilateral is $360°$. Hence it lies in a flat subspace of B (cf. [6]). But B has rank 1; that is, the maximal flat geodesic subspaces have R-dimension 1. Hence $P = P'$.

Terminology. An equidistant surface is hereafter called a *spinal* surface.

REMARK. It is clear that the coordinate system in (iii) can be selected to be standardizing.

Let $\operatorname{Aut} S$ denote the subgroup of B stabilizing the spinal surface S. Then $\operatorname{Aut} S$ stabilizes the spine P and indeed operates transitively on P. Thus S is the orbit of any of its slices under $\operatorname{Aut} S$, and $\operatorname{Aut} S$ is the direct product $\operatorname{Aut}_P A \times \operatorname{Aut} P$ where A denotes the slice through a point $p \in P$ and $\operatorname{Aut}_P A$ the subgroup of $\operatorname{Aut} B$ stabilizing A and fixing each point of the normal to A at p (i.e., the C-line L containing P), and $\operatorname{Aut} P$ is the one parameter subgroup of $\operatorname{Aut} B$ which operates transitively on P and moves each tangent vector to B at a point of P parallel to itself. Explicitly, if P is the R-line $\operatorname{Im} x_1/x_0 = 0$, $x_2 = \cdots = x_n = 0$ with respect to a standardizing coordinate system $x_0, \cdots, x_n$ then $\operatorname{Aut} P$ is given by the maps

$$\begin{cases} x_0' = ax_0 + bx_1, \; a, b \in R, \; a^2 - b^2 = 1 \\ x_1' = bx_0 + ax_1 \\ x_i' = x_i \qquad (i = 2, \cdots, n) \, . \end{cases}$$

In purely algebraic terms, $\operatorname{Aut} S$ is the centralizer in $\mathrm{PU}(H)$ of T_R where T is a maximal R-split torus in $PU(H)$, i.e., T_R is a one-parameter diagonalizable subgroup with positive real eigenvalues. In the terminology of transformation groups, a slice of S is a *slice* for $\operatorname{Aut} S$ operating on S.

LEMMA 3.2.3. *Let S be a spinal surface, P the spine of S, A a slice of S, $p = P \cap A$, L the unique C-line containing P, and let M, C denote the preimage subspaces in V of L, A respectively. Then*

(i) $M = (M \cap C) + C^{\perp}$ *and* $C = (M \cap C) + M^{\perp}$.

(ii) *Let x_0 and x_1 be independent linear functions on V vanishing on $M^{\perp}$ and let $y_1 = x_1/x_0$. Then y_1 is an S-function.*

(iii) *If moreover x_0 vanishes on $(M \cap C)^{\perp}$ and $x_1(M \cap C) = 0$ then the equation of S is $\arg y_1 = \theta$, where θ is a constant and the equations of P are:*

$$\arg y_1 = \theta,\ y_2 = y_3 = \cdots = y_n = 0$$

where $y_i = x_i/x_0$ and $x_2, \cdots, x_n$ is a base for the annihilator of $M (i = 2, \cdots, n)$.

Proof. (i) and (ii) are essentially a reformulation of previous remarks (cf. proof of Lemma 3.2.1). As for (iii), let e_0 be a nonzero element in $M \cap C$ and e_1 a nonzero element in $C^{\perp}$ such that $\pi(e_0 + e_1) \in P$. Then $P = \pi((e_0 + \mathrm{R}e_1) \cap V^-)$ and $y_1(\pi(e_0 + te_1)) = x_1(e_0 + te_1)/x_0(e_0 + te_1) = tx_1(e_1)/x_0(e_0) = tx_1(e_0 + e_1)/x_0(e_0 + e_1) = ty_1(\pi(e_0 + e_1))$. Thus on P, $\arg y_1 = \theta$, where $\theta = \arg y_1(\pi(e_0 + e_1))$. Since y_1 is an S-function, $\arg y_1 = \theta$ is the equation of S. The conclusion in (iii) is now evident.

LEMMA 3.2.4. *Let S_1 and S_2 each be spinal surfaces whose spines lie in the same C-line and intersect. Then S_1 and S_2 meet at a constant Riemannian angle at all points of $S_1 \cap S_2$.*

Proof. Let P_i denote the spine of S_i and let L denote the C-line containing $P_i (i = 1, 2)$. Since the geodesic joining two distinct points of B is unique, either $P_1 = P_2$ or $P_1 \cap P_2$ consists of a single point. We need only consider the latter case. Set $\{\mathcal{O}\} = P_1 \cap P_2$. Then $S_1 \cap S_2 = \mu_L^{-1}(\mathcal{O})$, a common slice. Set $A = S_1 \cap S_2$. Since $\mathrm{Aut}\, B$ acts transitively on (point, line) pairs, we can assume with no loss of generality that the spine of S_1 is given by the line $0 = x_1 = x_2 = \cdots = x_{n-1} \operatorname{Im} x_n/x_0 = 0$ with respect to a standardizing coordinate system $x_0, \cdots, x_n$ and that the common slice A is at $x_n/x_0 = 0$. Then by Lemma 3.2.3, both S_1 and S_2 are $\boldsymbol{R}$-linear in $x_1/x_0, \cdots, x_n/x_0$ coordinates.

Let C denote the preimage subspace of A in V and let H_C denote the restriction of the hermitian form H to C. For any $\pi(x) \in A$ there is a $g_C \in U(H_C)$ such that $g_C \mathcal{O} = \pi(x)$. Then g_C is given by equations

$$\begin{aligned} x'_0 &= a_{00}x_0 + a_{01}x_1 + \cdots + a_{-(n-1)}x_{n-1} \\ &\vdots \\ x'_{n-1} &= a_{(n-1)0}x_0 + \cdots + a_{(n-1)(n-1)}x_{n-1} . \end{aligned}$$

We complete these equations to get an element g of $U(H)$ by setting

$$x'_n = ax_n, \text{ where } a = a_{00}\,|a_{00}|^{-1} .$$

With respect to the nonhomogeneous coordinates $x_1/x_0, \cdots x_n/x_0$ on B, g acts affinely on L. Indeed the element g sends any p with $p \in L$ into the point $g(\mathscr{O}) + ap/(a_{00} + 0) = g(\mathscr{O}) + |a_{00}|^{-1}P$. Thus g sends vectors $\mathscr{O}P$ with $p \in L$ into vectors $g(\mathscr{O})g(p)$ *parallel* with respect to the coordinate system $x_1/x_0, \cdots, x_n/x_0$. Hence $g(p) \in S_i$ for $p \in S_i \cap L$ since S_i is $\boldsymbol{R}$*-linear* $(i = 1, 2)$. Consequently, g sends $(S_1 \cap L, S_2 \cap L)$ to $(S_1 \cap g(L), S_1 \cap g(L))$. Inasmuch as L is the normal to A at $\mathscr{O}$ with respect to the Riemannian metric, $g(L)$ is the normal to A at $g(\mathscr{O})$, and hence by definition $\sphericalangle(S_1 \cap g(L), S_2 \cap g(L))$ is the Riemannian angle formed by S_1, S_2 at $g(\mathscr{O})$. Since g induces an isometry on B, it preserves Riemannian angles. The proof of Lemma 3.2.4 is now complete.

LEMMA 3.2.5. *Let S be a spinal surface and let I be a $\boldsymbol{C}$-line. If I is orthogonal to a slice of S, then $I \cap S$ is a geodesic line.*

Proof. Let A be a slice of S such that I is orthogonal to A at a point q of A. Let P denote the spine of A and set $\mathscr{O} = A \cap P$. Let $y_1, y_2, \cdots, y_n$ be a standard nonhomogeneous coordinate system centered at $\mathscr{O}$ with $y_1 = y_2 = \cdots = y_{n-1} = 0$ on P and $y_n = 0$ on A. The proof of Lemma 3.2.4 shows that the $\boldsymbol{C}$-line I orthogonal to A at q is given by the equations

$$y_1 = y_1(q), \cdots, y_{n-1} = y_{n-1}(q) ,$$

and moreover, the geodesic lines in I through the point q are given by the additional equation

$$\arg y_n = \text{constant}.$$

In particular, $I \cap S$ is a geodesic line by Remark 3 of 3.1.

The common Riemannian angle between S_1 and S_2 along $S_1 \cap S_2$ is denoted as $\sphericalangle(S_1, S_2)$; it is called the angle formed by S_1, S_2.

LEMMA 3.2.6. *Let $S_i (i = 1, 2)$ be a spinal surface in B, and assume that z is both an S_1-function and an S_2-function. Then the Riemannian angle between S_1 and S_2 is given by the euclidean angle* *in the complex z-plane of the image sets $z(S_1)$ and $z(S_2)$.*

Proof. By definition, z is nonzero polynomial of degree 1 in nonhomogeneous coordinates on B which is constant on all slices. Let L be a $\boldsymbol{C}$-line containing the spine of S_1. Then L is orthogonal to $S_1 \cap S_2$ at the point $L \cap A$. By definition, as Riemannian angles, $\sphericalangle(S_1, S_2) = \sphericalangle(S_1 \cap L, S_2 \cap L)$. As a Riemannian space L has constant curvature and the Riemannian angle coincides with the euclidean. The function z restricted to L is a complex analytic map of a $\boldsymbol{C}$-line and is therefore conformal. Consequently

$$\sphericalangle(S_1 \cap L, S_2 \cap L) = \sphericalangle(z(S_1 \cap L), z(S_2 \cap L)) .$$

Since z is an S_i-function, $z(S_i \cap L) = z(S_i)(i = 1, 2)$. Consequently, $\sphericalangle(S_1, S_2) = \sphericalangle(Z(S_1), z(S_2))$.

LEMMA 3.2.7. *Let S be a spinal surface, and let I be a $\boldsymbol{C}$-line, and let μ denote the projection of S onto its spine.*

(i) *With respect to any nonhomogeneous coordinate system, $I \cap S$ is an (Riemannian-unbounded circular arc in the disc I or is empty.*

(ii) *If I meets the spine of S, then $I \cap S$ is a geodesic subspace.*

(iii) *$I \cap S$ is a geodesic line if and only if for any $p \in I \cap S$, the line $p\mu(p)$ and $I \cap S$ span an $\boldsymbol{R}$-k-plane $k \leqq 2$.*

(iv) *If $I \cap S$ is a geodesic line, then $I \cap S$ and the spine of S span an $\boldsymbol{R}$-k-plane with $k \leqq \inf(3, n)$.*

(v) *Let v_0, v_1, v_2 be elements in V such that $\pi(v_1)$ and $\pi(v_2)$ are distinct points of $I \cap S$ and $\pi(v_0) = \mu(\pi(v_1))$. Then $I \cap S$ is a geodesic line if and only if $\langle v_0, v_1\rangle\langle v_1, v_2\rangle\langle v_2, v_0\rangle \in \boldsymbol{R}$.*

Proof. Let P denote the spine of S. Let $y_1, \cdots, y_n$ be any standard nonhomogeneous coordinate system centered at a point $\mathscr{O} \in P$ with $y_n = 0$ on the slice through $\mathscr{O}$. By Lemma 3.2.3, the surface S is $\boldsymbol{R}$-linear in $\operatorname{Re} y_i$, $\operatorname{Im} y_i$ $(i = 1, \cdots, n)$, and for any $\boldsymbol{C}$-line I the same is also true. Hence $I \cap S$ is $\boldsymbol{R}$-linear. With respect to any other standard nonhomogeneous coordinate system, I is a disc and $I \cap S$ is the transform of a straight line by a fractional linear transformation. Hence $I \cap S$ is an unbounded circular arc.

Assertion (ii) follows from the fact that any straight line in the $y_1, \cdots, y_n$ coordinate which passes through $\mathscr{O}$ is a geodesic line.

To prove (iii), suppose first that $I \cap S$ is a geodesic line. Let p, p' be distinct points on $I \cap S$, and let $\mathscr{O}, \mathscr{O}'$ be the orthogonal projections of p, p' on P. We can choose e_0, v, v' in V which are mapped by π onto $\mathcal{O}$, p, p' respectively and standardizing coordinates $x_0, \cdots, x_n$ with dual base $e_0, e_1, \cdots, e_n$ so that $y_n = 0$ on the slice through $\mathcal{O}$, $y_1(v) \in \boldsymbol{R}$,

$$v = e_0 + a_1 e_1,\ v' = e_0 + b_1 e_1 + b_n e_n + f$$

where $a_1 \in \boldsymbol{R}$, $b_n \in \boldsymbol{R}$, and $f = b_2 e_2 + \cdots + b_{n-1} e_{n-1}$. We have $I \cap S \subset \pi(v + \boldsymbol{R}v')$. Since $I \cap S$ is a geodesic line, $\langle v, v' \rangle \in \boldsymbol{R}$. Clearly $\langle e_0, v \rangle = \langle e_0, v' \rangle = \langle e_0, e_0 \rangle \in \boldsymbol{R}$. Consequently $\pi((\boldsymbol{R}e_0 + \boldsymbol{R}v + \boldsymbol{R}v') \cap V^-)$ is an $\boldsymbol{R}$-k-plane spanned by $I \cap S$ and $p\mathcal{O}$ of dimension 2 or 1 according as $p \neq \mathcal{O}$ or $p = \mathcal{O}$. Set $e' = e_0 + b_n e_n$. Then $\pi(e') = \mathcal{O}'$ and $\langle e', v \rangle$, $\langle e', v' \rangle$, $\langle e', e_0 \rangle$ are all in $\boldsymbol{R}$. Hence $\pi((\boldsymbol{R}e_0 + \boldsymbol{R}v + \boldsymbol{R}v' + \boldsymbol{R}e') \cap V^-)$ is an R-k-plane with $k \leqq 3$ and $k \leqq 2$ if $n = 2$.

Conversely, if $I \cap S$ and $p\mu(p)$ span an $\boldsymbol{R}$-2-plane, then $I \cap S$ is the intersection of two geodesic subspaces and is therefore geodesic.

Assertion (v) is clearly a restatement of (iii)

LEMMA 3.2.8. *Let A and B be orthogonal $\boldsymbol{C}$-planes in $\boldsymbol{Ch}^n$ with* $\dim_{\boldsymbol{C}} A = k$ *and* $\dim_{\boldsymbol{C}} B = n - k$. *Let G be a geodesic $\boldsymbol{R}$-j-plane. If $A \cap B \subset G$ and* $\dim_{\boldsymbol{R}} A \cap G = k$, *then* $\dim_{\boldsymbol{R}} B \cap G = j - k$.

Proof. Let p denote the unique point in $A \cap B$, let T denote the tangent space to $\boldsymbol{Ch}^n$ at p, and let $\dot{A}$, $\dot{B}$, $\dot{G}$ denote the tangent space at p to A, B, G respectively. It suffices to prove that $\dim_{\boldsymbol{R}} \dot{B} \cap \dot{G} = k - j$. Clearly we can choose $e_1, \cdots, e_j, \cdots, e_n$ an orthonormal base of $\dot{T}$ with $e_1, \cdots, e_j$ a base for $\dot{G}$, and $e_1, \cdots, e_k$ a base for $\dot{A}$. Then $e_{k+1}, \cdots, e_n$ is a base for $\dot{B}$. Hence $\dot{B} \cap \dot{G}$ has a base $e_{k+1}, \cdots, e_j$. Thus $\dim_{\boldsymbol{R}} \dot{B} \cap \dot{G} = j - k$.

LEMMA 3.2.9. *Let S_i be a spinal surface with spine P_i and let L_i be the $\boldsymbol{C}$-line containing P_i $(i = 1, 2)$. Assume that $L_1 \cap L_2$ contains a point p_0 which is not on P_1 or on P_2.* (i) *Let $\mathcal{O} \in S_1 \cap S_2$. Then the tangent space to S_1 and S_2 at $\mathcal{O}$ are distinct.* (ii) *If S_1 and S_2 have a common slice, then $L_1 = L_2$ and $P_1 \cap P_2$ is not empty.*

Proof. Suppose not. Let A_i denote the slice of S_i through $\mathcal{O}$ $(i = 1, 2)$. Let $\dot{A}_i$ and $\dot{S}_i$ denote the tangent space to A_i and S_i at $\mathcal{O}$ respectively $(i = 1, 2)$. Then $\dot{A}_i$ is the maximum C-linear subspace of $\dot{S}_i$ $(i = 1, 2)$. To prove (i), suppose that $\dot{S}_1 = \dot{S}_2$. Hence $\dot{A}_1 = \dot{A}_2$. This implies that $A_1 = A_2$ since A_i is a $\boldsymbol{C}$-linear subspace $(i = 1, 2)$. Let μ_i denote the orthogonal projection of the ball onto L_i $(i = 1, 2)$. Then $\mu_i(\mathcal{O}) = A_i \cap L_i$ $(i = 1, 2)$. Consequently the geodesic line $p_0\mu_1(\mathcal{O})$ forms a right angle with the geodesic line $\mu_2(\mathcal{O})\mu_1(\mathcal{O})$. Similarly, $\mu_1(\mathcal{O})\mu_2(\mathcal{O})$ forms a right angle with $p_0\mu_2(\mathcal{O})$

at $\mu_2(\mathcal{O})$. Inasmuch as the sum of the three angles in the geodesic triangle $p_0\mu_1(\mathcal{O})\mu_2(\mathcal{O})$ is less than or equal to two right angles with equality only if the triangle is degenerate, either $p_0 = \mu_i(\mathcal{O})$ $(i = 1$ or $2)$ or $\mu_1(\mathcal{O}) = \mu_2(\mathcal{O})$. The first possibility is excluded since $\mu_i(\mathcal{O}) \in P_i$ $(i = 1, 2)$. Consequently $\mu_1(\mathcal{O}) = \mu_2(\mathcal{O}) \in P_1 \cap P_2$. In particular, $L_1 \cap L_2$ contains the two distinct points p_0 and $\mu_1(\mathcal{O})$. Therefore $L_1 = L_2$ since a C-line is uniquely determined by two points. Applying Lemma 3.2.4, one infers that $\sphericalangle(S_1, S_2) = 0$ at $\mu_1(\mathcal{O})$ as well as at $\mathcal{O}$. Hence $P_1 = P_2$ and $S_1 = S_2$.

To prove (ii), let A denote a common slice of S_1 and S_2. Choose $\mathcal{O} \in A$, and consider the geodesic triangle $p_0\mu_1(\mathcal{O})\mu_2(\mathcal{O})$. By the argument above, $\mu_1(\mathcal{O}) = \mu_2(\mathcal{O}) \in P_1 \cap P_2$ and $L_1 = L_2$.

LEMMA 3.2.10. *If the spines of two spinal surfaces S_1 and S_2 have two distinct common perpendicular lines which meet one spine in two distinct points, then $S_1 = S_2$. In particular, if two spinal surfaces have two common slices, they coincide.*

Proof. Let P_i denote the spine of S_i $(i = 1, 2)$ and let g^1 and g^2 be distinct geodesic lines orthogonal to both P_1 and P_2. Then the quadrilateral formed by g^1, P_1, g^2, P_2 has four right angles. Consequently, it must be degenerate and $P_1 = P_2$. Hence $S_1 = S_2$.

3.3. *Intersections of half-spaces.*

LEMMA 3.3.1. *Let S_i be a spinal surface with spine P_i $(i=1, 2)$. Let L_i denote the C-line containing P_i $(i = 1, 2)$ and assume that $L_1 \cap L_2$ contains a point p_0. Let S_1^+ denote the half-space bounded by S_1 and containing p_0, then*

(i) for any slice A_2 of S_2, $S_1^+ \cap A_2$ is convex (with respect to geodesics in Ch^n);

(ii) $S_1 \cap S_2$ is an unbounded connected $(2n - 2)$ manifold or is empty.

Proof. To prove (i) it suffices to prove $A_2' \cap S_1^+$ is convex for any C-line A_2' in A_2. Choose a standard nonhomogeneous coordinate system $y_1, y_2, \cdots, y_n$ centered at p_0 so that L_1 has the equation $y_2 = \cdots = y_n = 0$. Then the ball B is given by $|y_1|^2 + \cdots + |y_n|^2 < 1$, the spinal surface S_1 is given by the equation $|y_1 - a| = b$ where $|a|>b$, and S_1^+ is given by $|y_1 - a| > b$. Moreover the slice A_2 is a disc orthogonal to the C-line L_2. Consider now the orthogonal projection μ_1 of the ball B onto L_1. The restriction of μ_1 to A_2' is holomorphic and hence a conformal map if $L_1 \neq L_2$; if $L_1 = L_2$, μ_1 maps A_2 onto a point. In the former case, $\mu_1(A_2')$ is a circular disc lying in the disc L_1: $|y_1| < 1$; and $\mu_1(A_2') \cap S_1^+$ is therefore a convex subset of $\mu_1(A_2')$ with respect to the induced metrics, (which coincides with the Poincaré metric). The inverse image of $\mu_1(A_2' \cap S_1^+)$ in A_2 is therefore also convex, because conformal maps of Poincaré discs are isometries. If $L_1 = L_2$, then $\mu_1(A_2) \cap S_1^+ = \mu_1(A_2)$ or is empty, and correspondingly, $A_2 \cap S_1^+ = A_2$ or is empty. This proves (i).

To prove (ii) for $n = 2$, consider the projection μ_1 of the spinal surface S_1 onto its spine P_1. By Lemma 3.2.7 (i), for each slice A_1 of S_1, $A_1 \cap S_2$ is either an unbounded circular arc in A_1 or is empty. From this (ii) follows. For larger n, a similar inductive argument can be given. Proof of the lemma is now complete.

LEMMA 3.3.2. *Let Δ be a finite subset of isometries of $\boldsymbol{Ch}^n$. Let $p_0 \in \boldsymbol{Ch}^n$. For any $\gamma \in \Delta$, set*

$$\gamma^+ = \{x \in \boldsymbol{Ch}^n; d(x, p_0) \leqq d(\gamma x, p_0)\}$$
$$\hat{\gamma} = \{x \in \boldsymbol{Ch}^n; d(x, p_0) = d(\gamma x, p_0)\}$$
$$F(\Delta) = \bigcap_{\gamma \in \Delta} \gamma^+ .$$

Then $F(\Delta)$ and each of its k-dimensional faces is topologically a cell, $k = 0, 1, 2, \cdots, 2n$.

Proof. We prove the result for $n = 2$. For larger n, an inductive argument based on similar considerations can be given.

The region F is clearly star-shaped with respect to geodesics originating at p_0. Inasmuch as the boundary of F is made up of a finite number of piecewise smooth 3-surfaces, F is a 4-cell.

Let e^3 be a 3-face of F; that is, e_3 is a connected component of $\hat{\gamma}_0 \cap \bigcap_{\gamma \in \Delta} \gamma^+$.

Let μ denote the projection of the spinal surface $\hat{\gamma}_0$ onto its spine P. By Lemma 3.3.1, each slice of $\hat{\gamma}_0$ meets γ^+ in a convex set and consequently each fiber of the map μ is a cell. Since e^3 is connected, $\mu(e^3)$ is an interval. It follows that e^3 is a cell. The proofs for 2-cells and 1 cells are similar.

4. Arithmeticity of groups generated by C-reflections.

Let G be a semi-simple real Lie group and let Γ be a lattice subgroup of G.

DEFINITION. Γ is an *arithmetic* lattice in G if and only if there is an algebraic matrix group A defined over the field $\boldsymbol{Q}$ of rational numbers and containing $\mathrm{Ad}\, G$, the adjoint group of G such that

(i) $A_R = \mathrm{Ad}\, G \times K$ (direct) with K compact.

(ii) $\operatorname{Ad}\Gamma$ is commensurable with the projection of A_Z into G i.e., $\operatorname{Ad}\Gamma \cap (A_Z K \cap G)$ is of finite index in both Γ and $A_Z K \cap G$.

REMARK. Let G^* denote the Zariski-closure of $\operatorname{Ad} G$ in the full matrix algebra $\mathscr{E}$ over complex numbers and let k denote the field generated by $\boldsymbol{Q}[\operatorname{Tr}\operatorname{Ad}\Gamma]$. Let $\mathscr{O}$ denote the ring of algebraic integers of k and let $\operatorname{Gal} k$ denote the set of all monomorphisms of k into $\boldsymbol{C}$. It is known (cf. (2.5.1)) that the algebraic group G^* can be defined over the field k. Let I denote the ideal of polynomial functions (in the entries of matrices in $\mathscr{E}$) with coefficients in k which vanish on G^* and let ${}^\sigma I$ denote the image of I under $\sigma \in \operatorname{Gal} k$. One denotes by ${}^\sigma G^*$ the algebraic matrix group on which the ideal ${}^\sigma I$ vanishes; ${}^\sigma G^*$ is defined over the field ${}^\sigma k$.

In this paper, the group $\operatorname{Ad} G$ will usually be $\mathrm{PU}(n, 1)$ and $G^* = \mathrm{PGL}(n+1, \boldsymbol{C})$ which is a simple group. Whenever G^* is a simple group, the groups A and K in the definition of arithmetic group are easy to identify. Set $\mathscr{G} = \operatorname{Gal} k$, and

$$\mathscr{G}_R = \{\sigma \in \operatorname{Gal} k;\ \bar{\sigma} = \sigma\}$$
$$\mathscr{G}_- = \{\sigma \in \mathscr{G}_R;\ ({}^\sigma G^*)_R \text{ is compact}\}\ .$$

For $\sigma \in \mathscr{G} - \mathscr{G}_R$, ${}^\sigma G^* \times {}^{\bar\sigma} G^*$ is defined over $\boldsymbol{R}$, and, as is easy to see, $({}^\sigma G^* \times {}^{\bar\sigma} G)_R = ({}^\sigma G^*)_C$. It is well-known (and easy to prove) that a connected compact complex group is abelian. Thus $\sigma \in \mathscr{G} - \mathscr{G}_R$ implies that $({}^\sigma G^* \times {}^{\bar\sigma} G^*)_R$ is not compact. Consequently if we form

$$A = \prod_{\sigma \in \mathscr{G}} {}^\sigma G^*$$

and set

$$K = \prod_{\sigma \in \mathscr{G}_-} ({}^\sigma G^*)_R,\ B = \prod_{\sigma \in \mathscr{G} - \mathscr{G}_-} {}^\sigma G^* \tag{4.1.1}$$

then B is defined over $\boldsymbol{R}$ and the algebraic group A is defined over $\boldsymbol{Q}$; it is the group obtained from G^* by "restriction from k to $\boldsymbol{Q}$". Moreover, B_R is the product of all the noncompact factors of A_R. By definition, the subgroup $A_Z K \cap B$ is an arithmetic lattice in B_R, the fact that it is a lattice following from the theorem of Borel-Harish Chandra that A_R/A_Z has finite measure (cf. [2]).

If G^* is simple and $k = \boldsymbol{Q}[\operatorname{Tr}\operatorname{Ad}\Gamma]$, then $B_R = \operatorname{Ad} G$, $\mathscr{G}_- = \mathscr{G} - \{\text{identity}\}$ and thus ${}^\sigma k \subset \boldsymbol{R}$ for all $\sigma \neq$ identity. Since G is a real Lie group, $k \subset \boldsymbol{R}$, thus k is a totally real field with ${}^\sigma(G^*)_R$ compact for all $\sigma \in \mathscr{G}_-$.

The following lemma provides us with a test for arithmeticity of a lattice generated by complex reflections.

LEMMA 4.1. *Let H be a nondegenerate hermitian form $\Sigma a_{ij} z_i \bar{z}_j$*

on C^n, *let* $h = \boldsymbol{Q}\{\{a_{ij}\}; i, j = 1, \cdots, n\}$ *and let* $G = U(H)$, *the unitary group of* H. *Let* Γ *be a lattice subgroup of* G, *let* $k = \boldsymbol{Q}[\mathrm{Tr\,Ad}\,\Gamma]$ *and let* $\mathcal{O}$ *be the ring of algebraic integers in* k. *Let* E *denote the composite field of* h *and* k. *Assume*

(1) *The field* k *is totally real.*

(2) *For all* $x \in h$ *and* $\sigma \in \mathrm{Gal}\,h$, $\sigma(\bar{x}) = \overline{\sigma(x)}$.

(3) $\mathrm{Tr}\,[\mathrm{Ad}\,\Gamma] \subset \mathcal{O}$.

Then $\mathrm{Ad}\,\Gamma$ *is arithmetic in* $\mathrm{Ad}\,G$ *if and only if*:

For all $\sigma \in \mathrm{Gal}\,E$ *with* $\sigma \neq 1$ *on* k, ${}^\sigma H$ *is definite.*

Proof. Let G^* denote the Zariski-closure of $\mathrm{Ad}\,G$. Then $G^* = \mathrm{PGL}\,(n, \boldsymbol{C})$, a simple group. Let $A = \boldsymbol{R}_{k/\boldsymbol{Q}}G$, the group defined over $\boldsymbol{Q}$ by restricting the ground field from k to $\boldsymbol{Q}$. Then

$$A = \prod_{\sigma \in \mathscr{S}} {}^\sigma G^*$$

where $\mathscr{S} = \mathrm{Gal}\,k$. For any $\sigma \in \mathrm{Gal}\,E$ which is not the identity on k, ${}^\sigma G^*$ is defined over $\boldsymbol{R}$ by hypothesis (1) and ${}^\sigma H$ is a hermitian form by (2), $(G^*)_{\boldsymbol{R}} = \mathrm{PU}\,(H)$ and $({}^\sigma G^*)_{\boldsymbol{R}} = \mathrm{PU}\,({}^\sigma H)$.

Assume ${}^\sigma H$ is definite for $\sigma \neq 1$ on k. Then ${}^\sigma(G^*)_{\boldsymbol{R}}$ is compact for all $\sigma \in \mathrm{Gal}\,k$ with $\sigma \neq 1$. Let Γ' denote the projection of $A_{\boldsymbol{Z}}$ into the $\sigma = 1$ factor. Since $\prod_{\sigma \neq 1} ({}^\sigma G^*)_{\boldsymbol{R}}$ is compact, Γ' is a discrete subgroup. The $\boldsymbol{Q}$-structure on A arises from the $\boldsymbol{Q}$-structure on the field k regarded as a $\boldsymbol{Q}$-vector space, embedded into

$$\prod_{\sigma \in \mathscr{S}} {}^\sigma k \approx k \bigotimes_{\boldsymbol{Q}} \boldsymbol{C}$$

via the diagonal embedding $x \to \prod_{\sigma \in \mathscr{S}} {}^\sigma x$. Hence $\Gamma' = (G^*)_{\mathcal{O}}$. By hypothesis (3), $\Gamma \subset \Gamma'$. To prove that Γ is arithmetic, it remains only to prove that Γ'/Γ is a finite set. By hypothesis, G/Γ has finite Haar measure. Hence $\mathrm{Ad}\,G/\mathrm{Ad}\,\Gamma$ has finite Haar measure. We have $G^*_{\boldsymbol{R}} = \mathrm{Ad}\,G$, and $(\mathrm{Ad}\,G)/\Gamma'$ has finite measure. Moreover

$$\mu(\mathrm{Ad}\,G/\Gamma') \cdot \mu(\Gamma'/\Gamma) = \mu(\mathrm{Ad}\,G/\Gamma)$$

where μ denotes Haar measure. It follows that Γ'/Γ is finite. Hence Γ is arithmetic.

Conversely, assume that $\mathrm{Ad}\,\Gamma$ is arithmetic in $\mathrm{Ad}\,G$. Then ${}^\sigma\mathrm{Ad}\,\Gamma$ is bounded for all $\sigma \in \mathrm{Gal}\,k$ with $\sigma \neq 1$. Since the map of $\mathrm{GL}\,(n, \boldsymbol{C})$ to $\mathrm{PGL}\,(n, \boldsymbol{C})$ has compact kernel and since ${}^\sigma\mathrm{Ad}\,\gamma = \mathrm{Ad}\,\sigma\gamma$, the matrices ${}^\sigma\Gamma$ are bounded in $\mathrm{PU}\,({}^\sigma H)$ for any $\sigma \in \mathrm{Gal}\,E$ with $\sigma \neq 1$ on k. Hence the topological closure ${}^\sigma\Gamma$ is a compact group and as is well-known stabilizes a positive definite hermitian form on $\boldsymbol{C}^n$. But ${}^\sigma\Gamma$ is irreducible on $\boldsymbol{C}^n$ (cf. (2.3.3)) and therefore stabilizes aunique hermitian form up to a scalar factor. Consequently ${}^\sigma H$ is definite.

REMARK 1. The proof of Lemma 4.1 shows that the group $\operatorname{Ad}\Gamma$ is commensurable to $G^*_{\mathscr{O}}$ if G^* is simple and $\mathscr{O}$ is the ring of integers in $\boldsymbol{Q}[\operatorname{Tr}\operatorname{Ad}\Gamma]$.

By the same argument one can prove:

Let G be a semi-simple matrix group defined over a field k, let $\mathscr{O}$ denote the ring of algebraic integers in k. If

(1) *k is totally real*

(2) *${}^{\sigma}\operatorname{Tr}\operatorname{Ad}\Gamma$ is bounded for all $\sigma \in \operatorname{Gal} k$ with $\sigma \neq 1$, then G_0 is discrete (and an arithmetic lattice in G).*

REMARK 2. Let k be a subfield of $\boldsymbol{R}$ and $\mathscr{O}$ the ring of integers of k. Let G be an algebraic matrix group defined over k. If $G_{\mathscr{O}}$ is discrete, it is an arithmetic lattice in $G_{\boldsymbol{R}}$. The proof involves "weak approximation" i.e., a generalization of the Chinese remainder theorem to algebraic groups.

LEMMA 4.3. *Let $\mathscr{D}$ be a Coxeter diagram, let H be a non-degenerate hermitian form associated to $\mathscr{D}$, and let Γ be the group generated by $\boldsymbol{C}$-reflections associated to $(\mathscr{D}, H)$ and rational phase-shifts. Then for all $\gamma \in \Gamma$, $\operatorname{Tr}\gamma$ is an algebraic integer, and moreover, it is a sum of roots of unity if $p_1 = p_2 = \cdots$.*

Proof. Let n be the number of nodes of $\mathscr{D}$, let $e_1, \cdots, e_n$ denote the canonical base of $\boldsymbol{C}^n$, and let $\{p_i, q_{ij}, i, j = 1, \cdots, n, i \neq j\}$ denote the data of $\mathscr{D}$. Then

$$\langle e_i, e_j\rangle = H(e_i, e_j) = \alpha_{ij}\varphi_{ij},\ |\varphi_{ij}| = 1$$

and

$$\alpha_{ii} = 1, \quad \varphi_{ii} = 1$$
$$\alpha_{ij} = -\frac{c_{ij}}{s_{ij}},\ i \neq j$$

where

$$c_{ij} = \left(\frac{\cos(\pi/p_i - \pi/p_j) + \cos 2\pi/q_{ij}}{2}\right)^{1/2}$$
$$s_{ij} = (\sin\pi/p_i \sin\pi/p_j)^{1/2}\,.$$

By definition, the $\boldsymbol{C}$-reflection R_i in $e_i^{\perp}$ is:

$$R_i x = x + (\eta_i^2 - 1)\langle x, e_i\rangle e_i$$

that is, $R_i = 1 + e_i \otimes \beta_i$, where β_i is the linear map $x \to (\eta_i^2 - 1)\langle x, e_i\rangle$, and $e \otimes \beta$ denote the endomorphism of $\boldsymbol{C}^n$: $(e \otimes \beta)(v) = \beta(v)e$. In this notation, $(e \otimes \beta)(f \otimes \alpha) = \beta(f)e \otimes \alpha$. Thus

$$\operatorname{Tr} R_{i_1}R_{i_2}\cdots R_{i_m} = \sum_{l=1}^{m} \sum_{1 \leq j_1 < j_2 < \cdots < j_l \leq m} \beta_{i_{j_1}}(e_{i_{j_2}})\beta_{i_{j_2}}(e_{i_{j_3}})\cdots\beta_{i_{j_l}}(e_{i_{j_1}}) .$$

Thus $\operatorname{Tr}\gamma$ is a sum of terms of the type

$$\pm (\eta_{i_1}^2 - 1)\cdots(\eta_{i_p}^2 - 1)\alpha_{i_1 i_2}\alpha_{i_2 i_3}\cdots\alpha_{i_p i_1}\varphi_{i_1 i_2}\cdots\varphi_{i_p i_1}$$

or

$$\pm \prod_{j=1}^{p} (\eta_{i_j}^2 - 1)\frac{1}{\sin\dfrac{\pi}{p_{i_j}}} \cdot \prod_{j=1}^{p} c_{i_j i_{j+1}} .$$

Now

$$\begin{aligned}(\eta_j^2 - 1)\cdot\frac{1}{\sin\dfrac{\pi}{p_j}} &= \eta_j(\eta_i - \bar{\eta}_j)\frac{2\sqrt{-1}}{\eta_j - \eta_j} \\ &= 2\eta_j\sqrt{-1} .\end{aligned}$$

Hence $\operatorname{Tr}\gamma$ is a sum of terms of the form $i^m \prod_{j=1}^{p} \eta_{i_j}\cdot\prod_{j=1}^{p}(2c_{i_j i_{j+1}})$. Since $2c_{ij}$ is an algebraic integer and moreover, a sum of roots of unity, if $p_1 = p_2 = \cdots = p_n$ (by the $\cos(\theta/2)$ formula) the same is true of $\operatorname{Tr}\gamma$ provided $\prod_{j=1}^{p} \varphi_{i_j i_{j+1}}$ is a root of unity.

This last condition is assured by the hypothesis of rational phase shifts. Proof of the lemma is now complete.

LEMMA 4.4. *Let Γ be a group generated by C-reflections associated to a Coxeter diagram $\mathscr{D}$ with $n+1$ nodes and Hermitian form H. Assume*

(1) *H is of type (n plus, 1 minus).*

(2) *The group Γ^i generated by the n-reflections $R_1, \cdots, \widehat{R_i}, \cdots, R_{n+1}$ of the reflections $R_1, \cdots, R_{n+1}$ has a connected diagram and is finite.*

(3) *H has rational phase shifts.*

(4) $p_1 = p_2 = \cdots = p_{n+1}$.

Then $\operatorname{Ad}\Gamma$ *is arithmetic in* $\mathrm{PU}(H)$ *if and only if: For all automorphisms of C which are not the identity on the field* $\boldsymbol{Q}[\operatorname{Ad}\Gamma]$, *${}^{\sigma}H$ has positive determinant.*

Proof. Let $W = Ce_1 + \cdots Ce_{\hat{i}} + \cdots Ce_{n+1}$. By (2.3.3) and hypothesis (2), Γ^i is irreducible on W. Consequently Γ^i stabilizes a unique hermitian form on W modulo a scalar factor. Since Γ^i is a finite group, it stabilizes a positive definite hermitian form on W. Hence any hermitian form on W stabilized by Γ^i is definite. Then the restriction of ${}^{\sigma}H$ to W is definite for each automorphism σ of

C. Inasmuch as $\langle e_i, e_i \rangle = 1$ for all i, ${}^{\sigma}H$ is positive definite on W and is of type (n plus, 1 minus) or else positive definite for each automorphism σ of C; these two possibilities correspond to ${}^{\sigma}\Delta < 0$ and ${}^{\sigma}\Delta > 0$ respectively, where Δ denotes the determinant of the $(n+1) \times (n+1)$ matrix $H(e_i, e_j)$.

Let $k = \boldsymbol{Q}[\operatorname{Tr} \operatorname{Ad} \Gamma]$. By Lemma 4.2, $k = \boldsymbol{Q}[|\operatorname{Tr} \Gamma|^2]$. By hypothesis (4) and Lemma 4.3, k consists of sums of real parts of roots of unity. Hence k is totally real. By hypothesis (3) and (4) $H(e_i, e_j)$ lies in a field generated by roots of unity for $i, j = 1, \cdots, n+1$; let k denote the field generated by h_{ij}. Then for all $\sigma \in \operatorname{Gal} h$ and $x \in h$, $\sigma(\bar{x}) = \overline{\sigma(x)}$. Thus hypotheses (1), (2), and (3) of Lemma 4.1 are satisfied. Let E denote the field generated by h and k. We have seen that for all $\sigma \in \operatorname{Gal} E$, ${}^{\sigma}H$ is either of type $(n, 1)$ or positive definite according as ${}^{\sigma}\Delta < 0$ or ${}^{\sigma}\Delta > 0$. The conclusion of the lemma follows from Lemma 4.1.

5. Verification of discreteness.

In the search for nonarithmetic lattices among groups Γ generated by C-reflections, we can test a lattice for arithmeticity by the criterion of Lemma 4.4. But how do we test whether Γ is a lattice? Most particularly, is there a test for Γ to be discrete?

It may be of interest to describe an algorithm applicable to a wide class of Γ which is finite if and only if Γ is discrete.

It is based on the following general simple lemma.

LEMMA. *Let X be a connected simply connected metric space and Γ a group of isometries of X. Let F be a closed subset of X and Δ a finite subset of Γ satisfying*

(1) *F lies in the interior of $\Delta F = \bigcup_{\gamma \in \Delta} \gamma F$.*

(2) *If $\gamma_1, \gamma_2 \in \Delta$ and $\gamma_1 F \cap \gamma_2 F \neq \phi$, then $\gamma_1^{-1}\gamma_2 \in \Delta$.*

(3) *The induced metric on F modulo Δ is complete.*[1]

Then Δ generates Γ and Γ is discrete.

Proof. On the space $\Gamma \times F$ define the equivalence relation $(\gamma_1, x_1) \equiv (\gamma_2, x_2)$ if and only if $\gamma_1 x_1 = \gamma_2 x_2$ and $\gamma_1^{-1}\gamma_2 \in \Delta$. Let Y denote the quotient space mod$\equiv$ of $\Gamma \times F$ and let $\eta: \Gamma \times F \to Y$ the quotient map. Let π denote the map of Y to X induced by $(\gamma, x) \to \gamma x$.

Γ acts on Y in the obvious way and π is a Γ map. It is easy to see that ΓF is open and by (3) is closed in X. Hence $\Gamma F = X$.

Moreover, the map π is a covering map. To prove this, it suffices to prove it is an even covering at each interior point of F in view of hypothesis (1) and $\Gamma F = F$.

[1] I am indebted to Bernard Maskit for pointing out the necessity of this hypothesis.

On $\eta(\Delta \times F)$, the map π is a homeomorphism by definition of the quotient topology. Given $\eta(\gamma, x) \in \pi^{-1}(x')$ with $\gamma \in \Gamma$ and $x, x' \in F$, then $\gamma x = x'$ and hence $\eta(\gamma\Delta, F)$ maps homeomorphically onto $\gamma\Delta F$. The latter contains γF in its interior. Indeed $U = \bigcap_{\substack{\gamma \in \Gamma \\ \gamma x = x'}} \gamma\Delta F$ is a neighborhood of x'; since $\Gamma \subset \mathrm{Isom}\, X$, U contains a ball centered at x' of radius $d(F, x' - \Delta F)$. Then π maps each connected component of $\pi^{-1}(U')$ homeomorphically onto U'.

Since X is simply connected, π is a homeomorphism. Clearly Γ is discontinuous on Y. Hence it is discontinuous on X.

Suppose next that Γ is a group of isometries on $X = \boldsymbol{C}h^n$. Let F be a compact region in X and let Δ_1 be a finite subset of Γ satisfying

(1) F lies in the interior of $\Delta_1 F$.

(2) $\Delta_1 = \Delta_1^{-1}$.

Set $\Delta_1^2 = \{\gamma\gamma'; \gamma, \gamma' \in \Delta_1\}$.

$E_1 = \{\gamma \in \Delta_1^2 - \Delta_1;\ \gamma F \cap F \neq \phi\}$.

Inductively set

$$\Delta_{i+1} = \Delta_i \cup E_i$$
$$E_{i+1} = \{\gamma \in \Delta_{i+1}^2 - \Delta_i; \gamma F \cap F \neq \phi\}\ .$$

PROPOSITION. *Γ is discrete if and only if E_i is empty for some i, $(i = 1, 2, \cdots)$.*

Proof. $(\leftarrow)$. This assertion follows immediately from the lemma.

$(\rightarrow)$. This assertion follows immediately from the definition of discontinuous group.

This algorithm for proving discreteness is impractical because if the group Γ is not discrete, the algorithm never comes to a conclusion. However, a refinement of the lemma on which it is based leads under the added hypotheses of § 6, to a criterion which is more effective i.e., it implies discreteness in some cases, and enables us to prove nondiscreteness in others by suggesting where to look for pairs of $\boldsymbol{C}$-reflections with nonadmissible values.

6. Joined spaces and a discreteness criterion.

6.1. *Abutted families of polyhedra.*

For any n-dimensional polyhedron F we denote by $E_k(F)$ the set of its k-condimensional faces. We consider only *finite polyhedra F which are cells minus faces* and whose faces are also of this type. In particular, each $e \in E_2(F)$ lies on exactly two elements of $E_1(F)$. Write $E_{n+1}(F)$ for the empty set $\varnothing$ and set $E_0(F) = F$.

DEFINITION. Let X be a topological space. An *abutted* family $\mathscr{F}$ of polyhedra in X is a family $\mathscr{F}$ of polyhedra together with a distinguished subset $\mathscr{N}$ of $\mathscr{F} \times \mathscr{F}$ satisfying;

(1) If $(F, F') \in \mathscr{N}$, then $F \neq F'$ and $(F', F) \in \mathscr{N}$.

(2) If $(F, F') \in \mathscr{N}$, then $F \cap F' \in E_1(F) \cap E_1(F')$.

(3) If $(F, F') \in \mathscr{N}$ and $(F, F'') \in \mathscr{N}$ and $F \cap F' = F \cap F''$, then $F' = F''$.

(4) For each $e \in E_1(F)$, there is an $F' \in \mathscr{F}$ with $F \cap F' = e$ and $(F, F') \in \mathscr{N}$.

The F' in (4) is necessarily unique and is denoted $e(F)$. $\mathfrak{n}$ is called the *adjacency* of $\mathscr{F}$; two elements of $\mathscr{F}$ are called *adjacent* if and only if $(F, F') \in \mathscr{N}$.

For any F_0 and F in $\mathscr{F}$ and $e \in E_k(F_0) \cap E_k(F)$, we say that F_0 and F are e-connected, if there if a sequence $F_1, F_2, \cdots, F_n = F$ with $(F_i, F_{i+1}) \in \mathscr{N}$ $(i = 1, 2, \cdots, n-1)$ and $e \in E_k(F_0) \cap \cdots \cap E_k(F_1) \cap E_k(F_n)$; in case $k = n+1$, we say simply that F_0 and F are connected. We called the abutted family $\mathscr{F}$ *connected* if every two of its elements are connected.

We set for any $F \in \mathscr{F}$

$$\mathscr{N}(F) = \{e(F); e \in E_1(F)\} .$$

For any subset $\mathscr{S} \subset \mathscr{F}$, we write

$$\mathscr{N}(\mathscr{S}) = \{\mathfrak{h}(F); F \in \mathscr{S}\} .$$

A subfamily $\mathscr{S} \subset \mathscr{F}$ is called *open* if $\mathscr{N}(\mathscr{S}) = \mathscr{S}$. Clearly any open subfamily of $\mathscr{F}$ which contains F contains any connected subfamily $\mathscr{C}$ with $F \in \mathscr{C}$.

6.2. *The joined $\mathscr{F}$-space.*

Let $\mathscr{F}$ be an abutted family of polyhedra in a topological n-manifold X. Let $X \times \mathscr{F}$ denote the topological direct product of the spaces X and $\mathscr{F}$ where $\mathscr{F}$ is given the discrete topology. Set

$$D = \bigcup_{F \in \mathscr{F}} Fx\{F\} = \{(x, F); x \in F, F \in \mathscr{F}\} .$$

On the topological space D consider the relation $\equiv$ which is generated by the equivalences

$$(x, F) = (x', F') \text{ if } x = x' \text{ and } x \in E_1(F) \cap E_1(F') .$$

Set $Y = D \bmod \equiv$. Let η denote the canonical map of D to Y.

DEFINITION. The space Y is called the *joined $\mathscr{F}$-space*.

The projection $X \times \mathscr{F} \to X$ induces a well-defined continuous map π of Y into X. We call π the canonical map of the joined $\mathscr{F}$-space into X.

For any face e of an $F \in \mathscr{F}$, let F_e denote the union of the interiors (of F and) of all faces of F which contain e. Set

$$\mathscr{F}_e = \{F_e; F \in \mathscr{F}, e \subset F\} .$$

Then $\mathscr{F}_e$ is an abutted family of polyhedra on the space X_e which is the union of $\{\pi(F_e); F_e \in \mathscr{F}_e\}$, when we restrict adjacency in $\mathscr{F}_e$ from $\mathscr{F}$.

For any $e \in E_k(F)$, let $\mathscr{F}_{e,F}$ denote the maximum connected subfamily of $\mathscr{F}_e$ which contains F_e. Clearly $\mathscr{F}_{e,F'} = \mathscr{F}_{e,F}$ for all $F' \in \mathscr{F}_e$. Let $Y(e, F)$ denote the joined $\mathscr{F}_{e,F}$ space. It is easy to see that $Y(e, F)$ is a neighborhood in Y of $\{\eta(x, F); x \in$ interior of $e\}$.

Suppose that X is a topological manifold. Let $F \in \mathscr{F}$, $e \in E_k(F)$, and let $e^\perp$ be a closed small (topological) k ball transversal to e. Set $S_e = \partial e^\perp$. S_e is a $(k-1)$-sphere. Set

$$\mathscr{F}_e(S_e) = \{F \cap S_e; F \in \mathscr{F}_e\} ;$$
$$\mathscr{F}_{e,F}(S_e) = \{F \cap S_e; F \in \mathscr{F}_{e,F}\} .$$

DEFINITION. The abutted family $\mathscr{F}$ on X is called *smooth* if it satisfies the following conditions:

(1) The polyhedra of $\mathscr{F}$ are "nice" in the sense that for any $F \in \mathscr{F}$, $e \in E_k(F)$, and "nicely" embedded small transversal k ball $e^\perp$, $F \cap S_e$ is a polyhedron.

(2) $\mathscr{F}_e(S_e)$ is an abutted family of polyhedra on the $(k-1)$-sphere S_e, and combinatorially, this family is independent of the choice of the nicely embedded small transversal k ball $e^\perp$.

(3) Let $S(e, F)$ denote the joined $\mathscr{F}_{e,F}(S_e)$ space. Then $Y(e, F)$ is homeomorphic to the direct product $\mathrm{Int}\, e \times S(e, F)$, where Int denotes interior.

6.3. *A criterion for discreteness.*

THEOREM 6.3.1. *Let $\mathscr{F}$ be a connected abutted smooth family of compact polyhedra in the (connected) simply connected topological n-manifold X. Assume that X has a metric and that there exists a positive number r such that each $F \in \mathscr{F}$ contains a ball of radius r. Assume also that $\mathscr{F}$ satisfies the condition*

(CD2) *For any $F_0 \in \mathscr{F}$ and $e \in E_2(F_0)$, and any sequence $F_0, F_1, F_2, \cdots, F_n$ of successively adjacent polyhedra with $e \in E_2(F_0) \cap E_2(F_1) \cap \cdots \cap E_2(F_n)$, if $F_0 \cap F_n$ has a nonempty interior, then $F_0 = F_n$.*

Let Y denote the jointed $\mathscr{F}$-space. Then the canonical map $\pi\colon Y \to X$ is a homeomorphism of Y onto X.

Proof. We use induction on $n = \dim X$. If $\dim X = 1$, then X is homeomorphic to the real line, and the hypotheses imply that $\mathscr{F}$ is a locally finite covering of the line by intervals which meet only at their end points.

Now assume the result for spaces of dimension less than n. If $k = 2$, then by hypothesis (CD2) any maximal connected subfamily of $\mathscr{F}_e$ gives a cell decomposition with disjoint interiors of the circle S_e. For $k \geqq 2$ and $e \in E_k$, it is clear that the family $\mathscr{F}_e(S_e)$ inherits property (CD2) from $\mathscr{F}$. Thus any maximal connected subfamily of $\mathscr{F}_e(S_e)$ yields a finite cell decomposition with disjoint interiors of the $(k-1)$-sphere S_e.

It follows, using the smoothness, that for any $F \in \mathscr{F}$, $e \in E_k(F)$, and p an interior point of the k-polyhedron e, that the space $Y(e, F)$ is an n-manifold. Moreover the canonical map $\pi\colon Y \to X$ is a homeomorphism of $Y(e, F)$ onto a neighborhood of $\operatorname{Int} e$.

Consider now the canonical map $\pi\colon Y \to X$. The hypothesis that each $F \in \mathscr{F}$ contains an r-ball easily implies that the π image of each connected component of Y can contain no limit points and that it must be all of X. It follows at once that π is a covering map. Since X is simply connected, π is univalent on each connected component. Since $\mathscr{F}$ is a connected family, Y is connected and π is a homeomorphism of Y onto X.

THEOREM 6.3.2. (I) *Let F be a smooth polyhedron in the Riemannian manifold X. Let Δ be a finite subset of the isometry group* $\operatorname{Isom} X$ *and let Γ denote the subgroup of* $\operatorname{Isom} X$ *generated by Δ. Assume*

(1) $\Delta = \Delta^{-1}$.

(2) *There is a bijective map $\gamma \to e(\gamma)$ of Δ onto $E_1(F)$ satisfying $\gamma(F) \cap F = e(\gamma)$ for all $\gamma \in \Delta$.*

Set $\mathscr{N} = \{(\gamma F, \gamma\delta F); \gamma \in D\}$, $\mathscr{F} = \Gamma F$. Then $\mathscr{F}$ is a connected abutted family of polyhedra with adjacency $\mathscr{N}$. Moreover Γ operates discontinuously on the joined $\mathscr{F}$-space Y.

(II) *If, in addition, $\mathscr{F}$ satisfies the codimension* 2 *condition,* (CD2) *and X is simply connected, then*

(1) *Γ is a discrete subgroup of* $\operatorname{Isom} X$.

(2) *Let $\operatorname{Aut}_\Gamma F$ denote the stabilizer of F in Γ. Then a fundamental domain for $\operatorname{Aut}_\Gamma F$ in F is a fundamental domain for Γ in X* (i.e., *F is a fundamental domain* $\operatorname{mod} \operatorname{Aut}_\Gamma F$).

Proof. That $\mathscr{F}$ is a connected abutted smooth family of polyhedra with adjacency $\mathfrak{h}$ follows directly from definitions. Clearly $\mathscr{F}$ yields a decomposition of the space Y into polyhedra with disjoint interiors. Consider the action of Γ on Y. It follows at once that Γ operates discontinuously on Y, since it permutes the interiors of the polyhedra of $\mathscr{F}$. Under the additional hypotheses in (II), we can apply Theorem 6.3.1 to conclude that $\pi: Y \to X$ is a homeomorphism. Hence Γ is discontinuous on X.

REMARK. Let Γ be a discontinuous group of isometries on a Riemannian space such as Ch^n. Let $p_0 \in X$, and set

$$F = \{x \in X; d(x, p_0) \leqq d(x, \Gamma p_0)\} .$$

Then clearly $X = \Gamma F$. Also, for all $\gamma \in \Gamma$ with $\gamma p_0 \neq p_0$, the interiors of γF and F are disjoint. F is called a "Dirichlet[1] fundamental domain" for Γ. For any k, let $E_k(F)$ denote the set of codimension k faces of F. For each $e \in E_1(F)$, let $T(e)$ denote an element γ^{-1} of Γ such that

$$e \subset \{x \in X; d(x, p_0) = d(\gamma x, p_0)\} .$$

Set $\Delta = T(E_1(F))$. Then $T: E_1(F) \to \Delta$ is a bijective map and $\Delta = \Delta^{-1}$. Moreover, F satisfies the condition

$$\text{(CD1):} \qquad T(e)F \cap F = e \text{ for all } e \in E_1(F) ,$$

and also the codimension-two condition (CD2). Thus the converse of Theorem 6.2 is valid, so that (CD1) and (CD2) are necessary conditions for a Dirichlet fundamental domain.

THEOREM 6.3.3. *We continue the notation and hypotheses of Theorem* 6.3.2 I *and* II. *A presentation for the group* Γ *is given by the generators* Δ *with the following relations*:

For each $e \in E_2(F)$, *choose* $e_1 \in E_1(F)$ *with* $e \subset e_1$ *and let* r_e *denote the word of shortest positive length* $\gamma_1\gamma_2\cdots\gamma_n(\gamma_i \in \Delta)$ *such that*

(i) $e \subset \gamma_1\gamma_2\cdots\gamma_i F \cap \gamma_1\gamma_2\cdots\gamma_{i+1}F \in E_1(\gamma_1\cdots\gamma_i F)$ $(i = 1, \cdots, n-1)$.

(ii) $F \cap \gamma_1 F = e_1$.

(iii) $F \cap \gamma_1\gamma_2\cdots\gamma_n F$ *has a nonempty interior*.

Set $R_2 = \{r_e; e \in E_2(F)\}$. (*By condition* (CD2), *the words of* R_2 *yield elements of* $\mathrm{Aut}_\Gamma F$. *Set* $R_F = \{$*relations among words of* R_2 *as elements of* $\mathrm{Aut}_\Gamma F\}$. *Then*

(a) R_2 *generates* $\mathrm{Aut}_\Gamma F$.

(b) (Δ, R_F) *is a presentation for* Γ.

Proof. Let $\mathscr{G}$ denote the free group generated by the elements of Δ, let R denote the kernel of the canonical homomorphism of $\mathscr{G}$ onto Γ and let R' denote the preimage in $\mathscr{G}$ of $\mathrm{Aut}_\Gamma F$. One defines a homomorphism of R' into the fundamental group of $X - \Gamma E_2(F)$

[1] (Also known as Poincaré or normal fundamental domain.)

as follows. Choose a base point P_0 interior to F and a point $p_\gamma \in \gamma F$ for each $\gamma \in \Gamma$. For each $\gamma \in \Delta$, choose a path in $F \cup \gamma F$ from p_0 to p_γ which does not meet any codimension 2-face of F or γF. To each element in w in $\mathscr{G}$ there corresponds a path $\varphi(w)$ in $\Gamma F - \Gamma E_2(F) = X - \Gamma E_2(F)$ with initial point p_0; the path $\varphi(w)$ is a closed path if and only if $w \in R'$. Moreover $\varphi(w_1 w_2) = \varphi(w_1)\cdot\varphi(w_2)$. The resulting map θ: $R' \to \pi_1(X - \Gamma E_2(F))$ is a homomorphism.

Inasmuch as X is simply connected, $\pi_1(X - \Gamma E_2(F))$ is generated by Γ-conjugates of closed paths linking $E_2(F)$, and therefore $\{\theta(r_e);\ e \in E_2(F)\}$ generates $\pi_1(X - \Gamma E_2(F))$. It is easy to see that for $w \in R'$, any homotopy of $\theta(w)$ in $X - \Gamma E_2(F)$ leads to a path $\theta(w')$ where w' is obtained from w by successive substitution of subwords xz for $xyy^{-1}z (x, y, z \in \Delta)$, and vice-versa. It follows at once that the kernel of θ is (1) and θ is an isomorphism. In particular, $R_2 = \{r_e;\ e \in E_2(F)\}$ generates R'. This proves (a).

Let R_F denote the normal subgroup of R' generated by words in R_2 that represent 1 in $\mathrm{Aut}_\Gamma F$. Then R_F represents 1 in Γ and it is clear that $R_F = R$. Hence (Δ, R_F) is a presentation of Γ.

6.4. Branching and complex analytic joined spaces.

The foregoing results provide a criterion for deciding which of the subgroups of $U(2, 1)$ presented in § 9 are discrete. We shall require a generalization of the above discussion in order to treat the action of some nondiscrete subgroups of $U(2, 1)$. The results of the rest of this section will not be required until § 18.3.

PROPOSITION 6.4.1. *Let $\mathscr{F}$ be a connected abutted smooth family of compact polyhedra in the topological n-manifold X. Assume also that for any $F \in \mathscr{F}$ and $e \in E_k(F)$ the joined $\mathscr{F}_e(S_e)$ space is an $(k-1)$-sphere $(k = 1, \cdots, n)$ (which is a finite branched over of S_e). Then the joined $\mathscr{F}$-space Y is a topological n-manifold and $\mathscr{F}$ gives a polyhedral decomposition with disjoint interiors of Y.*

Proof. This proposition follows directly from definitions. As asserted above, $Y(e, F)$ is a neighborhood of $\eta(\mathrm{Int}\, e, F)$ in Y. By the hypothesis on $\mathscr{F}_e(S_e)$ and the smoothness of $\mathscr{F}$, Int $Y(e, F)$ is an n-manifold for every k-face e $(k = 1, 2, \cdots, n)$. It follows at once that Y is an n-manifold. That $\mathscr{F}$ gives a polyhedral decomposition of Y with disjoint interiors is obvious.

DEFINITION. Let $\mathscr{F}$ be an abutted smooth family on a space X. We say that $\mathscr{F}$ satisfies condition BR if

(BR) for all $F \in \mathscr{F}$ and $e \in E_2(\mathscr{F})$, $S(e, F)$, the joined $\mathscr{F}_{e,F}(S_e)$ space is a circle.

If $\mathscr{F}$ satisfies condition BR, then the canonical map of $S(e, F)$ onto S_e is an even covering map for every codimension 2-face e; we denote the degree of this map by $\beta(e)$. We call $\beta(e)$ the *branching order* of $\mathscr{F}$ around e. We call a codimension$_R$ 2-face e *branching* if $\beta(e) > 1$. Set

$$B_2(\mathscr{F}) = \{e; \beta(e) > 1\} .$$

For any face $e \in E_k(F)$, where $F \in \mathscr{F}$, set $B^2(e) = \{e'; e' \in E_2(F') \text{ with } F' \in \mathscr{F}_{e,F}, \beta(e') > 1\}$. Thus $B^2(e) \subset B_2(\mathscr{F})$ and $e \subset e'$ for all $e' \in B^2(e)$.

PROPOSITION 6.4.2. *Let X be a complex analytic manifold of n complex dimensions. Let $\mathscr{F}$ be an abutted smooth family of polyhedra on X, which satisfies condition* BR. *Assume*:

(1) *Each codimension$_R$ 2-face e such that $\beta(e) > 1$ lies on a hypersurface of* **C***-codimension* 1.

(2) *For any s branching codimension$_R$ 2-faces $\{e_1, \cdots, e_s\}$ whose hypersurfaces are distinct, $e_1 \cap \cdots \cap e_s$ is either empty or has dimension $2n - 2s$.*

(3) *Let $e \in E_k(F)$ with $F \in \mathscr{F}$, and let $e_1, \cdots, e_s$ be the distinct elements of $B^2(e)$ (thus $k \geqq 2s$). Then for any compact subset $K \subset e$, there exists an admissible complex analytic coordinate system $z = (z_1, \cdots, z_n)$ in a neighborhood U in X and a neighborhood W of $\eta(K, F)$ in $Y(e, F)$ so that $U \cap e_i \subset \{z; z_i = 0\}$, $K \subset U = \pi(W)$, and the restriction to W of the canonical map of $Y(e, F)$ into X is equivalent to the map $(w_1, \cdots, w_n) \to (z_1, \cdots, z_n)$ given by*

$$\begin{aligned} z_i &= w_i^{\beta(e_i)} \qquad (i = 1, \cdots, s) \\ z_i &= w_i \qquad (i = s+1, \cdots, n) . \end{aligned}$$

Then the joined $\mathscr{F}$-space has the structure of a complex analytic n-manifold such that the canonical map into X is holomorphic.

Proof. For any face e such that $\beta(e') = 1$ for all codimension 2-faces e' containing e, the (codim 2) condition of Theorem 6.3.1 is satisfied, and we can argue just as we did there that the canonical map of $Y(e, F)$ into X is a homeomorphism onto a neighborhood in X; we endow such $Y(e, F)$ with the structure pulled back from X.

For faces e lying in s branching codimension 2-faces with branching orders $m_1, m_2, \cdots, m_s$ the hypotheses give us the structure of a complex analytic manifold on $Y(e, F)$, such that the canonical map of $Y(e, F)$ into X has degree $m_1 m_2 \cdots m_s$. Putting together these complex analytic structures from the various $Y(e, F)$, it can be verified directly that we get the desired complex analytic structure on Y.

REMARK. Let X be a complex analytic manifold. Let $\mathscr{F}$ be an abutted smooth family of compact polyhedra on X which satisfy condition BR. Hypothesis (3) of Proposition 6.5 follows from the apparently weaker hypotheses in which we restrict the face $e \in E_k(F)$ to satisfy the condition: $B^2(e)$ is a maximal subset among $\{B^2(e'); \text{all faces } e'\}$. The verification of this observation comes from straightforwardly studying the situation of $Y(e', \mathscr{F})$ in $Y(e, \mathscr{F})$ when $e \subset e'$.

6.5. *Polyhedral Γ-spaces Y and the Γ-cover $Y^\#$.*

A *polyhedral Γ-space* Y is a topological space which is covered by a family $\mathscr{F}$ of polyhedra with disjoint interiors, together with a group Γ of transformations of Y which permute the polyhedra of $\mathscr{F}$. The polyhedral Γ-space is called a *joined* Γ-space if

(1) Γ acts transitively on $\mathscr{F}$ i.e.. $\mathscr{F} = \Gamma F$ where $F \in \mathscr{F}$.

(2) There is an injective map $E_1(F) \to \Gamma$ satisfying

  (1) $\Delta = \Delta^{-1}$ where $\Delta = T(E_1(F))$,

  (2) $T(e)F \cap F = e$ for all $e \in E_1(F)$.

  (3) Δ generates the group Γ.

We define the subset of $\mathscr{F} \times \mathscr{F}$:

$$\mathscr{N} = \{(\gamma F, \gamma\delta F), \gamma \in \Gamma, \delta \in \Delta\} .$$

Then $\mathscr{F}$ is clearly an abutted family of polyhedra on Y with adjacency $\mathscr{N}$, and the joined $\mathscr{F}$-space may be identified with Y.

Let Γ_F denote the stabilizer in Γ of the polyhedron F. We can define a joined Γ-space $Y^\#$ covered by a family $\mathscr{F}^\#$ of polyhedra and a Γ-map $\pi^\#$: $Y^\# \to Y$ such that $\pi^\#\mathscr{F}^\# = \mathscr{F}$ and $\Gamma_{F\#} = (1)$ for $F^\# \in \mathscr{F}^\#$. Namely, fix $F \in \mathscr{F}$, let Γ operate on the topological space $\Gamma \times F$ via left multiplication on the first factor. Set

$$F^\# = (1, F)$$

and define the Γ-stable equivalence relation on $\Gamma \times F$ generated by

$$(1, y) = (\delta, \delta^{-1}(y)) \text{ for all } y \in e, \delta = T(e)$$

where e varies over $E_1(F)$. Let $Y^\#$ denote the quotient topological space $\Gamma \times F \bmod \equiv$ and $\mathscr{F}^\# = \{\gamma F^\#; \gamma \in \Gamma\}$. The map $\pi^\#$: $(\gamma, y) \to \gamma y$ of $\Gamma \times F \to Y$ induces a well-defined continuous Γ map of $Y^\#$ to Y. Moreover the stabilizer $\Gamma_{F\#} = (1)$; for if $(1, y)$ is an interior point of

F^* and $\gamma \in \Gamma_{F^*}$, then $\gamma(1, y) \equiv (1, y')$ with y, y' interior points of F. Since the only identifications on interior points of $(1, F)$ come from equality, we conclude that $\gamma(1, y) = (1, y')$. Consequently $(\gamma, y) = (1, y')$ and $\gamma = 1$.

DEFINITION. $Y^\#$ is called the Γ-cover of the joined Γ-space Y.

On the associated family of polydedra $\mathscr{F}^\#$, Γ operates simply transitively, since $\Gamma_{F^\#} = (1)$.

Warning. Even if Y is a manifold, $Y^\#$ need not be. However, $Y^\#$ is connected, locally connected, and locally simply connected if Y is a manifold.

6.6. *The stabilizer of a face in a joined Γ-space.*

Let Y be a joined Γ-space, let $\mathscr{F}$ denote the associated abutted family of polyhedra on which Γ operates transitively, and let $\mathfrak{h}$ denote its adjacency. Let $F \in \mathscr{F}$ and $e \in E_k(F)$. In keeping with previous notation, set

$F_e = \cup \{\mathrm{Int}\, f;\, f \in E_k(F) (k = 0, 1, 2, \cdots) \text{ and } e \subset f\}$

$\mathscr{F}_e = \{F_e;\, F \in \mathscr{F} \text{ and } e \subset F\}$

$\mathscr{F}_{e,F} =$ the maximum connected subfamily of $\mathscr{F}_e$ containing F_e

$G(e, F) = \{\gamma \in \Gamma;\, (\gamma F)_e \in \mathscr{F}_{e,F}\}$

$\Gamma_{e,F} =$ stabilizer of e and of $\mathscr{F}_{e,F}$ in Γ.

Clearly $\Gamma_{(e,F)} \subset G(e, F)$, $\Gamma_{(e,F)} G(e, F) = G(e, F)$, and $\{G(e, F) F_e\} = \mathscr{F}_{(e,F)}$. Tracing back definitions, one sees that

$$G(e, F) = \{\gamma \in \Gamma;\, \gamma = \gamma_1\gamma_2\cdots\gamma_m,\ \gamma_i \in \Delta,\ e \in E_k(\gamma_1\gamma_2\cdots\gamma_i F)\ , \\ i = 1, 2, \cdots, m;\, m = 1, 2\}\ .$$

REMARK 1. If the underlying topological space Y is a n-manifold as it is in the cases of interest here, then $\mathscr{F}_{e,F} = \mathscr{F}_e$. Theorem 6.3.2 (I) describes the situation out of which our joined Γ-spaces will arise.

The family $\mathscr{F}_e$ is an abutted family of polyhedra and we denote $\mathscr{N} \cap (\mathscr{F}_e \times \mathscr{F}_e)$ by $\mathscr{N}$ also; it is the adjacency of $\mathscr{F}_e$.

PROPOSITION 6.6. *Let $e \in E_k(F)$ $(k = 0, 1, 2, \cdots, n)$ and let Γ' be a subgroup of $\Gamma(e, F)$. If*

$$\mathscr{N}(\mathscr{N}(\mathscr{F}_e) \subset \Gamma'(\mathscr{N}(F_e) \cup F_e)$$

then $\mathscr{F}_{e,F} = \Gamma'(\mathscr{N}(F_e) \cup F_e)$ and $\Gamma_{(e,F)} = \Gamma' S$ where $S = \{\gamma \in \Gamma;\, \gamma F \in \mathscr{N}(F),\ \gamma e = e\}$.

Proof.

$$\begin{aligned}\mathcal{N}(\Gamma'(\mathfrak{h}(F_e) \cup F_e)) &= \Gamma'(\mathcal{N}(\mathcal{N}(F_e) \cup \mathcal{N}(F_e)))\\ &\subset \Gamma'(\mathcal{N}(F_e) \cup \Gamma' F_e)\\ &\subset \Gamma'(\mathcal{N}(F_e) \cup F_e)\ .\end{aligned}$$

Hence $\Gamma'(\mathcal{N}(F_e \cup F_e))$ is an open subfamily of $\mathscr{F}_{e,F}$; since it contains F_e it contains the e-connected family $\mathscr{F}_{e,F}$. This proves the first assertion.

As mentioned above, $\mathscr{F}_{e,F} = G(e, F)F_e$ and $\Gamma_{(e,F)}G(e, F) = G(e, F)$. Let $\Gamma = \{\gamma \in \Gamma; \gamma F_e \in \mathcal{N}(F_e) \cup F_e\}$. Then we have $G(e, F) = \Gamma'T$. Hence $\Gamma_{(e,F)} = \Gamma'(T \cap \Gamma_{(e,F)})$. Clearly $T \cap \Gamma_{(e,F)} = S$. Consequently $\Gamma_{(e,F)} = \Gamma'S$.

REMARK 2. Set $F[e] = \{\gamma \in \Delta; \gamma F_e \in \mathfrak{h}(F_e)\}$, $T' = \{1\} \cup F[e]$, and let S' be any subset of T' such that $\Gamma'S' = \Gamma'T'$. Then $S = T'\Gamma_F \cap \Gamma_{(e,F)}$ and therefore $\Gamma'S = \Gamma'(T'\Gamma_F \cap \Gamma_{(e,F)}) = \Gamma'T'\Gamma_F \cap \Gamma_{(e,F)} = \Gamma'S'\Gamma_F \cap \Gamma_{(e,F)} = \Gamma'(S'\Gamma_F \cap \Gamma_{(e,F)})$.

REMARK 3. Set $S^\# = (F[e] \cap \Gamma_{(e,F)}) \cup \{1\}$. In the Γ-cover $Y^\#$ of the joined Γ-space Y, let $\mathscr{F}^\#$ and $\mathcal{N}^\#$ denote the associated abutted family and adjacency. For $F \in \mathscr{F}$ and $e \in E_k(F)$, set $F^\# = (1, F)$ and $e^\# = (1, e)$. Then $G(e^\#, F^\#) = G(e, F)$ and $\Gamma(e, F)G(e^\#, F^\#) = G(e^\#, F^\#)$ and $\Gamma_{(e^\#, F^\#)} = \Gamma'(S^\# \cup \{1\})$.

7. Fundamental domains for finite groups generated by C-reflections.

7.1. *An algorithm for finding the faces of a fundamental domain in Ch^2*

Let Γ_{12} be a finite group generated by two C-reflections $\{R_{e_1,p_1}, R_{e_2,p_2}\}$ in C^3 and preserving a hermitian form H of type (2, 1). We follow the notation of § 3 writing $V(H)$ or V for C^3 with H as inner product:

$$\begin{aligned}&\langle p, q\rangle = H(p, q) \qquad p, q \in V\\ &V^- = \{p \in V; \langle p, p\rangle < 0\}\\ &Ch^2 = \pi(V^-), \qquad X = Ch^2\ .\end{aligned}$$

Fix a point $p_0 \in Ch^2$ fixed by no $\gamma \in \Gamma_{12}$, $\gamma \neq 1$. For any $\gamma \in \Gamma_{12}$, set

$$\begin{aligned}&\gamma^+ = \{x \in X; d(x, p_0) \leqq d(\gamma x, p_0)\}\\ (7.1.1)\qquad &\hat{\gamma} = \{x \in X; d(x, p_0) = d(\gamma x, p_0)\}\\ &F_{12} = \bigcap_{\delta \in \Gamma_{21}} \gamma^+\ .\end{aligned}$$

It is easy to see that

(1) $\Gamma_{12}F_{12} = X$

(2) $\gamma F_{12} \cap F_{12}$ has empty interior if $\gamma \neq 1$.

These two properties define a *fundamental domain.*

In order to determine whether the domains we encounter in §9 satisfy condition (CD1), we must explicitly know the faces of the fundamental domain F_{12}. Some of the groups under consideration have order 600, and thus the explicit computation of (7.1.1) would be lengthy, even if we were merely seeking a fundamental domain for the action of F_{12} in C^2.

Despite the length of the computation, it is clearly finite. Therefore a computing machine can be used to settle the question.

Let $p_{12} = \pi(e_1^\perp \cap e_2^\perp)$, then p_{12} is the point in $\pi(V)$ fixed under R_{e_1,p_1} and R_{e_2,p_2}. Inasmuch as $0 \leqq |\langle e_1, e_2\rangle| < 1$, $e_1^\perp \cap e_2^\perp \subset V^- \cup \{0\}$ and thus $p_{12} \in \pi(V^-) = \boldsymbol{Ch}^2$. We call p_{12} the *apex* of F_{12} since $\gamma p_{12} = p_{12}$ for all $\gamma \in \Gamma_{12}$ and thus $p_{12} \in \hat{\gamma}$ for all $\gamma \in \Gamma_{12}$. Let S_r denote the boundary of the ball in $\boldsymbol{Ch}^2$ with center at p_{12} and radius r. Then S_r is Γ-stable. A machine can best be used to compute numbers. Thus, the machine can be programmed to compute the coordinates of the vertices of $S_r \cap F_{12}$ for any given value of r. If we select $r = d(p_{12}, p_0)$, then $p_0 \in S_r$ and $S_r \cap F_{12}$ is a Dirichlet fundamental domain for Γ_{12} acting on S_r (with distance on S_r defined by the ambient $\boldsymbol{Ch}^2$).

A machine can be programmed to find the vertices of $S_r \cap F_{12}$ as follows.

Set

$$d = \inf \{d(\gamma p_0, p_0); \gamma \in \Gamma_{12}\}$$
$$D_0 = \{\gamma \in \Gamma_{12}; d(\gamma p_0, p_0) < 2d\} .$$

(The 2 is merely a good empirical choice.) Let T_0 denote the set of distinct triplets of elements of D_0. In each distinct triplet $t = (\gamma_1, \gamma_2, \gamma_3)$ of elements in Γ, set

$$V_t = \hat{\gamma}_1 \cap \hat{\gamma}_2 \cap \hat{\gamma}_3 \cap S_r .$$

Set $V_0 = \cup \{V_t; t \in T_0, V_t \text{ is finite}\}$. For any $x \in X$, and for any subset $D \subset X$, set

$$D^\#(x) = \{\gamma \in D; d(\gamma x, p_0) = d(x, p_0)\}$$
$$E(x) = \{\gamma \in \Gamma_{12}; d(\gamma x, p_0) < d(x, p_0)\}$$

and let $\sigma(x)$ denote an element of Γ_{12} such that

$$d(\sigma(x)x, p_0) = \inf \{d(\gamma x, p_0); \gamma \in E(x)\} .$$

Set

$$\begin{aligned}
V_0^{\#} &= \{x \in V_0;\, d(x, p_0) \leqq d(\gamma x, p_0) \text{ for all } \gamma \in D_0\}\\
D_0^{\#} &= \cup\, \{D_0^{\#}(x);\, x \in V_0^{\#}\}\\
S_0 &= \{x \in V_0^{\#};\, E(x) \text{ is not empty}\}\\
E_0 &= \{\sigma(x);\, x \in S_0\}\\
D_1 &= D_0^{\#} \cup E_0 .
\end{aligned}$$

Recursively, define

$$\begin{aligned}
D_{i+1} &= D_i^{\#} \cup E_i\\
T_{i+1} &= \text{the set of distinct triplets of elements of } D_{i+1}\\
V_{i+1} &= \cup\, \{V_t;\, t \in T_{i+1},\ V_t \text{ is finite}\}\\
V_{i+1}^{\#} &= \{x \in V_{i+1};\, d(x, p_0) \leqq d(\gamma x, p_0) \text{ for all } x \in D_{i+1}\}\\
D_{i+1}^{\#} &= \cup\, \{D_{i+1}^{\#}(x);\, x \in V_{i+1}^{\#}\}\\
S_{i+1} &= \{x \in V_{i+1}^{*};\, E(x) \text{ is not empty}\}\\
E_{i+1} &= \{\sigma(x),\, x \in S_{i+1}\}\ .
\end{aligned}$$

Each element of $V_i^{\#}$ is a vertex of

$$\cap\, \{\gamma^+;\, \gamma \in D_i^{\#}\}\ .$$

Each $x \in V_i^{\#}$ with $E(x)$ not empty is cut off by $\hat{\gamma}$ for $\gamma \in E(x)$ ($\sigma(x)$ makes a deepest cut) and disappears from $V_{i+1}^{\#}$. After a finite number of cuts, the process stops. At the final stage, no more vertices of $V_n^{\#}$ can be cut off and $V_n^{\#}$ is the set of vertices of $F_{12} \cap S_r$. Let Δ_{12} denote the set of all $\gamma \in D_n^{\#}$ such that $\hat{\gamma}$ contains at least three points of $V_n^{\#}$. Then $\{\hat{\gamma};\, \gamma \in \Delta_{12}\}$ yield the 3-faces of F_{12} which meet the sphere S_r.

REMARK. It turns out in most cases that $\Delta_{12} \subset D_0$.

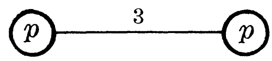
7.2. *Enumeration of* Γ_{12}.

In programming the algorithm of § 7.1 for the computation of the faces of $F_{12} \cap S_r$, it is convenient to have a simple way of enumerating the finite group Γ_{12}, for repeatedly one has to compute

$$\{d(\gamma x, p_0);\, \gamma \in \Gamma_{12}\}\ .$$

In § 2.1, we remarked that the canonical map of $\boldsymbol{C}^2 - 0$ to the complex projective line gives a representation ρ of Γ_{12} in SO(3, $\boldsymbol{R}$) the group of notations of the standard 2-sphere S^2. The kernel of ρ is the center Z of Γ_{12} whose generator and order is given by (2.2.3) and (2.2.4).

For the groups Γ_{12} with Coxeter diagram



the group $\rho(\Gamma_{12})$ is the group of rotational symmetries of the regular tetahedron, cube, and duodecahedron respectively according as $p =$ 3, 4, and 5. Each generating C-reflection R_i corresponds to a rotation of a face of the regular polyhedron $\mathscr{R}(p)$. Let $\Gamma_1 = \{\{R_1\}\}$, the subgroup generated by R_1.

Each element of Γ_{12} can be expressed as a product $\gamma \cdot R_1^n z^m$ where γ is a representative of a coset of $\Gamma_{12}/\Gamma_1 Z$, and z is a generator of Z. In turn $\Gamma_1 Z$ is the stabilizer in Γ of the face of $\mathscr{R}(p)$ stabilized by Γ_1. Hence $\sharp\Gamma_{12}/\Gamma_1 Z = 4, 6, 12$ according as $p = 3, 4, 5$. Explicit representatives are indicated in Figures 7.2 a, b, c: the face labeled 1 is stabilized by R_1 and the face labeled γ is the image of the face 1 under γ. Thus the computer runs through Γ_{12} by running

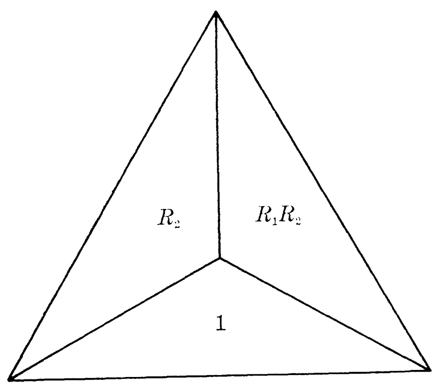


FIGURE 7.2a. $p=3$: Representatives for $\Gamma_{12}/\Gamma_1 Z$
The bottom face represents the coset $R_1^2R_2\Gamma_1 Z$.

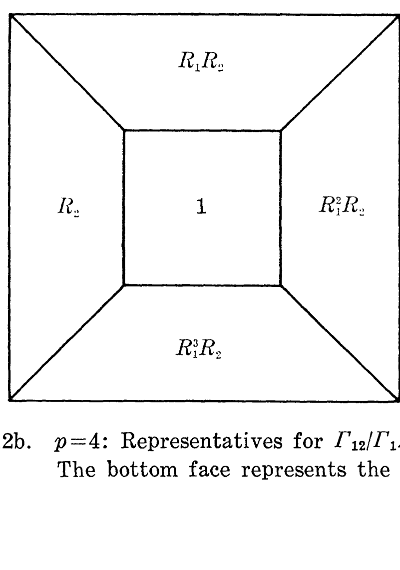


FIGURE 7.2b. $p=4$: Representatives for $\Gamma_{12}/\Gamma_1 Z$
The bottom face represents the coset $R_2^2\Gamma_1 Z$.

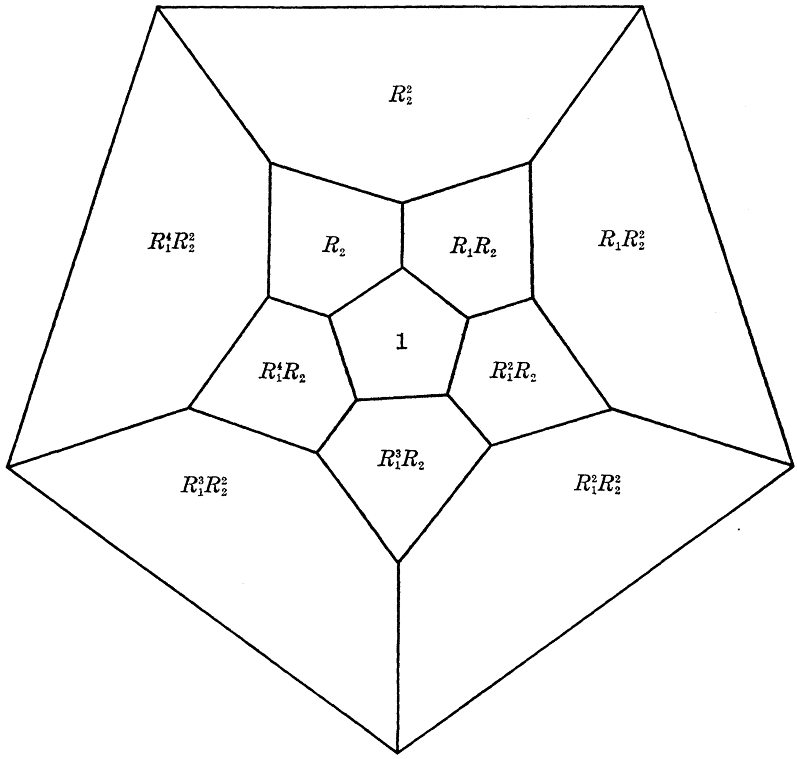
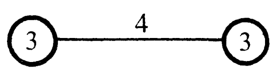


FIGURE 7.2c. $p=5$: Representatives for $\Gamma_{12}/\Gamma_1 Z$
The bottom face represents the coset $R_2^{-1}R_1^2R_2^2\Gamma_1 Z$.

through the 3-fold product $(\Gamma_{12}/\Gamma_1 Z, \Gamma_1, Z)$ of orders

$$\frac{12}{6-p} \times p \times \frac{2p}{6-p} = 24\left(\frac{p}{6-p}\right)^2 \quad \text{i.e.,}$$
$$24,\ 96,\ 600 \text{ according as } p = 3,\ 4,\ 5\,.$$

Similarly, if Γ' has the Coxeter diagram



then the representatives of $\Gamma'/\Gamma_1 Z$ can be read off the faces of the regular octohedron. Note that $\#\Gamma' = 72$ by (2.2.5).

In § 10.1, we give an explicit description of the Dirichlet fundamental domain F_{12} for Γ_{12} on Ch^2: its 3-dimensional faces lie in $\{\hat{\gamma};\ \gamma \in \Delta_{12}\}$, where

$$\Delta_{12} = \{R_i^{\pm 1}, (R_iR_j)^{\pm 1}, (R_iR_jR_i)^{\pm 1}, i \neq j, i, j = 1, 2, 3\}\,,$$

a set of 10 elements. By contrast the group Γ' has as Dirichlet fundamental domain a much more complicated region having 24 faces.

8. Solving a system of four equations of degree 2.

In §7.1, we described an algorithm for finding a fundamental domain for a finite group generated by two C-reflections in Ch^2. That algorithm can be executed readily by a computer, provided that there is a program for computing the intersection of surfaces such as $\hat{\gamma}_1 \cap \hat{\gamma}_2 \cap \hat{\gamma}_3 \cap S_r$. In §9, one deals with intersections $\hat{\gamma}_1 \cap \hat{\gamma}_2 \cap \hat{\gamma}_3 \cap \hat{\gamma}_4$. If we introduce nonhomogeneous coordinates y_1, y_2 on Ch^2 and express the equation of $\hat{\gamma}$ or of S_r in terms of $\operatorname{Re} y_1, \operatorname{Im} y_1, \operatorname{Re} y_2, \operatorname{Im} y_2$ the equations turn out to be real polynomial equations of degree 2. Strange to say, as of 1978 there seemed to be no reliable program in the computation centers' bibliography which could provide all solutions of four such simultaneous equations. Accordingly the implementation of the algorithm necessitates elimination of variables and reduction to a polynomial equation of degree six in a single variable. We sketch the solution.

For any $\gamma \in U(H)$, the surface $\hat{\gamma}$ is by definition $\{x \in Ch^2; d(\gamma x, p_0) = d(x, p_0)\}$. For convenience, we denote a point $x \in C^3 - \{0\}$ and $\pi(x) \in Cp^2$ by the same letter x. Thus the preimage of $\hat{\gamma}$ in C^3 satisfies (in view of $\langle \gamma x, \gamma x\rangle = \langle x, x\rangle$):

$$|\langle \gamma x, p_0\rangle| = |\langle x, p_0\rangle| \tag{8.1}$$

where $\gamma e_i = \sum_{j=1}^{3} \gamma_{ji} e_j$,

$$x = x_1 e_1 + x_2 e_2 + x_3 e_3, \text{ and } p_0$$

is selected so that $\langle e_1, p_0\rangle = \langle e_2, p_0\rangle = \langle e_3, p_0\rangle = h$ and $\langle p_0, p_0\rangle < 0$. Thus (8.1) is equivalent to

$$\begin{aligned} &|(\gamma x)_1 + (\gamma x)_2 + (\gamma x)_3| = |x_1 + x_2 + x_3| \\ &y_i = \frac{x_i}{x_1 + x_2 + x_3} \quad (i = 1, 2, 3)\ . \end{aligned} \tag{8.2}$$

Then $y_1 + y_2 + y_3 = 1$ and y_1, y_2 form a nonhomogeneous coordinate system on CP^2. Thus (8.2) is equivalent to

$$|\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3| = 1, \quad \alpha_i = \gamma_{1i} + \gamma_{2i} + \gamma_{3i}$$

or

$$\begin{aligned} &|(\alpha_1 - \alpha_3) y_1 + (\alpha_2 - \alpha_3) y_2 + \alpha_3| = 1 \\ &|a y_1 + b y_2 + c| = 1\ . \end{aligned} \tag{8.3}$$

The equation of a sphere with center at a point p

$$S_r = \{x \in Ch^2; d(x, p) = r\}$$

and radius r is given by

$$\cosh^{-1} \frac{|\langle x, p\rangle|}{(\langle x, x\rangle\langle p, p\rangle)^{1/2}} = r$$

$$|\langle x, p\rangle|^2 = (\cosh r)^2 \langle p, p\rangle\langle x, x\rangle = k\langle x, x\rangle, \ k < 0$$

$$|\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3|^2 = k \sum_{i,j=1}^{3} h_{ij} x_i \bar{x}_j \ .$$

Dividing both sides by $|x_1 + x_2 + x_3|^2$ and collecting terms, one gets

$$2\mathrm{Re}\,(g_1 y_1 + g_2 y_2 + g_5 y_1 \bar{y}_2) + g_3 y_1 \bar{y}_1 + g_4 y_2 \bar{y}_2 + g_6 = 0 \tag{8.4}$$

where g_3, g_4, g_6 are real.

Given two distinct surfaces of type (8.3):

$$\begin{aligned} |a_1 y_1 + b_1 y_2 + c_1| &= 1 \\ |a_2 y_1 + b_2 y_2 + c_2| &= 1 \ . \end{aligned} \tag{8.5}$$

Set

$$\begin{aligned} v_1 &= a_1 y_1 + b_1 y_2 \\ v_2 &= a_2 y_1 + b_2 y_2 \ . \end{aligned} \tag{8.5$'$}$$

If $v_2 = cv_1$, then (8.5) is equivalent to $|v_1 + c_1| = 1$, $|v_1 + c_2/c| = 1/|c|$, and these can be solved in the v_1-plane to give 2 solutions for v_1; only one solution is in $\boldsymbol{C}h^2$ by Lemma 3.4.1. The solution of a system of 3 equations of type (8.3) with a fourth of type (8.3) or (8.4) then reduces to intersecting these surfaces with the $\boldsymbol{C}$-line $v_1 =$ constant. The problem reduces to solving a quadratic equation in one real variable. Suppose therefore that v_1 and v_2 are independent. Then (8.5) is equivalent to

$$|v_1| = 1, \ |v_2| = 1 \ . \tag{8.6}$$

Expressing a third equation of type (8.3) in terms of v_1, v_2 yields

$$|a_3 v_1 + b_3 v_2 + c_3| = 1 \ .$$

Squaring both sides, we get using (8.6)

$$2\mathrm{Re}\ v_1(b_3 v_2 + c_3) + dv_2 + e = 0 \tag{8.7}$$

with e real.

Thus we need only solve the system

$$\begin{aligned} &|v_1| = 1 \\ &|v_2| = 1 \\ &\quad v_1(\bar{b}_1 \bar{v}_2 + \bar{c}_1) + \bar{v}_1(b_1 v_2 + c_1) + d_1 v_2 + \bar{d}_1 \bar{v}_2 + e_1 = 0 \\ &\quad v_1(\bar{b}_2 \bar{v}_2 + \bar{c}_2) + \bar{v}_1(b_2 v_2 + c_2) + d_2 v_2 + \bar{d}_2 \bar{v}_2 + e_2 = 0 \end{aligned}$$

with e_1, e_2 real.

Elimination of $\bar{v}_1$ from these last two equations[2] yields

$$v_1(G_1v_2 - \bar{G}_1\bar{v}_2 + G_3) - F_1v_2^2 - F_2v_2 - F_3\bar{v}_2 - F_4 = 0 ,$$

and solving for v_1

$$v_1 = \frac{F_1v_2^2 + F_2v_2 + F_3\bar{v}_2 + F_4}{G_1v_2 - \bar{G}_1\bar{v}_2 + G_3} . \tag{8.8}$$

Since $|v_1| = 1$, we get

$$|F_1v_2^2 + F_2v_2 + F_3\bar{v}_2 + F_4| = |G_1v_2 - \bar{G}_1\bar{v}_2 + G_3| .$$

Squaring both sides yields an equation

$$2\mathrm{Re}\,(P_1v_2^3 + P_2v_2^2 + P_3v_2) + P_4 = 0$$

with P_4 real. Multiplying the above equation by $2\mathrm{Re}\,(\bar{P}_1v_2^3 + \bar{P}_2v_2^2 + \bar{P}_3v_2) + P_4$, one obtains an equation of the form

$$2\mathrm{Re}\,(Q_1v_2^6 + Q_2v_2^5 + Q_3v_2^4 + \cdots + Q_6v_2 + Q_7) = 0 \tag{8.9}$$

with $Q_i \in R$ for $i = 1, \cdots, 7$. Set

$$v_2 = z + iw .$$

Since $|v_2| = 1$, $\mathrm{Re}\, v^n$ is a polynomial in z and thus (8.8) becomes

$$R_1z^6 + R_2z^5 + R_3z^4 + R_4z^3 + R_5z^2 + R_6z + R_7 = 0 \tag{8.10}$$

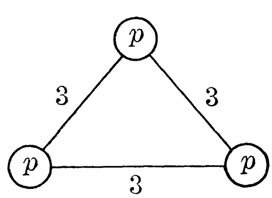
with R_i real, $i = 1, \cdots, 7$.

One can use a packaged program to solve (8.9) to any desired degree of accuracy for z. Thereafter one gets, in turn, w, v_2, v_1 (from (8.8)), and y_1, y_2 (from (8.5)′). One admits only common solutions which lie in $\pi(V^-)$.

9. $\Gamma(\varphi)$ and its automorphisms.

9.1. *$\Gamma(\varphi)$ and some of its elements.*

Let $\Gamma(\varphi)$ denote the group generated by C-reflections corresponding to the diagram



[2] The solution presented here is an improvement of the original solution. I am indebted to Dr. Sidnie Feit for this solution.

where the corresponding hermitian form H_φ is given by

$$\langle e_1, e_2\rangle = \langle e_2, e_3\rangle = \langle e_3, e_1\rangle = -\frac{1}{2\sin\dfrac{\pi}{p}}\varphi\ ,$$

$\varphi \in \boldsymbol{C}$, $|\varphi| = 1$. Select p so that $1/p + 1/p + 2/3 > 1$, i.e., $p < 6$. Set $\alpha = 1/(2\sin\pi/p)$,

$$\begin{aligned} V &= \boldsymbol{C}e_1 + \boldsymbol{C}e_2 + \boldsymbol{C}e_3 \\ \Delta &= \det(\langle e_i, e_j\rangle) \qquad (i, j = 1, 2, 3) \\ &= \det\begin{pmatrix} 1 & -\alpha\varphi & -\alpha\bar{\varphi} \\ -\alpha\varphi & 1 & -\alpha\varphi \\ -\alpha\varphi & -\alpha\bar{\varphi} & 1 \end{pmatrix} \\ &= 1 - 3\alpha^2 - \alpha^3(\varphi^3 + \bar{\varphi}^3)\ . \end{aligned}$$

Set $\varphi = e^{i\theta}$. Then $\Delta = 1 - 3\alpha^2 - 2\alpha^3\cos 3\theta$. Recalling that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, we find that for $\cos\theta = 1/2\alpha$,

$$\Delta = 1 - 3\alpha^2 - 2\alpha^3\Big(4\cdot\frac{1}{8\alpha^3} - 3\cdot\frac{1}{2\alpha}\Big) = 0\ ,$$

$$\Delta < 0 \text{ if and only if } \cos\theta > \frac{1}{2\alpha}\ ;$$

that is, $\cos\theta > \sin\pi/p$ or $|\theta| < \pi/2 - \pi/p$. Thus the hermitian form H has signature (two $+$, one $-$) only for $p > 2$. Therefore for $p = 3, 4, 5$ and

$$\arg(\varphi^3) < 3\Big(\frac{\pi}{2} - \frac{\pi}{p}\Big)$$

the group $\Gamma(\varphi)$, which preserves the hermitian form H_φ is embedded in $U(2, 1)$. *Hereafter we impose these conditions on p and φ.* We write $V(\varphi)$ for the vector space V with H_φ as inner product; when there is no ambiguity, we write V for $V(\varphi)$.

Set $\eta = \exp(\pi\sqrt{-1}/p)$, $R_i(x) = x + (\eta^2 - 1)\langle x, e_i\rangle e_i$ $(i = 1, 2, 3)$. The $\boldsymbol{C}$-reflection R_i depends on φ and we sometimes write it as $R_i(\varphi)$. Since $(\eta^2 - 1)\alpha = (\eta^2 - 1)i/(\eta - \eta^{-1}) = \eta i$ we find

$$R_1 = \begin{pmatrix} \eta^2 & -\eta i\bar{\varphi} & -\eta i\varphi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ -\eta i\varphi & \eta^2 & -\eta i\bar{\varphi} \\ 0 & 0 & 1 \end{pmatrix} \quad R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\eta i\bar{\varphi} & -\eta i\varphi & \eta^2 \end{pmatrix}$$

$$R_1R_2 = \begin{pmatrix} 0 & -\eta^3 i\bar{\varphi} & -\eta^2\bar{\varphi}^2 - \eta i\varphi \\ -\eta i\varphi & \eta^2 & -\eta i\bar{\varphi} \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_1R_2R_3 = \begin{pmatrix} \eta^3 i\bar{\varphi}^3 - \eta^2 & -\eta^2\varphi^2 & -\eta^4\bar{\varphi}^2 - \eta^3 i\varphi \\ -\eta i\varphi - \eta^2\bar{\varphi}^2 & 0 & -\eta^3 i\bar{\varphi} \\ -\eta i\bar{\varphi} & -\eta i\varphi & \eta^2 \end{pmatrix}$$

$$(R_1R_2R_3)^2 = \begin{pmatrix} -\eta^6\bar{\varphi}^6 - \eta^5 i\bar{\varphi}^3 + \eta^4 + \eta^3\varphi^3 i & 0 & -\varphi\eta^3 i(\eta^4\bar{\varphi}^6 - \eta^2 + \eta^3 i\varphi^3) \\ -\eta^5 i\bar{\varphi}^5 + \eta^4\bar{\varphi}^2 + \eta^3 i\varphi & \eta^3 i\varphi^3 & \varphi^2\eta^4(-1 + \eta^2\bar{\varphi}^6 + \eta i\bar{\varphi}^3) \\ \eta^4\bar{\varphi}^4 - \eta^2\varphi^2 + \eta^3 i\bar{\varphi} & 0 & -\eta^4 - \eta^5 i\bar{\varphi}^3 \end{pmatrix}$$

$$(R_3R_2R_1)^2 = \begin{pmatrix} -\eta^4 - \eta^5 i\varphi^3 & 0 & \eta^4\varphi^4 - \eta^2\bar{\varphi}^2 - \eta^3 i\varphi \\ \bar{\varphi}^2\eta^4(-1 + \eta^2\varphi^6 + \eta i\varphi^3) & \eta^3 i\bar{\varphi}^3 & -\eta^5 i\varphi^5 + \eta^4\varphi^2 + \eta^3 i\bar{\varphi} \\ -\bar{\varphi}\eta^3 i(\eta^4\varphi^6 - \eta^2 + \eta^3 i\varphi^3) & 0 & -\eta^6\varphi^6 - \eta^5 i\varphi^3 + \eta^4 + \eta^3\bar{\varphi}^3 i \end{pmatrix}$$

$$R_2R_1R_2R_3 = \begin{pmatrix} \eta^3 i\bar{\varphi}^3 - \eta^2 & -\eta^2\varphi^2 & -\eta^3\varphi(\eta\bar{\varphi}^3 + i) \\ -\eta^2\varphi^2 & i\eta^2(\eta\varphi^3 + i) & -\eta^3\bar{\varphi}(\eta\varphi^3 + i) \\ -\eta i\bar{\varphi} & -\eta i\varphi & \eta^2 \end{pmatrix}.$$

The characteristic polynomial of $R_1R_2R_3$ is:

$$-\lambda^3 + \lambda^2[\eta^3 i\bar{\varphi}^3] - \lambda[\eta^4 - (\eta^3 i\varphi^3 + \eta^4) + (\eta^5 i\bar{\varphi}^3 - \eta^4 - (\eta^5 i\bar{\varphi}^3 - \eta^4)] + \eta^6$$

and its roots satisfy

$$\lambda^3 - \lambda^2[\eta^3 i\bar{\varphi}^3] - \lambda[\eta^3 i\varphi^3] - \eta^6 = 0$$

which factors

$$(\lambda^2 - \eta^3 i\varphi^3)(\lambda - \eta^3 i\bar{\varphi}^3) = 0$$

yielding as eigenvalues $\lambda = \eta^3 i\bar{\varphi}^3$ and the roots of $\lambda^2 = \eta^3 i\varphi^3$. Similarly, the eigenvalues of $R_3R_2R_1$ satisfy

$$\lambda^3 - \lambda^2[\eta i\varphi^3] - \lambda[\eta i\bar{\varphi}^3] - \eta^6 = 0\ ;$$

this yields $\lambda = \eta^3 i\varphi^3$ and the roots of $\lambda^2 = \eta^3 i\varphi^{-3}$.

The eigenvalues of $(R_1R_2R_3)^2$ are therefore

(9.1.1) $$-\eta^6\bar{\varphi}^6,\ \eta^3 i\varphi^3,\ \eta^3 i\varphi^3$$

and the eigenvalues of $(R_3R_2R_1)^2$ are

(9.1.2) $$-\eta^6\varphi^6,\ \eta^3 i\bar{\varphi}^3,\ \eta^3 i\bar{\varphi}^{-3}\ .$$

We next compute the image of $(R_1R_2R_3)^2 - \eta^3 i\varphi^3 \times 1_3$ where 1_3 denotes the identity 3×3 matrix.

(9.1.3) $$(R_1R_2R_3)^2 - \eta^3 i\varphi^3 \cdot 1_3$$

$$= (-\eta^2 i\bar{\varphi}^3 + i\varphi^3 + \eta)\bar{\varphi}^2\eta^3 \begin{pmatrix} -\bar{\varphi}\eta i & 0 & -\bar{\varphi}\eta i \cdot \varphi\eta i \\ 1 & 0 & 1 \cdot \varphi\eta i \\ \varphi\bar{\eta} i & 0 & \varphi\bar{\eta} i \cdot \varphi\eta i \end{pmatrix}.$$

Set

$$v_{123}=\begin{pmatrix}-\bar\varphi\eta i\\ 1\\ \varphi\bar\eta i\end{pmatrix}\quad v_{231}=\begin{pmatrix}\varphi\bar\eta i\\ -\bar\varphi\eta i\\ 1\end{pmatrix}\quad v_{312}=\begin{pmatrix}1\\ \varphi\bar\eta i\\ -\bar\varphi\bar\eta i\end{pmatrix}$$

$$v_{321}=\begin{pmatrix}\bar\varphi\bar\eta i\\ 1\\ -\varphi\eta i\end{pmatrix}\quad v_{132}=\begin{pmatrix}-\varphi\eta i\\ \bar\varphi\bar\eta i\\ 1\end{pmatrix}\quad v_{213}=\begin{pmatrix}1\\ -\varphi\eta i\\ \bar\varphi\bar\eta i\end{pmatrix}.$$

v_{ijk} is the normal in $V(\varphi)$ to the multiplicity-two eigenspace of $(R_iR_jR_k)^2$ where i, j, k denotes a permutation of 1, 2, 3.

$$\begin{aligned}\langle v_{123}, v_{123}\rangle &= \langle -\bar\varphi\eta i e_1 + e_2 + \varphi\bar\eta i e_3,\ -\bar\varphi\eta i e_1 + e_2 + \varphi\bar\eta i e_3\rangle\\ &= 3-\frac{i}{\eta-\eta}[-\bar\varphi\eta i(\varphi-\bar\varphi\eta i\bar\varphi)+(\varphi\bar\eta i\bar\varphi-\bar\varphi\eta i\varphi)+\varphi\bar\eta i(\varphi\bar\eta i\varphi+\bar\varphi)]\\ &= 3-\frac{i}{\eta-\bar\eta}[-\eta i-\bar\varphi^3\eta^2+\bar\eta i-\eta i-\varphi^3\bar\eta^2+\bar\eta i]\\ &= 3-\frac{i}{\eta-\bar\eta}[-i(\eta-\bar\eta)\cdot 2-\varphi^3\bar\eta^2-\bar\eta^3\eta^2]\\ &= 1+\frac{i}{\eta-\bar\eta}(\varphi^3\bar\eta^2+\bar\varphi^3\eta^2)\end{aligned}$$

$$(9.1.4)\qquad = 1+\frac{\cos\left(\frac{2\pi}{p}-3\theta\right)}{\sin\frac{\pi}{p}}\,,\quad 3\theta=\arg\varphi^3\,.$$

$$\langle v_{321}, v_{321}\rangle = 1+\frac{i}{\eta-\bar\eta}(\bar\varphi^3\bar\eta^2+\varphi^3\eta^2)$$

$$(9.1.5)\qquad = 1+\frac{\cos\left(\frac{2\pi}{p}+3\theta\right)}{\sin\frac{\pi}{p}}\,.$$

We have

$$\langle v_{123}, v_{123}\rangle > 0 \text{ if and only if}$$

$$(9.1.4)'\qquad -\left(\frac{\pi}{2}-\frac{\pi}{p}\right)<\arg\varphi^3<3\left(\frac{\pi}{2}-\frac{\pi}{p}\right)$$

and

$$\langle v_{321}, v_{321}\rangle > 0 \text{ if and only if}$$

$$(9.1.5)'\qquad -3\left(\frac{\pi}{2}-\frac{\pi}{p}\right)<\arg\varphi^3<\frac{\pi}{2}-\frac{\pi}{p}\,.$$

Similar computation shows that the characteristic polynomial of $R_2R_1R_2R_3$ is

$$-\lambda^3 + \lambda^2[\eta^3 i\bar{\varphi}^3 + \eta^3 i\varphi^3 - \eta^2] - \lambda[-\eta^6 - \eta^5 i - \varphi^5 i\bar{\varphi}] + \eta^8 = 0$$

so that its eigenvalues are

$$\eta^3 i\varphi^3,\ \eta^3 i\bar{\varphi}^3,\ -\eta^2\ . \tag{9.1.6}$$

9.2. *Automorphisms of $\Gamma(\varphi)$.*

Let J and J' denote the linear maps of V onto itself given by:

$$J\colon x_1e_1 + x_2e_2 + x_3e_3 \longrightarrow x_1e_2 + x_2e_3 + x_3e_1$$
$$J'\colon x_1e_1 + x_2x_2 + x_3e_3 \longrightarrow x_1e_3 + x_2e_2 + x_3e_1\ .$$

Then J is an isometry of $V(\varphi)$ to $V(\varphi)$, and J' is an isometry of $V(\varphi)$ to $V(\bar{\varphi})$ with J' of order two. Since $H_\varphi(e_1, e_2) = -\alpha\varphi = H_{\bar{\varphi}}(e_2, e_1) = H_{\bar{\varphi}}(e_3, e_2)$.

Define

$$\kappa\colon \sum_{i=1}^{3} x_ie_i \longrightarrow \sum_{i=1}^{3} \bar{x}_ie_i\ .$$

Then κ is a semi-linear map of $V(\varphi)$ to $V(\bar{\varphi})$ which is an isometry since,

$$\begin{aligned} H_{\bar{\varphi}}(\kappa(\sum_i x_ie_i), \kappa(\sum_i x_ie_i)) &= H_{\bar{\varphi}}(\sum \bar{x}_ie_i, \sum \bar{x}_ie_i) \\ &= \sum_{i,j} \bar{x}_ix_jH_{\bar{\varphi}}(e_i, e_j) \\ &= \sum_{i,j} x_i\bar{x}_jH_\varphi(e_i, e_j) \\ &= H_\varphi(\sum x_ie_i, \sum x_ie_i)\ . \end{aligned}$$

Regarding the index as an integer modulo 3, we have

$$\begin{aligned} &J(R_i(\varphi)) = R_{i+1}(\varphi) \\ &J'(R_i(\varphi)) = R_{1-i}(\bar{\varphi}) \\ &\kappa(R_i(\varphi)) = \overline{R_i(\varphi)} = R_i(\bar{\varphi})^{-1} \qquad i = 1, 2, 3\ . \end{aligned}$$

Set $a_{13} = J'\kappa$. Then a_{13} is a semi-linear isometry of $V(\varphi)$ to $V(\varphi)$ with

$$a_{13}\colon R_i(\varphi) \longrightarrow R_{1-i}(\varphi)^{-1} \qquad 1 = 1, 2, 3\ .$$

Set $a_{21} = J\alpha_{13}J^{-1}$, $a_{32} = J^{-1}a_{13}J$.

The group of isometries of $V(\varphi)$ generated by a_{12} and J is of order 6 — these are the isometries which permute e_1, e_2, e_3. (For $\varphi = 1$, there is the additional isometry κ.) The subgroup of linear isometries is of order 3. All these isometries induce automorphisms of $\Gamma(\varphi)$.

10. The region $F(\varphi)$.

10.1. *A fundamential domain for* Γ_{ij}.

Set $p_0 = e_1 + e_2 + e_3$. Then

$$\langle p_0, e_1\rangle = \langle p_0, e_2\rangle = \langle p_0, e_3\rangle = 1 - \alpha(\varphi + \bar{\varphi}) .$$

As in § 3, we take as our model of hermitian hyperbolic space Ch^2 the image in projective space CP^2 under the map π: $V - \{0\} \to CP^2$ of the negative cone

$$V^- = \{z \in V; \langle z, z\rangle < 0\} \qquad (= \text{the ball in } C^2)$$

with metric

$$d(z_1, z_2) = \cosh^{-1}\left(\frac{\langle z_1, z_2\rangle\langle z_2, z_1\rangle}{\langle z_1, z_1\rangle\langle z_2, z_2\rangle}\right)^{1/2} .$$

Since $\langle p_0, p_0\rangle = 3(1 - \alpha(\varphi + \bar{\varphi}))$, $\langle p_0, p_0\rangle < 0$ for $|\arg\varphi| < \pi/2 - \pi/p$.

We denote by p_0 also, the image of p_0 in Ch^2. For any $\varphi \in \Gamma$, set

$$\gamma^+ = \{x \in Ch^2; d(x, p_0) \leqq d(\gamma x, p_0)\}$$
$$\hat{\gamma} = \{x \in Ch^2; d(x, p_0) = d(\gamma x, p_0)\} .$$

Set $\Gamma_{ij} = \{\{R_i, R_j\}\}$ $(i \neq j;\ i, j \in (1, 2, 3))$ where $\{\{a, b\}\}$ denotes the group generated by $\{a, b\}$. Γ_{ij} is a finite group. Since Γ_{ij} is irreducible on $Ce_1 + Ce_2$, $\gamma p_0 \neq p_0$ if $\gamma \in \Gamma_{ij}$, $\gamma \neq 1$. Set

$$F_{ij} = \bigcap_{\gamma \in \iota\, ij} \gamma^+ .$$

Then F_{ij} is a fundamental domain for Γ_{ij}. Set $e_i^\perp = \{z \in V; \langle z, e_i\rangle = 0\}$ and denote also by $e_i^\perp$ the subset $\pi(e_i^\perp \cap V^-)$ of Ch^2 $i = 1, 2, 3$. Then $e_i^\perp$ is the fixed point set of R_i whether in V or in Ch^2.

Set $p_{ij} = \pi(e_i^\perp \cap e_j^\perp)$. The equation of $e_i^\perp$ is given by:

$$e_1^\perp\colon 0 = \langle x_1e_1 + x_2e_2 + x_3e_3, e_1\rangle = x_1 - \alpha\bar{\varphi}x_2 - \alpha\varphi x_3$$
$$e_2^\perp\colon 0 = \langle x_1e_1 + x_2e_2 + x_3e_3, e_2\rangle = -\alpha\varphi x_1 + x_2 - \alpha\bar{\varphi}x_3$$
$$e_3^\perp\colon 0 = \langle x_1e_1 + x_2e_2 + x_3e_3, e_3\rangle = -\alpha\bar{\varphi}x_1 - \alpha\varphi x_2 + x_3$$

(10.1.1)

$$p_{12}\colon x_1 = x_3\frac{\begin{vmatrix}\alpha\varphi & -\alpha\bar{\varphi}\\ \alpha\bar{\varphi} & 1\end{vmatrix}}{\begin{vmatrix}1 & -\alpha\bar{\varphi}\\ -\alpha\varphi & 1\end{vmatrix}} = x_3\frac{\alpha\varphi + \alpha^2\bar{\varphi}^2}{1 - \alpha^2},\quad x_2 = x_3\frac{\alpha\bar{\varphi} + \alpha^2\bar{\varphi}^2}{1 - \alpha^2} .$$

Thus $p_{12} = \pi(\varphi\alpha(1 + \alpha\bar{\varphi}^3), \bar{\varphi}\alpha(1 + \alpha\varphi^3), 1 - \alpha^2)$. Set $\xi = \varphi\alpha(1 + \alpha\bar{\varphi}^3)$. Then $\langle p_{12}, p_{12}\rangle = (1 - \alpha^2)\Delta < 0$, $p_{12} = \pi(\xi, \bar{\xi}, 1 - \alpha^2)$, $p_{23} = \pi(1-\alpha^2, \xi, \bar{\xi})$, $p_{31} = \pi(\bar{\xi}, 1 - \alpha^2, \xi)$. We also write $p_{12} = (\xi, \bar{\xi}, 1 - \alpha^2)$ and similarly for p_{23}, p_{31}.

For any $\gamma \in \Gamma_{12}$, the equation of $\hat{\gamma}$ is given by

$$|\langle x, p_0\rangle| = |\langle \gamma x, p_0\rangle|$$

since $\langle x, x\rangle = \langle \gamma x, \gamma x\rangle$. Since $R_1 p_{12} = p_{12} = R_2 p_{12}$, we have $\gamma p_{12} = p_{12}$ for all $\gamma \in \Gamma_{12}$ and thus $p_{12} \in \hat{\gamma}$ for all $\gamma \in \Gamma_{12}$. F_{12} is a curvilinear solid angle with apex at p_{12}. We must know the 3-faces of F_{12}. In principle, the 3-faces of F_{12} in some neighborhood of the apex p_{12} can be computed by hand thanks to Lemma 3.2.9. For by that lemma, the tangent planes to each $\hat{\gamma}$ at p_{12} are distinct for all $\gamma \in F_{12}$. Hence the fundamental domain $\dot{F}_{12}$ for Γ_{12} acting on the tangent space to Ch^2 at p_{12} determines the faces of F_{12} near p_{12}. The computation of the faces of $\dot{F}_{12}$ involves the solution of a system of $24(p/6 - p)^2$ linear inequalities ($p = 3, 4, 5$) and in principal this can be done by hand.

For any $r > 0$, set

$$S_r = \{x \in Ch^2; d(x, p_{12}) = r\}\,.$$

Then S_r is stable under Γ_{12}. A fundamental domain F_{12}^* for Γ_{12} on S_r can be calculated (for any fixed φ) via computer.

In Figure 1, we exhibit the domain F_{12}^* with $r = d(p_{12}, p_0)$. The faces of F_{12}^* correspond to the 10 elements

$$R_1^{\pm 1},\ R_2^{\pm 1},\ (R_1R_2)^{\pm 1},\ (R_2R_1)^{\pm 1},\ (R_1R_2R_1)^{\pm 1}$$

of the finite group Γ_{12}. In Figures 2, 3, and 4, we show the 3-faces of F_{12} which lie on the surfaces $\hat{R}_1$, $\widehat{R_1R_2}$, and $R_1\hat{R}_2R_1$. The definition of the vertices s_{ij}, t_{ik} are given in § 11.

From the remark following Theorem 6.3.2, we know that F_{12} satisfies condition

$$\text{(CD1):} \qquad \gamma^{-1}(\hat{\gamma}^{-1} \cap F_{12}) = \hat{\gamma} \cap F_{12}$$

for all $\gamma \in \{R_1^{\pm 1}, R_2^{\pm 1}, (R_1R_2)^{\pm 1}, (R_2R_1)^{\pm 1}, (R_1R_2R_1)^{\pm 1}\}$ and also condition (CD2). This last condition yields for each two faces of F_{12} an identity corresponding to the circuit $F_1\gamma_1F$, $\gamma_2\gamma_2F$, $\cdots$ where

$$e = \hat{\gamma}_0 \cap \hat{\gamma}_1^{-1} \cap F_{12},\ \gamma_1^{-1}(e) = \hat{\gamma}_1 \cap \hat{\gamma}_2^{-1} \cap F_{12},\ \gamma_2^{-1}\gamma_1^{-1}e = \hat{\gamma}_2 \cap \hat{\gamma}_3^{-1} \cap F_{12},\ \cdots,$$

and $\gamma_1\gamma_2\cdots\gamma_n F_{12} \cap F_{12}$ has a nonempty interior; namely

$$\gamma_1\gamma_2\cdots\gamma_n = 1\,.$$

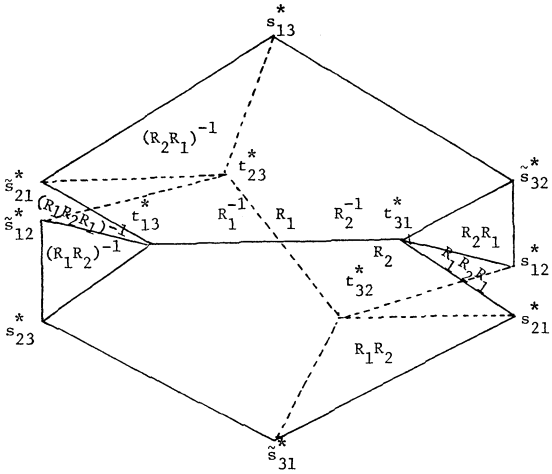


FIGURE 10.1. The intersection F^*_{12} of F_{12} with a 3-sphere S_r centered at p_{12} of radius $r=d(p_{12}, p_0)$.

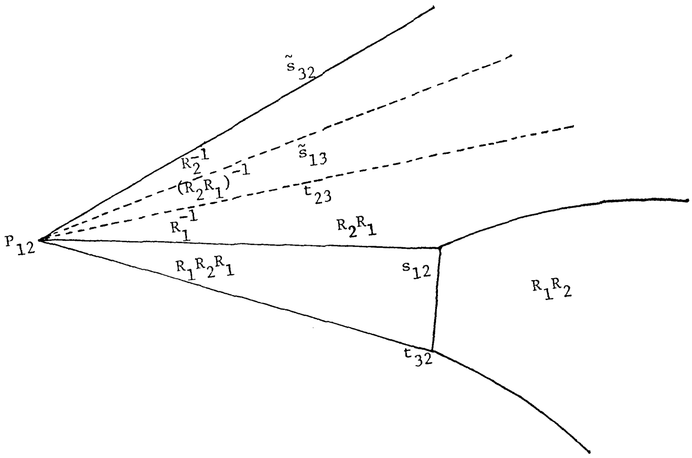


FIGURE 10.2. $F_{12}\cap \hat{R}_1$

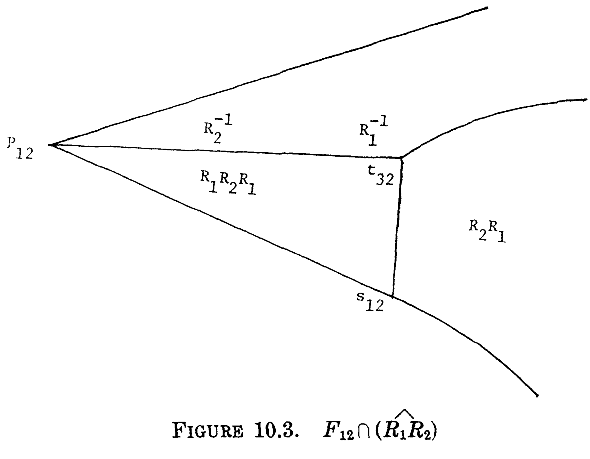


FIGURE 10.3. $F_{12}\cap (\widehat{R_1R_2})$

From Figure 1, one can read off the elements of $E_2(F_{12})$ which contain p_{12}; they correspond to the twenty edges of $F_{12} \cap S_r$. For any γ such that $\hat{\gamma} \cap F_{12} \in E_1(F_{12})$, set

$$^*\gamma = \hat{\gamma} \cap F_{12} \; .$$

Then along with the knowledge of $E_1(F_{12})$ one can determine the circuits around each $e \in E_2(F_{12})$ and the corresponding identity. For ${}^*R_i \cap {}^*R_i^{-1}$, the circuit yields the identity $R_i^p = 1$ $(i = 1, 2)$. The remaining eighteen 2-faces of F_{12} occur in circuits of three terms each

$$\begin{tikzcd}
 & {}^*R_1 \cap {}^*(R_2R_1) \arrow[dr, "R_1"] & \\
{}^*(R_2R_1)^{-1} \cap {}^*R_2^{1-1} \arrow[ur, "(R_2R_1)^{-1}"] & & {}^*R_2 \cap {}^*R_1^{-1} \arrow[ll, "R_2"']
\end{tikzcd} \qquad (R_2R_1)^{-1}R_2R_1 = 1$$

$$\begin{tikzcd}
 & {}^*R_2 \cap {}^*(R_1R_2) \arrow[dr, "R_2"] & \\
{}^*(R_1R_2)^{-1} \cap {}^*R_1^{-1} \arrow[ur, "(R_1R_2)^{-1}"] & & {}^*R_1 \cap R{}^*{}_2^{-1} \arrow[ll, "R_1"']
\end{tikzcd} \qquad (R_1R_2)^{-1}R_1R_2 = 1$$

$$\begin{tikzcd}
 & {}^*R_1 \cap {}^*(R_1R_2R_1) \arrow[dr, "R_1"] & \\
{}^*(R_1R_2R_1)^{-1} \cap {}^*(R_1R_2)^{-1} \arrow[ur, "(R_1R_2R_1)^{-1}"] & & {}^*R_1R_2 \cap {}^*R_1^{-1} \arrow[ll, "R_1R_2"']
\end{tikzcd} \qquad (R_1R_2R_1)^{-1}R_1R_2R_1 = 1$$

$$\begin{tikzcd}
 & {}^*R_2 \cap {}^*(R_1R_2R_1) \arrow[dr, "R_2"] & \\
{}^*(R_1R_2R_1)^{-1} \cap {}^*(R_2R_1)^{-1} \arrow[ur, "(R_1R_2R_1)^{-1}"] & & {}^*(R_2R_1)^{-1} \cap {}^*R_2^{-1} \arrow[ll, "R_2R_1"']
\end{tikzcd} \qquad (R_1R_2R_1)^{-1}R_2R_1R_2 = 1$$

$$\begin{tikzcd}
 & {}^*(R_1R_2) \cap {}^*(R_1R_2R_1) \arrow[dr, "R_1R_2"] & \\
{}^*(R_1R_2R_1)^{-1} \cap {}^*R_2^{-1} \arrow[ur, "(R_1R_2R_1)^{-1}"] & & {}^*R_2 \cap {}^*(R_1R_2)^{-1} \arrow[ll, "R_2"']
\end{tikzcd} \qquad (R_1R_2R_1)^{-}R_2R_1R_2 = 1$$

$$\begin{tikzcd}
 & {}^*(R_2R_1) \cap {}^*(R_1R_2R_1) \arrow[dr, "R_2R_1"] & \\
{}^*(R_1R_2R_1)^{-1} \cap {}^*R_1^{-1} \arrow[ur, "(R_1R_2R_1)^{-1}"] & & {}^*R_1 \cap {}^*(R_2R_1)^{-1} \arrow[ll, "R_1"']
\end{tikzcd} \qquad (R_1R_2R_1)^{-1}R_1R_2R_1 = 1 \; .$$

REMARK. In § 13, we shall show the eighteen mappings of (10.1.2) can be verified easily. Thus these circuits yield the presentation

$$R_1^p = 1 = R_2^p, \quad R_1R_2R_1 = R_2R_1R_2$$

of the finite group Γ_{12}.

The isometry a_{12} corresponding to the automorphism $R_1 \to R_2^{-1}$, $R_2 \to R_1^{-1}$ of Γ_{12} permutes the elements of $E_1(F_{12})$ sending $^*(R_1R_2)$ to $^*(R_2^{-1}R_1^{-1}) = {}^*(R_1R_2)^{-1}$, $^*(R_2R_1)$ to $^*(R_2R_1)^{-1}$, and $^*(R_1R_2R_1)$ to $^*(R_1R_2R_1)^{-1}$.

The isometry a_{12} sends $E_2(F_{12})$ to $E_2(F_{12})$ and thus we get two additional isometries

$$\begin{aligned} {}^*R_1 \cap {}^*(R_1R_2R_1) &\cong {}^*(R_1R_2R_1)^{-1} \cap {}^*(R_1R_2)^{-1} \cong {}^*(R_1R_2) \cap {}^*(R_1R_2R_1) \\ {}^*R_2 \cap {}^*(R_1R_2R_1) &\cong {}^*(R_1R_2R_1)^{-1} \cap {}^*(R_2R_1)^{-1} \cong {}^*(R_2R_1) \cap {}^*(R_1R_2R_2) . \end{aligned} \tag{10.1.3}$$

10.2. *The domain* $F(\varphi)$, $|\arg \varphi| < \pi/2 - \pi/p$.

We continue the preceding notation. Set

$$F(\varphi) = F_{12} \cap F_{23} \cap F_{31} .$$

When there is no ambiguity, we write F for $F(\varphi)$. In § 12, we shall determine certain vertices of F. For the present we note that for all φ with $|\arg \varphi| < \pi/2 - \pi/p$, F is stable under the isometries J and $\{a_{ij};\ (i \neq j,\ i, j = 1, 2, 3)\}$ and that J' and K' are isometries of $F(\varphi)$ to $F(\bar{\varphi})$. We denote by Isom F the group of six isometries of $V(\varphi)$ generated by J and a_{12}.

It will turn out that the combinatorial scheme of F and its faces remains unchanged for $|\arg (\varphi^3)| < \pi/2 - \pi/p$, is unbounded for $\arg (\varphi^3) = \pi/2 - \pi/p$, and becomes bounded, and combinatorially constant for $1/3(\pi/2 - \pi/p) < |\arg \varphi| < \pi/2 - \pi/2$; but it is combinatorially different than the case $|(\arg (\varphi^3))| < \pi/2 - \pi/p$.

11. 2-faces of $F(\varphi)$ not containing an apex.

For any permutation (i, j, k) of $(1, 2, 3)$, set

$$I_{ijk} = (\widehat{R_iR_j})^{-1} \cap \widehat{R_jR_k}$$

and using indices modulo 3

$$\begin{aligned} I_j &= I_{j-1,j,j+1} \\ I'_j &= I_{j+1,j,j-1} . \end{aligned}$$

LEMMA 11.1. *I_{ijk} is the common slice of $(\widehat{R_iR_j})^{-1}$ and $\widehat{R_jR_k}$.*

Proof. Let J and J' denote the isometries defined in § 9.2. Then clearly

$$JI_2 = I_3J^{-1}I_2 = I_1 .$$

Moreover, $J'R_i(\varphi) = R_{1-i}(\bar{\varphi})$ so that

$$J'I_{123}(\varphi) = I_{321}(\bar{\varphi}) . \tag{11.1}$$

Consequently, it suffices to prove that I_2 is a common slice of $(\widehat{R_1R_2})^{-1}$ and $\widehat{R_2R_3}$. We have

$$(R_1R_2)^{-1} = R_2^{-1}R_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \bar\eta i\varphi & \bar\eta^2 & \bar\eta i\bar\varphi \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar\eta^2 & \bar\eta i\bar\varphi & \bar\eta i\varphi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \bar\eta^2 & \bar\eta i\bar\varphi & \bar\eta i\varphi \\ \bar\eta^3 i\varphi & 0 & -\bar\eta^2\varphi^2+\bar\eta i\bar\varphi \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_2R_3 = \begin{pmatrix} 1 & 0 & 0 \\ -\eta i\varphi & \eta^2 & -\eta i\bar\varphi \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\eta i\bar\varphi & -\eta i\varphi & \eta^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -\eta i\varphi-\eta^2\bar\varphi^2 & 0 & -\eta^3 i\bar\varphi \\ -\eta i\bar\varphi & -\eta i\varphi & \eta^2 \end{pmatrix}.$$

In the homogeneous coordinates dual to e_1, e_2, e_3, we get the equations

$$(\widehat{R_1R_2})^{-1}\colon |(\bar\eta^2 + \bar\eta^3 i\varphi)x_1 + \bar\eta i\bar\varphi x_2 + (1 + \bar\eta i\varphi - \bar\eta^2\varphi^2 + \bar\eta i\bar\varphi)x_3| = 1$$
$$\widehat{R_2R_3}\colon |(1 - \eta i\bar\varphi - \eta^2\bar\varphi^2 - \eta i\varphi)x_1 - \eta i\varphi x_2 + (\eta^2 - \eta^3 i\bar\varphi)x_3| = 1$$

multiplying the equation of $(\widehat{R_1R_2})^{-1}$ by $-\eta^3 i\bar\varphi$ it takes the form

$$(\widehat{R_1R_2})^{-1}\colon |(1 - \eta i\bar\varphi)x_1 + \eta^2\bar\varphi^2 x_2 + (\eta^2 - \eta^3 i\bar\varphi + \eta i\varphi + \eta^2\bar\varphi^2)x_3| = 1$$

or

$$|(1 - \eta i\bar\varphi - \eta^2\bar\varphi^2)x_1 + (\eta^2 - \eta^3 i\bar\varphi + \eta i\varphi)x_3 + \eta^2\bar\varphi^2(x_1 + x_2 + x_3)| = 1 .$$

Similarly

$$\widehat{R_2R_3}\colon |(1 - \eta i\bar\varphi - \eta^2\bar\varphi^2)x_1 + (\eta^2 - \eta^3 i\bar\varphi + \eta i\varphi)x_3 - \eta i\varphi(x_1 + x_2 + x_3)| = 1.$$

Therefore, set

$$z = \frac{(1 - \eta i\bar\varphi - \eta^2\bar\varphi^2)x_1 + (\eta^2 - \eta^3 i\bar\varphi + \eta i\varphi)x_3}{x_1 + x_2 + x_3}$$

to obtain as equations

$$(11.2) \qquad \begin{aligned} &(\widehat{R_1R_2})^{-1}\colon |z + \eta^2\bar\varphi^2| = 1 \\ &\widehat{R_2R_3}\colon |z - \eta i\varphi| = 1 . \end{aligned}$$

These equations have 2 solutions (see figure):

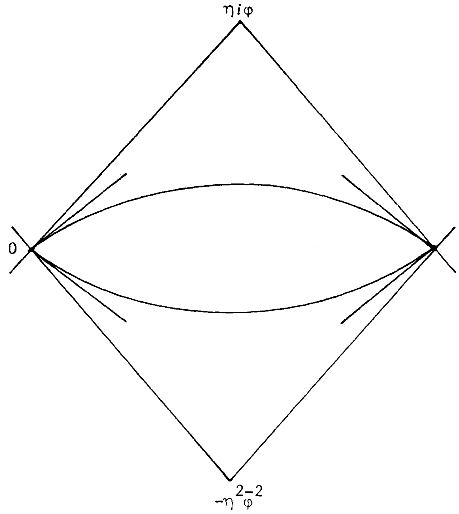


FIGURE 11.1

$$z = 0 \text{ and } z = \eta i\varphi - \eta^2\bar{\varphi}^2 .$$

We note that $1 - \eta i\bar{\varphi} - \eta^2\bar{\varphi}^2 = -\bar{\eta} i\bar{\varphi}(\eta^2 - \eta^3 i\bar{\varphi} + \eta i\varphi)$ and thus

$$z = (\eta^2 - \eta^3 i\bar{\varphi} + \eta i\varphi)\frac{(-\bar{\eta} i\bar{\varphi}x_1 + x_3)}{x_1 + x_2 + x_3} .$$

The above two solutions yield two possibilities for the preimage of I_2 and by Lemma 3.4.1, exactly *one* of two possibilities.

(11.3a) Case a: $-\bar{\eta} i\bar{\varphi}x_1 + x_3 = 0$

Case b: $-\bar{\eta} i\bar{\varphi}x_1 + x_3 = \dfrac{\eta i\varphi - \eta^2\bar{\varphi}^2}{\eta^2 - \eta^3 i\bar{\varphi} + \eta i\varphi}(x_1 + x_2 + x_3) .$

Case b is equivalent to

$$(1 - \eta i\bar{\varphi} - \eta^2\bar{\varphi}^2)x_1 + (\eta^2 - \eta^3 i\bar{\varphi} + \eta i\varphi)x_3 = (\eta i\varphi - \eta^2\bar{\varphi}^2)(x_1 + x_2 + x_3)$$

or

$$(1 - \eta i\bar{\varphi} - \eta i\varphi)x_i - (\eta i\varphi - \eta^2\bar{\varphi}^2)x_2 + (\eta^2 - \eta^3 i\bar{\varphi} + \eta^2\bar{\varphi}^2)x_3 = 0$$

or

$$(11.36) \qquad \begin{aligned} &(1 - \eta i\varphi - \eta i\bar{\varphi})x_1 - \eta i\varphi(1 + \eta i\bar{\varphi}^3)x_2 \\ &\qquad + \eta^2(1 - \eta i\bar{\varphi} + \bar{\varphi}^2)x_3 = 0 . \end{aligned}$$

Applying J, the equation for the preimage of I_2' is exactly one of the two possibilities

(11.3a)$'$ Case a: $-\bar{\eta} i\varphi x_3 + x_1 = 0$

(11.3b)′ Case b: $(1-\eta i\varphi)x_3-\eta i\bar{\varphi}(1-\eta i\varphi^3)x_2+\eta^2(1-\eta i\varphi)x_1=0$.

The next lemma determines the values of φ for which Cases a and b apply.

LEMMA 11.2. *Let v_{ijk} be the point in V defined in* § 9.1. *(i, j, k) being a permutation of* (1, 2, 3)

(1) *The equation of $\pi^{-1}(I_2)$ is given by $-\bar{\eta}i\bar{\varphi}x_1+x_3=0$ and v_{123} is the normal in V to $\pi^{-1}(I_2)$ for*

$$-\left(\frac{\pi}{2}-\frac{\pi}{p}\right)<\arg(\varphi^3)<3\left(\frac{\pi}{2}-\frac{\pi}{p}\right);$$

and $\pi^{-1}(I_2)$ is given by (11.3b) *for* $-3(\pi/2-\pi/p)<\arg\varphi^3\leqq-(\pi/2-\pi/p)$.

(2) *The equation of $\pi^{-1}(I_2')$ is given by $x_1-\bar{\eta}i\varphi x_3=0$ and v_{321} is the normal in V to $\pi^{-1}(I_2')$ for*

$$-3\left(\frac{\pi}{2}-\frac{\pi}{p}\right)<\arg(\varphi^3)<\frac{\pi}{2}-\frac{\pi}{p};$$

and $\pi^{-1}(I_2)$ is given by (11.3b)′ *for* $\pi/2-\pi/p\leqq\arg\varphi^3<3(\pi/2-\pi/p)$.

Proof. Consider the orthogonal subspace $v_{123}^{\perp}$ to v_{123} in V with respect to the inner product H. Then $x_1e_1+x_2e_2+x_3e_3$ is in $v_{123}^{\perp}$ if and only if

$$\langle x_1e_1+x_2e_2+x_3e_3,\ -\eta i\bar{\varphi}e_1+e_2+\bar{\eta}i\varphi\rangle=0$$
$$x_1[\bar{\eta}i\varphi-\alpha(\varphi-\eta i\bar{\varphi}^2]+x_2[1-\alpha(\bar{\eta}^i-\eta i)]+x_3[-\eta i\bar{\varphi}-\alpha(\bar{\eta}i\varphi^2+\bar{\varphi}]=0.$$

Replace α by $i/(\eta-\bar{\eta})$; we find

$$x_1[\varphi(-\bar{\eta}^2i-\eta\bar{\varphi}^3]+x_3[\bar{\varphi}(-\eta^2i+\bar{\eta}\varphi^3]=0\ .$$

Note that $\varphi(\bar{\eta}^2i+\bar{\eta}^3)=\bar{\eta}i\bar{\varphi}\cdot\bar{\varphi}(-\eta^2i+\bar{\eta}\varphi^3)$. Hence we get

$$v_{123}^{\perp}\colon -\bar{\eta}i\bar{\varphi}x_1+x_3=0\ .$$

By (9.1.4), $\langle v_{123}, v_{123}\rangle>0$ for $-(\pi/2-\pi/p)<\arg\varphi^3<3(\pi/2-\pi/p)$. Hence for such φ, $v_{123}^{\perp}\cap V^-$ is not empty and $\pi(v_{123}^{\perp}\cap V^-)$ is a $\boldsymbol{C}$-line in $\boldsymbol{C}h^2$. Its equation is precisely that of I_2. Hence $I_2=\pi(v_{123}^{\perp}\cap V^-)$. For the remaining values of φ, I_2 is given by (11.3a). This proves (1). (2) follows from (1) by applying the isometry $J'\colon V(\varphi)\to V(\bar{\varphi})$.

REMARK 1. In Case a, the preimage of I_2 in V^- has an equation independent of x_2. It follows at once that I_2 is stable under the $\boldsymbol{C}$-reflection $R_2\colon x\to x+(\eta^2-1)\langle x, e_2\rangle e_2$, since $x_1(e_2)=x_3(e_2)=0$. Clearly I_2 is not pointwise fixed under R_2. Being stable under R_2, it must be orthogonal to $\pi(e_2^\perp)$, the fixed point set of R_2. From this it follows that the C-reflection in the C-line I_2 commutes with R_2. But by (9.1.3), $(R_1R_2R_3)^2$ multiplies each element of $v_{123}^\perp$ by the scalar $\eta^3 i\varphi^3$. Thus I_2 is the fixed point set of $(R_1R_2R_3)^2$. Consequently R_2 commutes with $(R_1R_2R_3)^2$ in PU(H) the projective unitary group of H. We shall prove in § 14 that they commute in U(H).

REMARK 2. In the projective space $(V-\{0\})/C^*$, the intersection of the two 3-surfaces containing $(\widehat{R_1R_2})^{-1}$ and $\widehat{R_2R_3}$ meet in two complex lines, one meeting the ball V^-/C^* and one not. As $\arg\varphi^3$ passes through the value $-(\pi/2-\pi/p)$, we get $\eta i\varphi-\eta^2\bar{\varphi}^2=0$ and the two complex lines coincide. The intersection of I_2 with the boundary of the ball is thus a single point of tangency for $\bar{\varphi}^3=\bar{\eta}i$.

12. The vertices p_{ij}, s_{ij}, s_{ij}, t_{ik}.

In this and the next two sections, we verify that a region $\Omega(\varphi)$ related to the region $F(\varphi)$ satisfies the codimension-1 condition (CD1). In order to achieve this, we need information about certain k-faces of $F(\varphi)$ for $k=0, 1, 2, 3$. The apexes p_{ij}, $i\neq j\in\{1, 2, 3\}$, and the k-faces containing an apex have been discussed in § 10. It remains to discuss those k-faces which contain no apex. We do this in stages.

First we define points s_{ij}, $\tilde{s}_{ij}$, t_{ik} lying on the intersection of four spinal surfaces containing 3-faces of $F(\varphi)$. In § 13, we calculate the images of these points under γ for the $\hat{\gamma}$ which contain them. In § 14.3, we define the region $\Omega(\varphi)$ and in § 14.4 we verify the (CD1) condition for all its 3-faces.

The shape of $F(\varphi)$ undergoes a change when $\arg(\varphi^3)$ increases from values less than $\pi/2-\pi/p$ to greater values. Accordingly, we ultimately consider two cases.

Case 1. $|\arg(\varphi^3)|\leqq\pi/2-\pi/p$

Case 2. $\pi/2-\pi/p\leqq|\arg\varphi^3|<3(\pi/2-\pi/p)$.

In point of fact, parts of $F(\varphi)$ remain combinatorially unchanged for $-3(\pi/2-\pi/p)<\arg\varphi^3<\pi/2-\pi/p$, while other parts remain unchanged for $\arg\varphi^{-3}$ in the above range. Define for any i modulo 3, and for any φ

$$s_{i,i+1}=I_i\cap I'_{i+1},\ \tilde{s}_{i,i-1}=I_i\cap I'_{i-1}\,. \tag{12.1}$$

By definition therefore

$$s_{12}=(\widehat{R_3R_1})^{-1}\cap\widehat{R_1R_2}\cap(\widehat{R_3R_2})^{-1}\cap\widehat{R_2R_1}\,.$$

The intersection on the right hand side is symmetric under interchange of indices 1 and 2. Thus we can define $s_{21}=s_{12}$ without contradiction, taking care not to confuse s_{21} with

$$\tilde{s}_{21} = (\widehat{R_2R_1})^{-1} \cap \widehat{R_1R_3} \cap (\widehat{R_1R_2})^{-1} \cap \widehat{R_2R_3} .$$

More generally, set

$$s_{ij} = \widehat{R_iR_j} \cap \widehat{R_jR_i} \cap (\widehat{R_kR_i})^{-1} \cap (\widehat{R_kR_j})^{-1}$$
$$\tilde{s}_{ij} = (\widehat{R_iR_j})^{-1} \cap (\widehat{R_jR_i})^{-1} \cap \widehat{R_iR_k} \cap \widehat{R_jR_k} .$$

Also, define for any distinct i, k modulo 3,

$$t_{ik} = I_{ijk} \cap e_j^{\perp} . \tag{12.2}$$

LEMMA 12.1. *Assume* $|\arg(\varphi^3)| \leqq \pi/2 - \pi/p$. *Then*

$$s_{12} = \pi(\bar{\eta}i\varphi e_1 + \bar{\eta}i\bar{\varphi}e_2 + e_3)$$
$$\tilde{s}_{21} = \pi(-\eta i\varphi e_1 - \eta i\bar{\varphi}e_2 + e_3)$$
$$t_{32} = \pi(\alpha a(\varphi)\overline{a(\varphi)}e_1 + \overline{a(\varphi)}e_2 + a(\varphi)e_3)$$
$$t_{23} = \pi(\alpha a(\varphi)\overline{a(\varphi)}e_1 + \overline{a(\bar{\varphi})}e_2 + \overline{a(\bar{\varphi})}e_3)$$

where $\alpha = 1/(2\sin\pi/p)$, $a(\varphi) = \bar{\varphi}(1 - \eta i\varphi^3)$. *Moreover, all the above points are in the ball* $\pi(V^-)$.

Proof. By § 9.2, $\kappa(R_i(\varphi)) = R_i(\bar{\varphi})^{-1}$ for $i = 1, 2, 3$. Hence $\kappa(s_{12}(\varphi)) = \tilde{s}_{21}(\bar{\varphi})$ and $\kappa(t_{23}(\varphi)) = t_{32}(\bar{\varphi})$. Thus it suffices to verify the lemma for s_{12} and t_{32}. By definition $s_{12} = I_1 \cap I_2'$. By Lemma 11.2, the equation of the preimage of $I_2(\varphi)$ is:

$$I_2(\varphi)\colon\quad x_3 - \bar{\eta}i\bar{\varphi}x_1 = 0 .$$

Applying J^{-1}, we get as the equation for I_1:

$$I_1(\varphi)\colon\quad x_2 - \bar{\eta}i\bar{\varphi}x_3 = 0 . \tag{12.3}$$

By (11.2), $I_2'(\varphi) = J'I_2(\bar{\varphi})$. Hence the equation of the preimage of $I_2'(\varphi)$ is given by

$$I_2'(\varphi)\colon\quad x_1 - \bar{\eta}i\varphi x_3 = 0 .$$

Hence a preimage of $s_{12}(\varphi)$ is given by

$$x_1 = \bar{\eta}i\bar{\varphi},\ x_2 = \bar{\eta}i\varphi,\ x_3 = 1 ,$$

verifying the assertion for s_{12}.

As for t_{32}, by definition

$$t_{32} = I_{312} \cap e_1^{\perp} .$$

The equation of I_{312} is given by (12.1.1):

$$I_1\colon\quad x_2 - \bar{\eta}i\bar{\varphi}x_3 = 0 .$$

By (10.1.1), the equation for $e_1^{\perp}$ is

$$e_1^{\perp}\colon \quad x_1 - \alpha\bar{\varphi}x_2 - \alpha\varphi x_3 = 0 \ .$$

Solving these two equations simultaneously yields

$$\begin{aligned}
\pi^{-1}(t_{32}) &= x_3(\alpha\varphi(\bar{\eta}i\bar{\varphi}^3 + 1),\ \bar{\eta}i\bar{\varphi},\ 1) \\
&= x(\alpha(2 + \bar{\eta}i\bar{\varphi}^3 - \eta i\varphi^3,\ \varphi(1 + \bar{\eta}i\bar{\varphi}^3),\ \bar{\varphi}(1 - \eta i\varphi^3)) \\
&= x(\alpha a\bar{a},\ \bar{a},\ a)
\end{aligned}$$

where $a = a(\varphi) = \bar{\varphi}(1 - \eta i\varphi^3)$ and $x \in \boldsymbol{C} - \{0\}$.

It remains only to verify that the points are in the ball $\pi(V^-)$.

NOTATION. For convenience, we shall denote by s_{ij} and $\tilde{s}_i$, vectors in V. For example, we write

$$s_{12} = \bar{\eta}i\varphi e_1 + \bar{\eta}i\bar{\varphi}e_2 + e_3,\ s_{21} = -\eta i\varphi e_1 - \eta i\bar{\varphi}e_2 + e_3$$

and similarly we sometimes denote by t_{ik} the vector in V representing it as in the formula of Lemma 12.1. When there is risk of confusion, we write s_{ij} (in the ball) to indicate $\pi(s_{ij})$.

For the vector s_{12},

$$\begin{aligned}
\langle s_{12}, s_{12}\rangle &= 3 - \frac{i}{\eta-\eta}[\bar{\eta}i\varphi(-\eta i\varphi\cdot\varphi + \bar{\varphi}) + \bar{\eta}i\bar{\varphi}(-\eta i\bar{\varphi}\cdot\bar{\varphi} + \varphi) \\
&\qquad + (-\eta i\bar{\varphi}\cdot\varphi - \eta i\varphi\cdot\bar{\varphi})] \\
&= 3 - \frac{i}{\eta-\bar{\eta}}[-2i(\eta - \bar{\eta}) + \varphi^3 + \bar{\varphi}^3] \\
&= 1 - \frac{i}{\eta-\bar{\eta}}[\varphi^3 + \bar{\varphi}^3] \\
&= 1 - \frac{\cos\arg\varphi^3}{\sin\dfrac{\pi}{p}} \ . \qquad (12.4)
\end{aligned}$$

Thus $\langle s_{12}, s_{12}\rangle < 0$ if and only if $|\arg\varphi^3| < \pi/2 - \pi/p$. Since $\kappa(s_{12}(\varphi)) = \tilde{s}_{21}(\bar{\varphi})$, we see that $\langle\tilde{s}_{21}, \tilde{s}_{21}\rangle = \langle s_{12}, s_{12}\rangle$ and thus $\tilde{s}_{21} \in \pi(V^-)$ for $|\arg\varphi^3| < \pi/2 - \pi/p$.

Before verifying that $t_{32} \in V^-$, we note the identity

$$\begin{aligned}
a^2\bar{\varphi} + \bar{a}^2\varphi &= \bar{\varphi}^3(1 - 2\eta i\varphi^3 - \eta^2\varphi^6) + \varphi^3(1 + 2\bar{\eta}i\bar{\varphi}^3 - \bar{\eta}^2\bar{\varphi}^6) \\
&= -2\eta i + 2\bar{\eta}i + (1 - \eta^2)\varphi^3 + (1 - \bar{\eta}^2)\bar{\varphi}^3 \ .
\end{aligned}$$

Hence

$$\begin{aligned}
\alpha(a^2\bar{\varphi} + \bar{a}^2\varphi) &= \frac{i}{\eta - \bar{\eta}}\left(2\frac{\eta - \bar{\eta}}{i} + \eta(\bar{\eta} - \eta)\varphi^3 + \bar{\eta}(\eta - \bar{\eta})\bar{\varphi}^3\right) \\
&= 2 - \eta i\varphi^3 + \bar{\eta}i\bar{\varphi}^3 \\
&= a\bar{a} \ .
\end{aligned}$$

Then

$$\begin{aligned}\langle t_{32}, t_{32}\rangle &= \langle \alpha a\bar{a}e_1 + \bar{a}e_2 + ae_3, \alpha a\bar{a}e_1 + \bar{a}e_2 + ae_3\rangle \\ &= \alpha a\bar{a}[\alpha a\bar{a} - \alpha\varphi a - \alpha\bar{\varphi}\bar{a}] \\ &\qquad + \bar{a}[a - \alpha^2 a\bar{a}\bar{\varphi} - \alpha\bar{a}\varphi] \\ &\qquad + a[\bar{a} - \alpha^2 a\bar{a}\varphi - \alpha a\bar{\varphi}] \\ &= 2a\bar{a} - \alpha^2 a\bar{a} - \alpha(a^2\bar{\varphi} + \bar{a}^2\varphi) \\ &= a\bar{a}(1 - \alpha^2 a\bar{a}) + a\bar{a} - \alpha(a^2\bar{\varphi} + \bar{a}^2\varphi) \\ &= a\bar{a}(1 - \alpha^2 a\bar{a})\ .\end{aligned}$$

Thus $\langle t_{32}, t_{32}\rangle < 0$ if and only if $1 - \alpha^2 a\bar{a} < 0$; that is, $\alpha^{-2} < a\bar{a}$. This is equivalent to

$$\left(\frac{\eta - \bar{\eta}}{i}\right)^2 < (1 - \eta i\varphi^3)(1 + \bar{\eta}i\bar{\varphi}^3)$$

$$2 - \eta^2 - \bar{\eta}^2 < 2 - \eta i\varphi^3 + \bar{\eta}i\bar{\varphi}^3$$

$$-2\cos\frac{2\pi}{p} < 2\sin\left(\frac{\pi}{p} + 3\theta\right), \quad \theta = \arg\varphi$$

$$-\sin\left(\frac{\pi}{2} - \frac{2\pi}{p}\right) < \sin\left(\frac{\pi}{p} + 3\theta\right)$$

$$-\left(\frac{\pi}{2} - \frac{\pi}{p}\right) < 3\theta\ .$$

Thus $t_{32} \in V^-$ for $-(\pi/2 - \pi/p) < \arg\varphi^3 < 3(\pi/2 - \pi/p)$. Applying $t_{23}(\varphi) = \kappa(t_{32}(\bar{\varphi}))$, we find that $t_{23} \in V^-$ for $-3(\pi/2 - \pi/p) < \arg\varphi^3 < \pi/2 - \pi/p$.

The next lemma will allow us to determine how the cell complex $F(\varphi)$ changes as $\arg\varphi^3$ attains the values $\pi/2 - \pi/p$.

LEMMA 12.2. *For* $|\arg(\varphi^3)| \leqq \pi/2 - \pi/p$, *and for any integer* i *modulo* 3,

(i) $s_{i,i+1} = \bar{\eta}\sqrt{-1}\varphi e_i + \bar{\eta}\sqrt{-1}\bar{\varphi}e_{i+1} + e_{i-1}$
$\tilde{s}_{i+1,i} = -\eta\sqrt{-1}\varphi e_i - \eta\sqrt{-1}\bar{\varphi}e_{i+1} + e_{i-1}$
$t_{i,i-1} = a(\varphi)e_i + \bar{a}(\varphi)e_{i-1} + \alpha a(\varphi)\bar{a}(\varphi)e_{i+1}$
$t_{i,i+1} = a(\bar{\varphi})e_i + \bar{a}(\bar{\varphi})e_{i+1} + \alpha a(\bar{\varphi})\bar{a}(\bar{\varphi})e_{i-1}$

where $\alpha = (2\sin\pi/p)^{-1}$, $a(\varphi) = \bar{\varphi}(1 - \eta i\varphi^3)$, *and* $\bar{a}(\varphi) = \overline{a(\varphi)}$.

(ii) $s_{i,i+1} = \tilde{s}_{i,i-1}$ *if and only if* $\arg\varphi^3 = -(\pi/2 - \pi/p)$
$s_{i,i+1} = \tilde{s}_{i-1,i+1}$ *if and only if* $\arg\varphi^3 = \pi/2 - \pi/p$

(iii) $s_{i,i+1} = t_{i-1,i+1}$ *if and only if* $\arg\varphi^3 = -(\pi/2 - \pi/p)$
$\tilde{s}_{i+1,i} = t_{i+1,i-1}$ *if and only if* $\arg\varphi^3 = \pi/2 - \pi/p$.

Proof. By Lemma 12.2

$$\tilde{s}_{13} = \pi(-\eta i\bar{\varphi}e_1 + e_2 - \eta i\varphi e_3)$$

and $s_{12} = \pi(\bar{\eta} i\varphi e_1 + \bar{\eta} i\bar{\varphi} e_2 + e_3) = \pi(\eta^2 e_1 + e_2 - \eta i\varphi e_3)$. Thus $s_{12} = \tilde{s}_{13}$ if and only if $\varphi^3 = -\eta i$. By applying J and J^{-1}, we get the first assertion of (i), and by applying κ one gets the second.

As for the remaining values of φ, the next lemma gives an explicit formula for the points s_{ij}, $\tilde{s}_{ij}$, and t_{ik} in terms of the vector v_{ijk} of V listed in § 9.1.

LEMMA 12.3. *Assume that* $\pi/2 - \pi/p \leqq |\arg \varphi^3| < 3(\pi/2 - \pi/p)$. *Then for any permutation* ijk *of* (1, 2, 3),

$$s_{ij} = \begin{cases} v_{kij} & \text{if } \langle v_{kij}, v_{kij}\rangle < 0 \\ v_{kji} & \text{if } \langle v_{kji}, v_{kji}\rangle < 0 \end{cases}$$

$$\tilde{s}_{ij} = \begin{cases} v_{ijk} & \text{if } \langle v_{ijk}, v_{ijk}\rangle < 0 \\ v_{jik} & \text{if } \langle v_{jik}, v_{jik}\rangle < 0 \end{cases}$$

$$t_{ik} = \begin{cases} v_{ijk} & \text{if } \langle v_{ijk}, v_{ijk}\rangle < 0 \\ \text{by the formula (12.2)} & \text{if } \langle v_{ijk}, v_{ijk}\rangle > 0 \,. \end{cases}$$

That is,

$$s_{12} = \tilde{s}_{13} = t_{32} = v_{312}$$

$$\text{for } -3\left(\frac{\pi}{2} - \frac{\pi}{p}\right) < \arg \varphi^3 \leqq -\left(\frac{\pi}{2} - \frac{\pi}{p}\right)$$

$$\tilde{s}_{21} = s_{31} = t_{23} = v_{213}$$

$$\text{for } 3\left(\frac{\pi}{2} - \frac{\pi}{p}\right) > \arg \varphi^3 \geqq 3\left(\frac{\pi}{2} - \frac{\pi}{p}\right) .$$

Proof. The computation of the indicated points in the indicated range for φ proceeds from equation (11.3.b) for I_{ijk}, and the verification entails straightforward solutions of two linear equations. The condition on φ that $v_{ijk} \in V^-$ can be read off (9.1.4) and (9.1.5).

REMARK 1. For $|\arg \varphi| \geqq \pi/2 - \pi/p$, $\tilde{s}_{ki} = s_{ij}$ if and only if $\langle v_{ijk}, v_{ijk}\rangle \leqq 0$.

REMARK 2. As a consequence of the given relations in the group Γ, we will see in § 14 that $\langle v_{ijk}, v_{ikj}\rangle = 0 = \langle v_{ijk}, v_{jik}\rangle$. This can be verified directly of course. Since H has signature (two positive, one negative) $\langle v_{ijk}, v_{ijk}\rangle < 0$ implies $\langle v_{jik}, v_{jik}\rangle = \langle v_{ikj}, v_{ikj}\rangle > 0$. If $\langle v_{ijk}, v_{ijk}\rangle = 0$, then $v_{ijk} \in \boldsymbol{C} v_{ikj} + \boldsymbol{C} v_{jik}$.

13. Towards verification of (CD1): images of points.

LEMMA 13.1. *For all distinct* i, j *and all values of* φ *with* $|\arg \varphi^3| < 3(\pi/2 - \pi/p)$

$$\pi(R_i s_{ij}) = \pi(\tilde{s}_{ik}) .$$

Proof. By symmetry, it suffices to prove that

$$\pi(R_1 s_{12}) = \pi(\tilde{s}_{13}) .$$

If $|\arg \varphi^3| \leqq \pi/2-\pi/p$, then by Lemma 12.1, $s_{12} = \bar{\eta} i \varphi e_1 + \bar{\eta} i \bar{\varphi} e_2 + e_3$ and the components of $R_1 s_{12}$ are given by

$$\begin{pmatrix} \eta^2 & -\eta i \bar{\varphi} & -\eta i \varphi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{\eta} i \varphi \\ \bar{\eta} i \bar{\varphi} \\ 1 \end{pmatrix} = \begin{pmatrix} \bar{\varphi}^2 \\ \bar{\eta} i \bar{\varphi} \\ 1 \end{pmatrix} = \bar{\eta} i \bar{\varphi} \begin{pmatrix} -\eta i \bar{\varphi} \\ 1 \\ \eta i \varphi \end{pmatrix}$$

so that $\pi(R_1 s_{12}) = \pi(\tilde{s}_{13})$.

If $\pi/2 - \pi/p \leqq \arg \varphi^3 < 3(\pi/2 - \pi/p)$, then by Lemma 12.3, $s_{12} = v_{321}$ since $\langle v_{321}, v_{321} \rangle = \langle v_{213}, v_{213} \rangle < 0$. Thus $R_1 s_{12}$ is given by (cf. § 9.1 for v_{321} and v_{132}).

$$\begin{pmatrix} \eta^2 & -\eta i \bar{\varphi} & -\eta i \varphi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{\eta} i \bar{\varphi} \\ 1 \\ -\eta i \varphi \end{pmatrix} = \begin{pmatrix} -\eta^2 \varphi^2 \\ 1 \\ -\eta i \varphi \end{pmatrix} = -\eta i \varphi \begin{pmatrix} -\eta i \varphi \\ \bar{\eta} i \bar{\varphi} \\ 1 \end{pmatrix} .$$

Thus $R_1 s_{12} = -\eta i \varphi v_{132} = -\eta i \varphi \tilde{s}_{13}$ and again $\pi(R_1 s_{12}) = \pi(\tilde{s}_{13})$. The result for $-3(\pi/2 - \pi/p) < \arg \varphi^3 \leqq \pi/2 - \pi/p$ can be deduced from the above by applying the isometry J': $V(\varphi) \to V(\bar{\varphi})$.

LEMMA 13.2. *For all distinct i, j and for all φ with $|\arg \varphi^3| < 3(\pi/2 - \pi/p)$*

$$\pi(R_k \tilde{s}_{ji}) = \pi(s_{ij}) .$$

Proof. It suffices to prove the result for $R_3 \tilde{s}_{21}$. If $|\arg \varphi^3| < \pi/2 - \pi/p$, the computation is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\eta i \bar{\varphi} & -\eta i \varphi & \eta^2 \end{pmatrix} \begin{pmatrix} -\eta i \varphi \\ -\eta i \bar{\varphi} \\ 1 \end{pmatrix} = \begin{pmatrix} -\eta i \varphi \\ -\eta i \bar{\varphi} \\ -\eta^2 \end{pmatrix} = -\eta^2 \begin{pmatrix} \bar{\eta} i \varphi \\ \bar{\eta} i \bar{\varphi} \\ 1 \end{pmatrix} .$$

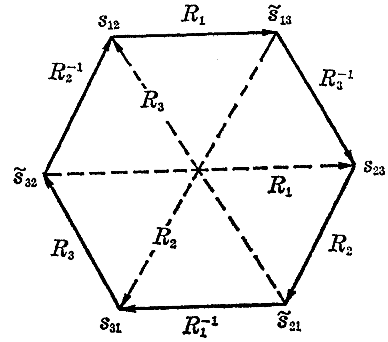
If $\pi/2 - \pi/p \leqq \arg \varphi^3 < 3(\pi/2 - \pi/p)$, then by Lemma 12.3, $\tilde{s}_{21} = v_{213}$ and $s_{12} = v_{321}$. The computation is (cf. § 9.1).

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\eta i \bar{\varphi} & -\eta i \varphi & \eta^2 \end{pmatrix} \begin{pmatrix} 1 \\ -\eta i \varphi \\ \bar{\eta} i \bar{\varphi} \end{pmatrix} = \begin{pmatrix} 1 \\ -\eta i \varphi \\ -\eta^2 \varphi^2 \end{pmatrix} = -\eta i \varphi \begin{pmatrix} \bar{\eta} i \bar{\varphi} \\ 1 \\ -\eta i \varphi \end{pmatrix} .$$

For the remaining values of φ, the result can be deduced from the foregoing by means of the isometry J': $V(\varphi) \to V(\bar{\varphi})$.

The information in Lemmas 13.1 and 13.2 yields the hexagonal diagram

(13.1)



LEMMA 13.3. (i) *For each index i modulo 3,*

$$R_iR_{i+1} = J \text{ on } I_i \text{ if } s_{i,i+1} \neq \tilde{s}_{i,i-1}$$
$$R_{i+1}R_i = J^{-1} \text{ on } I'_{i+1} \text{ if } s_{i,i+1} \neq \tilde{s}_{i-1,i+1}$$

(ii) $(R_iR_jR_k)^2$ *fixes each point of* I_{ijk} *if*

$$\langle v_{ijk}, v_{ijk}\rangle > 0$$

(iii) $R_kR_iR_jR_is_{ij} = s_{ij}$ (*in the ball*)
(all *permutations* (i, j, k) *of* $(1, 2, 3)$ *in* (ii) *and* (iii)).

Proof. (i) By symmetry, it suffices to consider the case $i = 1$. Then by definition (12.1),

$$s_{12} = I_1 \cap I'_2,\ \tilde{s}_{13} = I_1 \cap I'_3 .$$

By the hypothesis $s_{12} \neq \tilde{s}_{13}$, I_1 is the unique C-line containing s_{12} and $\tilde{s}_{13}$. From the hexagonal diagram,

$$R_1R_2s_{12} = s_{23},\ R_1R_2\tilde{s}_{13} = \tilde{s}_{21} .$$

Hence $R_1R_2I_1 = I_2$, the unique C-line containing s_{23} and $\tilde{s}_{21}$. Since the restrictions of R_1R_2 and J to I_1 are isometries, which coincide on s_{12} and $\tilde{s}_{13}$, the isometry $J^{-1}R_1R_2$ fixes these two points and therefore every point of the geodesic line joining them, and therefore every point of the C-line joining them. It follows that R_1R_2 and J coincide on I_1. That $R_2R_1 = J^{-1}$ on I'_1 comes from applying the complex conjugation isometry κ_2: $V(\bar{\varphi}) \to V(\varphi)$:

$$\kappa(s_{12}(\bar{\varphi})) = \tilde{s}_{21}(\varphi),\ \kappa(\tilde{s}_{13}(\bar{\varphi})) = s_{31}(\varphi)$$
$$\kappa(R_1(\bar{\varphi})) = R_1^{-1}(\varphi),\ \kappa(R_2(\bar{\varphi})) = R_2^{-1}(\varphi)$$
$$\kappa(I_1(\bar{\varphi})) = I'_1(\varphi) .$$

At $\bar{\varphi}$, $R_1R_2s_{12} = s_{23}$, $R_1R_2\tilde{s}_{13} = \tilde{s}_{21}$. Applying κ, we get at φ: $R_1^{-1}R_2^{-1}\tilde{s}_{21} = \tilde{s}_{32}$, $R_1^{-1}R_2^{-1}s_{31} = s_{12}$. Hence $R_2R_1\tilde{s}_{32} = \tilde{s}_{21}$, $R_2R_1s_{12} = s_{31}$. It follows that R_2R_1 coincides with J^{-1} on I_2'. If $\langle v_{123}, v_{123}\rangle > 0$ (resp. $\langle v_{321}, v_{321}\rangle > 0$), we can apply Remark 1 following Lemma 12.3. The information in (i) then yields the triangular diagrams

(13.2)
$$\begin{tikzcd} & I_1 \arrow[dr, "R_1R_2"] & \\ I_3 \arrow[ur, "R_3R_1"] & & I_2 \arrow[ll, "R_2R_3"] \end{tikzcd} \qquad \begin{tikzcd} & I_1' \arrow[dl, "R_1R_3"'] & \\ I_3' \arrow[rr, "R_3R_2"'] & & I_2' \arrow[ul, "R_2R_1"'] \end{tikzcd}$$

On I_2, $R_1R_2 \cdot R_3R_1 \cdot R_{23} = J^3 =$ identity, so that $(R_1R_2R_3)^2$ fixes each point of I_2 if $\langle v_{123}, v_{123}\rangle > 0$. Similarly $(R_3R_2R_1)^2$ fixes each point of I_2' if $\langle v_{321}, v_{321}\rangle > 0$.

(iii) follows at once from the hexagonal diagram.

REMARK. Assertion (ii) of Lemma 13.3 has been pointed out before in Remark 1 following Lemma 11.2; it comes from the fact that v_{ijk} is the eigenvector of $(R_iR_jR_k)^2$ corresponding to the eigenvalue of multiplicity 1. Thus $v_{ijk}^{\perp} \cap V^-$ is not empty if and only if $\langle v_{ijk}, v_{ijk}\rangle > 0$.

Consequences in another direction of the information in the hexagonal diagram are given in the next lemma.

LEMMA 13.4. *For any distinct i, j from* $\{1, 2, 3\}$,

$$\begin{aligned} s_{ij} &= \hat{R}_i \cap \hat{R}_j \cap R_i\hat{R}_j \cap \widehat{R_jR_i} \cap R_i\hat{R}_jR_i \cap \hat{R}_k^{-1} \cap (\widehat{R_kR_i})^{-1} \cap (\widehat{R_kR_j})^{-1} \\ \tilde{s}_{ij} &= \hat{R}_i^{-1} \cap \hat{R}_j^{-1} \cap (\widehat{R_iR_j})^{-1} \cap (R_j\hat{R}_i)^{-1} \cap (R_i\hat{R}_jR_i)^{-1} \cap \hat{R}_k \cap \widehat{R_iR_k} \cap \widehat{R_jR_k} \\ t_{ik} &= \hat{R}_j \cap \hat{R}_j^{-1} \cap \widehat{R_jR_k} \cap (R_i\hat{R}_j)^{-1} \cap R_j\hat{R}_kR_j \cap (R_j\hat{R}_iR_j)^{-1} . \end{aligned}$$

Proof. The definition of the above points are given in (12.1) and (12.2), explicit formulae for them being given in Lemmas 12.2 and 12.3.

It is easy to verify that the number $|\langle p_0, s_{ij}\rangle|$ is invariant under the cyclic permutation automorphism J and also under the isometries J' and κ of $V(\varphi)$ to $V(\bar{\varphi})$ and hence under the group Isom F of the six isometries of $V(\varphi)$ to $V(\varphi)$ generated by J and a_{12} (cf. § 10.2). Under Isom F, the six points $\{s_{ij}, \tilde{s}_{ji}; i \neq 1\}$ are permuted transitively and the point p_0 is fixed. Hence for any permutation (ijk) of (1, 2, 3),

$$d(p_0, s_{ij}) = d(p_0, \tilde{s}_{ik}) .$$

By Lemma 13.1, $R_is_{ij} = \tilde{s}_{ik}$ (in the ball). Hence $d(\tilde{s}_{ij}, p_0) = d(s_{ik}, p_0) = d(R_is_{ij}, p_0)$. By definition therefore, $s_{ij} \in \hat{R}_i$. By symmetry $s_{ij} \in \hat{R}_j$.

The proof $s_{ij} \in \hat{R}_k^{-1}$ is similar. Also

$$R_j R_i s_{ij} = R_j \tilde{s}_{ik} = s_{ki}$$

so that $d(s_{ij}, p_0) = d(s_{ki}, p_0) = d(R_j R_i s_{ij}, p_0)$. Hence $s_{ij} \in \widehat{R_j R_i}$. Also

$$R_i R_j R_i s_{ij} = R_i(s_{ki}) = R_i(s_{ik}) = \tilde{s}_{ij}$$

so that $d(s_{ij}, p_0) = d(\tilde{s}_{ij}, p_0) = d(R_i R_j R_i s_{ij}, p_0)$. Hence $s_{ij} \in \widehat{R_i R_j R_i}$. The remaining assertions for s_{ij} and $\tilde{s}_{ij}$ result from applying Isom F to the above results:

As for t_{ik}, by definition,

$$t_{ik} = I_{ijk} \cap e_j^{\perp} = \hat{R}_i \cap \hat{R}_j^{-1} \cap R_j R_k \cap (\widehat{R_i R_j})^{-1} \ .$$

Since R_j fixes each point of $e_j^{\perp}$, $R_j R_k R_j(t_{ik}) = R_j R_k t_{ik}$. If $\langle v_{ijk}, v_{ijk} \rangle > 0$, then I_{ijk} is given by (11.3a) and Lemma 13.3 (i) yields $R_j R_k t_{ik} = t_{ji}$. Thus

$$d(R_j R_k R_j t_{ik}, p_0) = d(t_{ji}, p_0) = d(t_{ik}, p_0)$$

and $t_{ik} \in R_j \widehat{R_k} R_j$. On the other hand, if $\langle v_{ijk}, v_{ijk} \rangle < 0$, then $t_{ik} = v_{ijk} = s_{jk}$, and

$$R_j R_k t_{ik} = R_j R_k \tilde{s}_{jk} = R_j s_{ki} = s_{ik} = v_{jki} = t_{ji} \ .$$

Thus by the same argument, $t_{ik} \in R_j \widehat{R_k} R_j$ is this case too. That $t_{ik} \in (R_j \hat{R}_i R_j)^{-1}$ can be deduced from the foregoing by applying the isometry a_{ik}. The proof of Lemma 13.4 is now complete.

LEMMA 13.5. *For all permutation ijk of* (1, 2, 3) *and for all* φ *with* $|\arg \varphi^3| < 3(\pi/2 - \pi/p)$

$$R_j R_k t_{ik} = t_{ji} \qquad (\textit{in the ball}) \ .$$

Proof. This was demonstrated in the proof of Lemma 13.4.

14. Some identities in Γ, some lines in F, and (CD1) for $\Omega(\varphi)$.

14.1. *Relations in* Γ. We mentioned in Remark 2 following Lemma 12.3 that $\langle v_{ijk}, v_{jik} \rangle = 0 = \langle v_{ijk}, v_{ikj} \rangle$ for all permutations (ijk) of (1, 2, 3). The group relations proved in the next lemma explain these orthogonality relations and the relation $\langle e_j, v_{ijk} \rangle = 0$ as well; they arise from the relation $R_i R_j R_i = R_j R_i R_j$.

LEMMA 14.1. *For any permutation* (ijk) *of* (1, 2, 3) *and for all* φ,
(i) $R_j(R_i R_j R_k)^2 = (R_i R_j R_k)^2 R_j$.

(ii) *Let* $a = R_iR_jR_kR_j$, $b = R_j^{-1}R_iR_jR_k$, $c = R_k^{-1}R_iR_kR_j$ *and* $\Gamma' = \{\{a, b\}\}$, *the subgroup of* Γ *generated by* $\{a, b\}$. *Then* Γ' *is abelian.*

(iii) $(R_iR_jR_k)^2$ *and* $(R_iR_kR_j)^2$ *commute.*

(iv) $(R_iR_jR_k)^2$ *and* $(R_jR_iR_k)^2$ *commute.*

(v) $a^3 = (R_iR_jR_k)^2 \cdot (R_iR_kR_j)^2$.

Proof.

$$\begin{aligned} R_jR_iR_jR_kR_iR_jR_k &= R_iR_jR_iR_kR_iR_jR_k \\ &= R_iR_jR_kR_iR_kR_jR_k \\ &= R_iR_jR_kR_iR_jR_kR_j . \end{aligned}$$

This proves (i).

By (i) $ab = (R_iR_jR_k)^2 = ba$. This proves (ii).

$$\begin{aligned} ca &= R_k^{-1}R_iR_kR_jR_iR_jR_kR_j \\ &= R_k^{-1}R_iR_kR_jR_iR_kR_jR_k \\ &= (R_iR_kR_j)^2, \text{ by (i).} \end{aligned}$$

Moreover $b^{-1}a = R_k^{-1}R_j^{-1}R_i^{-1}R_jR_iR_jR_kR_j = R_k^{-1}R_j^{-1}R_i^{-1} \cdot R_iR_jR_iR_kR_j = R_k^{-1}R_iR_kR_j = c$. Hence $\{\{a, b\}\} = \{a, c\}$ and $(R_iR_kR_j)^2 \in \Gamma'$. This and (ii) imply (iii). Applying the isomorphism $\kappa: \Gamma(\bar{\varphi}) \to \Gamma(\varphi)$ to the relation (iii) in $\Gamma(\bar{\varphi})$ yields that $(R_i^{-1}R_j^{-1}R_k^{-1})^2$ and $(R_i^{-1}R_k^{-1}R_j^{-1})^2$ commute in $\Gamma(\varphi)$. This implies (iv).

We have from above

$$(R_iR_jR_k)^2(R_iR_kR_j)^2 = abca = ab(b^{-1}a)a = a^3 .$$

This proves (iii).

14.2. *Geodesic lines.*

LEMMA 14.2.1. *Assume* $-(\pi/2 - \pi/p) < \arg \varphi^3 < 3(\pi/2 - \pi/p)$. *For all* i,

(i) $I_i \cap \hat{R}_i$ *and* $I_i \cap \hat{R}_i^{-1}$ *are geodesic lines.*

(ii) $e_i^\perp \cap \widehat{R_iR_{i+1}}$ *and* $e_i^\perp \cap (\widehat{R_{i-1}R_i})^{-1}$ *are geodesic lines.*

Proof. I_i is a slice of $\widehat{R_iR_{i+1}}$ by Lemma 11.1. Clearly $e_i^\perp$ is a slice of $\hat{R}_i$. We have $e_i^\perp$ is orthogonal to I_i at $t_{i-1,i+1}$; for by Lemma 11.2 (i), v_{123} is the normal to I_2, and $\langle e_2, v_{123}\rangle = 0$ as is easily verified. Hence $I_i \cap \hat{R}_i$ is a geodesic line by Lemma 3.2.5. The same argument applies to $I_i \cap \widehat{R_i^{-1}}$. This proves (i).

Since the C-line e_i is orthogonal to the slice $I_i = (\widehat{R_{i-1}R_i})^{-1} \cap R_i\hat{R}_{i+1}$, (ii) also follows from Lemma 3.2.5.

REMARK 1. For $-3(\pi/2 - \pi/p) < \arg \varphi^3 < \pi/2 - \pi/p$, we get by the map $J'\colon V(\bar{\varphi}) \to V(\varphi)$, that:

(i)′ $I_i' \cap \hat{R}_i$ and $I_i' \cap \hat{R}_i^{-1}$ are geodesic lines.

(ii)′ $e_i^{\perp} \cap \widehat{R_i R_{i-1}}$ and $I_i' \cap (\widehat{R_{i+1} R_i})^{-1}$ are geodesic lines.

Assertions (ii) and (ii)′ are in fact valid for $|\arg \varphi^3| < 3(\pi/2 - \pi/p)$. For $e_i^{\perp} \cap \widehat{R_1 R_2}$ is real analytic curve in the ball depending analytically on the parameter φ throughout the interval $|\arg \varphi^3| < 3(\pi/2 - \pi/p)$. Since it is a geodesic line on an open subinterval of φ, and since the condition of being a geodesic line is real analytic (with respect to standard nonhomogeneous coordinates centered at p_{12}, it amounts to being a real line), it follows that $e_1^{\perp} \cap R_1 R_2$ is a geodesic line for all φ with $|\arg \varphi^3| < 3(\pi/2 - \pi/p)$.

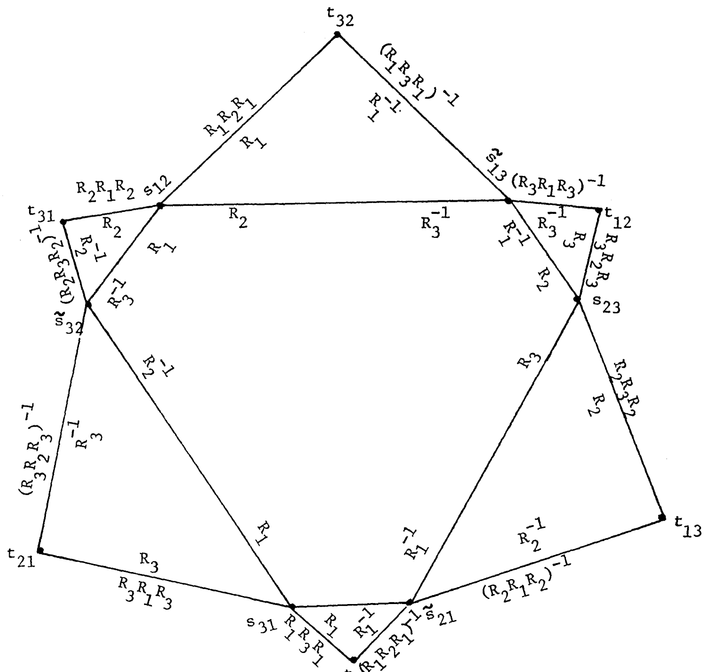


FIGURE 14.1. A schematic drawing of $\Delta_{ijk} = I_{ijk} \cap F(\varphi)$ for $|\arg \varphi^3| < \pi/2 - \pi/p$. As $\varphi^3 \to \bar{\eta} i$, Δ_{321} approaches the point at ∞. For $\arg \varphi^3 > \pi/2 - \pi/p$, Δ_{321} becomes a single (finite) point in the ball so long as $\arg \varphi^3 < 3(\pi/2 - \pi/p)$. For $-3(\pi/2 - \pi/p) < \arg \varphi^3 \leqq -(\pi/2 - \pi/p)$, Δ_{123} is a single point.

LEMMA 14.2.2. *Assume* $\langle v_{ijk}, v_{ijk} \rangle > 0$, *then*

(i) $R_k \hat{R}_j R_k \cap I_{ijk} = \hat{R}_j \cap I_{ijk}$

(ii) $(R_i \hat{R}_j R_i)^{-1} \cap I_{ijk} = \hat{R}_j^{-1} \cap I_{ijk}$

for all permutation (i, j, k) *of* $(1, 2, 3)$.

Proof. Set $l_j = \hat{R}_j \cap I_{ijk}$. Then for any $x_1 \in l_1$, set $x_2 = Jx_1$, $x_3 = J^2x_1$. Then $d(x_1, p_0) = d(x_2, p_0) = d(x_3, p_0)$ since $Jp_0 = p_0$. By Lemma 13.3, $J = R_iR_{i+1}$ on I_i. Hence $d(R_1R_2R_1x_1, p_0) = d(R_2R_1R_2x_1, p_0) = d(R_2x_2, p_0) = d(x_2, p_0)$ since $x_2 \in \hat{R}_2$. Hence $d(R_1R_2R_1x_1p_0) = d(x_1, p_0)$. It follows that $l_1 \subset R_1\hat{R}_2R_1 \cap I_{312}$. By Lemma 3.2.7, $R_1\hat{R}_2R_1 \cap I_{312}$ is a circular arc. Consequently $l_1 = R_1\hat{R}_2R_1 \cap I_{312}$. By symmetry $R_j\hat{R}_kR_j \cap I_{ijk} = R_j \cap I_{ijk}$. Since $R_jR_kR_j = R_kR_jR_k$, (i) follows. (ii) follows from (i) by applying the automorphism a_{ik}.

LEMMA 14.2.3. *For all permutations* (ijk) *of* (123) *and for all* φ *with* $|\arg \varphi^3| < 3(\pi/2 - \pi/p)$, $R_j\hat{R}_kR_j \cap e_j^\perp$ *is a geodesic line.*

Proof. Assume first that $\langle v_{ijk}, v_{ijk}\rangle > 0$. By Lemmas 14.2 and 14.1, $R_j\hat{R}_kR_j \cap I_{ijk}$ is a geodesic line l_j which contains t_{ik}. By Lemma 3.2.7. (iv) l_j and the spine of $R_j\hat{R}_kR_j$ span a geodesic $\boldsymbol{R}$-2-plane G. The $\boldsymbol{C}$-line $e_j^\perp$ is orthogonal to I_{ijk} at t_{ik}. By Lemma 3.2.8, $R_j\hat{R}_kR_j \cap e_j^\perp$ is the geodesic line, for it coincides with $G \cap e_j^\perp$.

By the analytic continuation argument of Remark 1 following Lemma 14.2, $R_j\hat{R}_kR_j \cap e_j^\perp$ is a geodesic line for all φ with $|\arg \varphi^3| < 3(\pi/2 - \pi/p)$.

LEMMA 14.2.4. *For all permutations* (ijk) *of* (123) *and for all* φ *with* $|\arg \varphi^3| < 3(\pi/2 - \pi/p)$,

(i) $\hat{R}_i^{-1} \cap I_{ijk}$ *is a geodesic line.*

(ii) $\hat{R}_i^{-1} \cap I_{ijk} = \hat{R}_k \cap I_{ijk}$.

Proof. By Lemma 14.2.1(i)$'$, $\hat{R}_i^{-1} \cap I_i'$ is a geodesic line. By Remark 2 following Lemma 12.3, I_i' is orthogonal to I_{i+1} at $\tilde{s}_{i+1,i}$. By the argument above based on Lemma 3.2.8, $\hat{R}_i^{-1} \cap I_{i+1}$ is a geodesic line. By symmetry one infers that $\hat{R}_i^{-1} \cap I_{ijk}$ is a geodesic line as asserted in (i). By applying the isometry $a_{ik}: V(\bar{\varphi}) \to V(\varphi)$, one infers that $\hat{R}_k \cap I_{ijk}$ is a geodesic line. By Lemma 13.4, both $\hat{R}_i^{-1} \cap I_{ijk}$ and $\hat{R}_k \cap I_{ijk}$ contain the points s_{jk} and $\tilde{s}_{ji}$. Since geodesic lines in $\boldsymbol{Ch}^n$ are unique, (ii) follows.

14.3. *The region* $\Omega(\varphi)$.

We have defined in § 10.2

$$F(\varphi) = F_{12} \cap F_{23} \cap F_{31}\,.$$

From the results of § 10.1, we have

$$s_{ij} \in F_{12},\ \tilde{s}_{ij} \in F_{12}$$

for all distinct i, j (modulo 3). Also

$$t_{13},\ t_{23},\ t_{31},\ t_{32} \in F_{12}\ .$$

By symmetry it follows that for $i \neq j$,

$$s_{ij},\ \tilde{s}_{ij} \in F_{12} \cap F_{23} \cap F_{31}$$

and also

$$t_{ik},\ t_{ki} \in F_{ij} \cap F_{jk}$$

and as a matter of fact for all permutations of i, j, k, the points p_{ij}, i_{ij}, $\tilde{s}_{ij}$, t_{ik} are in $F(\varphi)$.

We would like to assert that the above points make up all the vertices of the cell $F(\varphi)$. To verify this, we would have to show that the faces of the region F_{12} intersect the edges of $F(\varphi)$ in no points other than the above vertices. Proving this would entail estimates on the derivatives of $(\cosh d(x, p_0))^2 - (\cosh d(\gamma x, p_0))^2$ and their behavior on the faces of $F(\varphi)$. We circumvent this difficulty in the following way.

Let $\Delta_{ij} = \{R_i^{\pm 1}, (R_iR_j)^{\pm 1}, (R_iR_jR_i)^{\pm 1}; i \neq j = 1, 2, 3\}$, and let $\Delta = \Delta_{12} \cup \Delta_{23} \cup \Delta_{31}$. As in § 10 set $F_{12} = \bigcap_{\gamma \in \Delta_{12}} \gamma^+$. Let $E_k(F_{12})$ denote the set of codimension k-faces of F_{12}. The set $E_2^{12}(F_{12})$ consists of twenty 2-faces; set ${}^*e_i^\perp = {}^*R_i \cap {}^*R_i^{-1}$ (where ${}^*\gamma = \hat{\gamma} \cap F_{12}$ for $\gamma \in \Delta_{12}$), $i = 1, 2$. Then for

(i) any $e \in E_2(F_{12})$, $e \neq {}^*e_1^\perp$ or ${}^*e_2^\perp$, the information of § 14.2 shows that $e \cap F(\varphi)$ lies in a curvilinear triangle with vertex at p_{12} and opposite edge a geodesic line.

(ii) $e_1^\perp \cap F(\varphi)$ lies in a geodesic quadrilateral $p_{12}t_{32}p_{31}t_{23}$ bounded the geodesic lines

$$e_1^\perp \cap \widehat{R_1R_2R_1},\ e_1^\perp \cap (\widehat{R_2R_1})^{-1},\ e_1^\perp \cap \widehat{R_3R_1R_3},\ e_1^\perp \cap (\widehat{R_2R_1})^{-1}\ .$$

For each $e \in E_2(F_{12})$, define the $\tilde{e}$ as the above curvilinear triangle if $e \neq e_i^\perp$ and as the geodesic quadrilateral for $e = e_1^\perp$.

By use of the automorphism J, we define $\tilde{e}$ for $e \in E_2(F_{12}) \cup E_2(F_{23}) \cup E_3(F_{23})$. (It is easy to verify that the resulting $\tilde{e}_2^\perp = J^*e_1^\perp$ coincides with the geodesic quadrilateral obtainable from ${}^*e_2^\perp \in E_2(F_{12})$.) For any $\gamma \in \Delta$, we define $\tilde{\gamma}$ as the region of $\hat{\gamma}$ that is bounded by the 2-faces $\tilde{e}$ lying in $\hat{\gamma}$. Finally, we define $\Omega(\varphi)$ as the 4-dimensional region which is bounded by $\{\tilde{\gamma}; \gamma \in \Delta\}$. It is clear that $F(\varphi) \subset \Omega(\varphi)$. Computer calculations for certain values of φ of interest to us, shows that in all such cases $F(\varphi) = \Omega(\varphi)$.

Indeed, for all cases in which $\Omega(\varphi)$ satisfies (CD1) and (CD2), $\Omega(\varphi)$ is a fundamental domain mod Aut Ω by Theorem 6.3.2. On the other hand, if $D = \bigcap_{\gamma \in \Gamma} \gamma^+$, then $\Gamma D = X$. Therefore $\Omega \subset (\mathrm{Aut}\,\Omega)D \subset \mathrm{Aut}\,\Omega \cdot F = F$. Consequently $F(\varphi) = \Omega(\varphi)$ whenever $\Omega(\varphi)$ satisfies (CD1) and (CD2).

Henceforth we focus attention on $\Omega(\varphi)$ rather than $F(\varphi)$. The 3-faces $\tilde{R}_1$, $\widetilde{R_1R_2}$, $R_1\widetilde{R_2R_1}$ in case $|\arg\varphi^3| < \pi/2 - \pi/p$ are shown below. The 3-faces $\tilde{R}_1^{-1}$, $(\widetilde{R_2R_1})^{-1}$, $(R_1\widetilde{R_2R_1})^{-1}$ can be obtained from the complex conjugation map $V(\bar{\varphi}) \to V(\varphi)$, and the faces $\tilde{R}_2$, $\widetilde{R_2R_1}$ can be obtained by applying to these the isometry $a_{12}\colon V(\varphi) \to V(\varphi)$. The remaining faces can be obtained from the symmetry J.

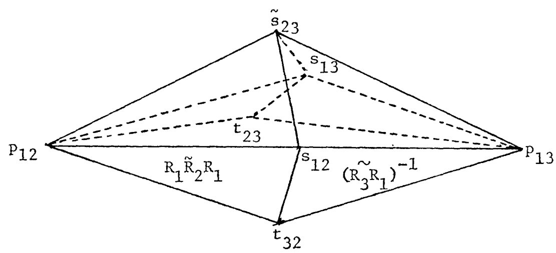


FIGURE 14.2. The nine faces of $\tilde{R}_1$ (labeled by the intersecting 3-faces) for $|\arg\varphi^3| < \pi/2 - \pi/p$.

$R_1\widetilde{R_2R_1}(p_{12}, s_{12}, t_{32})$, $\widetilde{R_2R_1}(p_{12}, s_{12}, \widetilde{s_{32}})$, $\tilde{R}_2^{-1}(p_{12}, \bar{s}_{32}, s_{31})$, $(\widetilde{R_2R_1})^{-1}(p_{12}, s_{31}, t_{23})$

$(\widetilde{R_3R_1})^{-1}(p_{31}, s_{12}, t_{32})$, $\tilde{R}_3^{-1}(p_{31}, s_{12}, \bar{s}_{32})$, $\widetilde{R_3R_1}(p_{31}, \bar{s}_{32}, s_{31})$, $\widetilde{R_3R_1}R_3(p_{31}, s_{31}, t_{23})$

$\tilde{R}_1^{-1}(p_{12}, t_{32}, p_{31}, t_{23})$

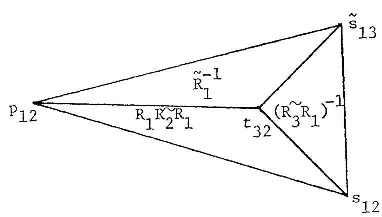


FIGURE 14.3. The four faces of $\widetilde{R_1R_2}$ for $-(\pi/2-\pi/p) < \arg\varphi^3 < 3(\pi/2-\pi/p)$.

$R_1\tilde{R}_2R_1(p_{12}, s_{12}, t_{32})$, $R_1^{-1}(p_{12}, \widetilde{s_{31}}, t_{32})$, $\tilde{R}_2(p_{12}, s_{12}, \bar{s}_{13})$, $(\widetilde{R_3R_1})^{-1}(t_{32}, s_{12}, \widetilde{s_{13}})$

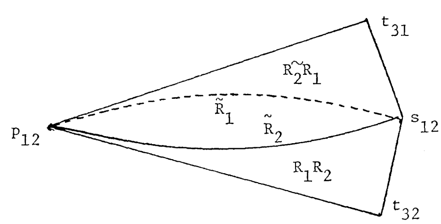


FIGURE 14.4. The four faces of $R_1\widetilde{R_2R_3}$ for $|\arg\varphi^3| < \pi/2 - \pi/p$.

$\tilde{R}_1(p_{12}, s_{12}, t_{32})$, $\widetilde{R_1R_2}(p_{12}, s_{12}, t_{32})$, $R_2(p_{12}, s_{12}, t_{31})$, $\widetilde{R_2R_1}(p_{12}, s_{12}, t_{31})$

14.4. *Condition* (CD1) *for* $\Omega(\varphi)$.

PROPOSITION. *For all φ with $|\arg\varphi| < \pi/2 - \pi/p$, the region $\Omega(\varphi)$ satisfies condition* (CD1).

Proof. It suffices to prove that:

$$\gamma\colon E_k(\tilde{\gamma}) = E_k(\tilde{\gamma}^{-1}) \tag{14.3.1}$$

for $k = 0, 1, 2$ and all $\gamma \in \varDelta$. For this will imply that γ maps the boundary of $\tilde{\gamma}$ onto the boundary of $\tilde{\gamma}^{-1}$ and therefore $\tilde{\gamma}$ onto $\tilde{\gamma}^{-1}$. Actually it suffices to prove (14.3.1) only for those 0, 1, 2-faces which do not contain the apex p_{12}. For by §9, each F_{ij} is a fundamental domain for the finite group Γ_{ij} and therefore satisfies (CD1)$(i \neq j$, $i, j \in 1, 2, 3)$. The (CD1) condition for F_{12}, together with (14.3.1) for $e \in E_k(\tilde{\gamma})$; $\gamma \in \varDelta_{12}$, with $p_{12} \notin e$, $k = 0, 1, 2$, yield $\gamma \cdot \tilde{\gamma} = \tilde{\gamma}^{-1}$ for all $\gamma \in \varDelta_{12}$; by symmetry one gets

$$\gamma \cdot \tilde{\gamma} = \tilde{\gamma}^{-1}\,, \quad \text{all} \quad \gamma \in \varDelta\,. \tag{CD1}$$

From the definition of $\Omega(\varphi)$, one sees that the only 2-faces of $\Omega(\varphi)$ which do not contain an apex are for any permutation (ijk) of $(1, 2, 3)$.

$$I_{ijk} \qquad \text{for} \quad |\arg \varphi^3| < \frac{\pi}{2} - \frac{\pi}{p} \tag{14.3.2}$$

$$I_{123}, I_{231}, I_{312} \qquad \text{for} \quad \left(\frac{\pi}{2} - \frac{\pi}{p}\right) \leqq \arg \varphi^3 < 3\left(\frac{\pi}{2} - \frac{\pi}{p}\right) \tag{14.3.3}$$

$$I_{321}, I_{132}, I_{213} \qquad \text{for} \quad -3\left(\frac{\pi}{2} - \frac{\pi}{p}\right) < \arg \varphi^3 \leqq -\left(\frac{\pi}{2} - \frac{\pi}{p}\right) \tag{14.3.4}$$

and that each vertex and 1-face not containing an apex lies on the above 2-faces.

Set

$$\varDelta_{ijk} = I_{ijk} \cap \Omega(\varphi) \quad \text{for} \quad \langle v_{ijk}, v_{ijk}\rangle > 0\,.$$

From the results of §14 and §10, one sees that every $e \in E_1(\Omega)$ which lies in some $\tilde{\gamma} \in E_1(\Omega)$ but does not lie in any $f \in E_2(F_{12})$ is a geodesic line segment. Inasmuch as the geodesic joining any two points in Ch^n is unique, to prove (CD1) for Ω, it suffices to prove (14.3.1) for $k = 0$; for reasons of symmetry, we need only consider $\gamma \in \varDelta_{12}$, and $\gamma = R_1, R_1R_2$, or $R_1R_2R_1$. The requisite information

$$R_1t_{32} = t_{32},\ R_1s_{12} = \tilde{s}_{13},\ R_1\tilde{s}_{32} = s_{23},\ R_1s_{31} = \tilde{s}_{21},\ R_1t_{23} = t_{23}$$
$$R_1R_2s_{12} = s_{23},\ R_1R_2t_{32} = t_{13},\ R_1R_2\tilde{s}_{13} = \tilde{s}_{21}$$
$$R_1R_2R_1t_{32} = t_{13},\ R_1R_2R_1s_{12} = \tilde{s}_{21},\ R_1R_2R_1t_{31} = t_{23}$$

is given in §13.

15. $(\widehat{R_iR_j})^{-1}, \widehat{R_jR_k})$ and related angles.

In order to determine for which values of the parameter φ the region $\Omega(\varphi)$ satisfies condition (CD2), we compute the angles at which the 3-faces of $\Omega(\varphi)$ meet at those $e \in E_2(\Omega)$ which do not contain an apex of Ω. Such 2-faces are Δ_{ijk} corresponding to all permutations (ijk) of (123) if $|\arg \varphi^3| < \pi/2 - \pi/p$, to the even permutations if $\pi/2 - \pi/p \leqq \arg \varphi^3 \leqq \arg \varphi^3 \leqq 3(\pi/2 - \pi/p)$ and to the odd permutations if $-3(\pi/2 - \pi/p) < \arg \varphi^3 \leqq -\pi/2 - \pi/p$.

LEMMA 15.1. (i) *For any permutation* (ijk) *of* $(1, 2, 3)(R_i\hat{R}_j)^{-1}$ *and* $\widehat{R_jR_k}$ *meet in* I_{ijk} *at a constant angle if* $\langle v_{ijk}, v_{ijk}\rangle > 0$. *Moreover*

(ii) $\sphericalangle((\widehat{R_1R_2})^{-1}, \widehat{R_2R_3}) = \pi/2 - \pi/p + \arg \varphi^3$ *for* $\arg \varphi^3 \geqq -(\pi/2 - \pi/p)$

(iii) $\sphericalangle(\widehat{R_3R_2})^{-1}, \widehat{R_2R_1} = \pi/2 - \pi/p - \arg \varphi^3$ *for* $\arg \varphi^3 \leqq \pi/2 - \pi/p$.

Proof. By Lemma 11.1, I_{ijk} is a common slice of $(\widehat{R_iR_j})^{-1}$ and $\widehat{R_jR_k}$. By Lemma 3.2.9 (ii) the spines of $(\widehat{R_iR_j})^{-1}$ and $\widehat{R_jR_k}$ lie in the same $\boldsymbol{C}$-line. Hence by Lemma 3.2.4, (i) follows. The angle $\sphericalangle((\widehat{R_1R_2})^{-1}$, $\widehat{R_2R_3})$ can be computed as $\sphericalangle(z((\widehat{R_1R_2})^{-1}), z(\widehat{R_2R_3}))$ for any $\boldsymbol{C}$-valued S-function ($S = (\widehat{R_1R_2})^{-1}, \widehat{R_2R_3}$) by Lemma 3.2.6. By (11.2), for such a function z, we get as the images of spinal surfaces

$$\begin{aligned} (\widehat{R_1R_2})^{-1}&\colon |z + \eta^{2-2}\varphi| = 1 \\ R_2\hat{R}_3&\colon \quad |z - \eta i\varphi| = 1 \,. \end{aligned}$$

The angle θ at which the normals to the two circles at 0 intersect is clearly

$$\begin{gathered} (\eta i\varphi/-\eta^2\bar{\varphi}^2) \\ \theta = \arg(\eta i\varphi = \eta^{2-2}\varphi) = \arg(-\bar{\eta} i\varphi^3)\,. \end{gathered}$$

Hence the angle at which the two circles intersect is $|\pi - \theta| = \pi/2 - \pi/p + \arg \varphi^3$ for $\arg \varphi^3 \geqq -(\pi/2 - \pi/p)$. This proves (ii). (iii) is obtained from (ii) by applying the isometry $J'\colon V(\bar{\varphi}) \to V(\varphi)$, wnich interchanges 1 and 3.

LEMMA 15.2(i). *In the geodesic triangle* Δ_{ijk}

$$\sphericalangle\, t_{ik}s_{jk}\tilde{s}_{ji} = \begin{cases} \dfrac{1}{2}\left(\dfrac{\pi}{2} - \dfrac{\pi}{p} - \arg \varphi^3\right), & \textit{for even } (ijk) \\ \dfrac{1}{2}\left(\dfrac{\pi}{2} - \dfrac{\pi}{p} + \arg \varphi^3\right), & \textit{for odd } (ijk) \end{cases}$$

where (ijk) *is a permutation of* (123).

Proof. By symmetry, it suffices to consider the geodesic triangle

Δ_{312}. The calculation is somewhat analagous to the one used in proving Lemma 15.1. We shall first determine the equations of the lines $\tilde{R}_1 \cap \Delta_{212}$ and $\tilde{R}_2 \cap \Delta_{312}$. From the matrices of §9, we get as the equations of $\hat{R}_1$ and $\hat{R}_2$

$$\hat{R}_1\colon |\eta^2 x_1 + (1-\eta i\bar{\varphi})x_2 + (1-\eta i\varphi)x_3| = |x_1 + x_2 + x_3|$$
$$\hat{R}_2\colon |(1-\eta i\varphi)x_1 + \eta^2 x_2 + (1-\eta i\bar{\varphi})x_3| = |x_1 + x_2 + x_3| \, .$$

From (11.3a) we get as the equation of I_{312}

$$I_{312}\colon x_3 = -\eta i\varphi x_2 \, .$$

Substituting for x_3 and setting $z = x_2/(x_1 + x_2 + x_3)$

$$I_{312} \cap \hat{R}_1\colon |z(1-\eta i\bar{\varphi})(1-\eta i\varphi - \eta^2\varphi^2) + \eta^2| = 1$$
$$I_{312} \cap \hat{R}_2\colon |z(-1+\eta i\varphi + \eta^2\varphi^2) + \eta^2| = 1 \, .$$

Setting $b = 1 - \eta i\varphi - \eta^2\varphi^2$, we get

$$I_{312} \cap \hat{R}_2\colon |z + \eta^2/b(1-\eta i\bar{\varphi})| = 1/|b(1-\eta i\bar{\varphi})|$$
$$I_{312} \cap \hat{R}_1\colon |z - (1-\eta i\varphi)/b| = 1/|b| \, .$$

The common solution corresponding to the point $s_{12} = \bar{\eta} i\varphi e_1 + \bar{\eta} i\bar{\varphi} e_2 + e_3$ is

$$s_{12}\colon z = \bar{\eta} i\bar{\varphi}/(1 + \bar{\eta} i(\varphi + \bar{\varphi})) \, .$$

The map of the $\boldsymbol{C}$-line I_{312} into $\boldsymbol{C}$ given by the function z is holomorphic and therefore conformal. It follows from elementary geometry of circles that one of the two angles between the two circles is the arc of

$$w = \frac{\dfrac{\bar{\eta} i\bar{\varphi}}{1+\bar{\eta} i(\varphi+\bar{\varphi})} + \dfrac{\eta^2}{b(1-\eta i\bar{\varphi})}}{\dfrac{\bar{\eta} i\bar{\varphi}}{1+\bar{\eta} i(\varphi+\bar{\varphi})} - \dfrac{1-\eta i\varphi}{b}} \cdot \frac{\dfrac{1}{b}}{\dfrac{1}{|b(1-\bar{\eta} i\varphi)|}}$$

After clearing denominators and simplifying numerators we get

$$w = \frac{(1+\bar{\eta} i\bar{\varphi} + \bar{\eta}^2)(1-\eta i\bar{\varphi})}{(1+\bar{\eta} i\varphi + \eta^2)(1-\eta i\bar{\varphi})} \, .$$

Note that $1 + \eta i\bar{\varphi} + \bar{\varphi}^2 = \bar{\varphi}^2(1 + \eta i\varphi + \eta^2)$. Hence

$$\begin{aligned} w^2 &= \varphi^{-4}\frac{(1-\eta i\bar{\varphi})(1+\bar{\eta} i\bar{\varphi})}{(1-\eta i\bar{\varphi})(1-\eta i\bar{\varphi})} \\ &= \varphi^{-4}\frac{\bar{\eta} i\varphi(1-\eta i\bar{\varphi})}{1-\eta i\bar{\varphi}} \\ &= \bar{\eta} i\bar{\varphi}^3 \, . \end{aligned}$$

It follows that $\sphericalangle\, t_{32}s_{12}\tilde{s}_{13} = (\pi/2 - \pi/p - \arg \varphi^3)/2$. From this the lemma follows.

REMARK 1. The angles above can be computed by applying the analogue of Napier's rule for a right triangle ABC with hypotenuse c:

$$\cot A \cot B = \cosh 2c$$

in the constant curvature $\boldsymbol{C}$-line I_{ijk}. The resulting computation is longer than the one presented here.

REMARK 2. Note that the triangle Δ_{ijk} is isosceles since R_j carries one side into another by Lemmas 14.2.1(i) and 13.4. Hence

$$\sphericalangle\, t_{ik}s_{jk}\tilde{s}_{ji} = \sphericalangle t_{ik}\tilde{s}_{ji}s_{jk} = \frac{1}{2} \sphericalangle\, ((\widehat{R_k R_j})^{-1}, \widehat{R_j R_i})$$

for $|\arg \varphi^3| < \pi/2 - \pi/p$.

LEMMA 15.3. $\sphericalangle\, t_{13}p_{12}t_{31} = ((6 - p)/2p)\pi (p = 3, 4, 5)$.

Proof. Let $\Delta_i = F_{12} \cap e_i^{\perp} (i = 1, 2)$. From the description of the fundamental domain F_{12} in §10.1, we see that F_{12} has only two 2-dimensional faces fixed under a $\boldsymbol{C}$-reflection; namely Δ_1 and Δ_2. Consequently, from $X = \Gamma_{12}F_{12}$ we infer

$$e_1^{\perp} = \bigcup \{\gamma\Delta_2;\ \gamma\Delta_2 \subset e_2^{\perp}\} \cup \{\gamma\Delta_1;\ \gamma\Delta_1 \subset e_2^{\perp}\}\ .$$

Clearly $\gamma\Delta_1 \subset e_2^{\perp}$ implies $\gamma e_2^{\perp} = e_2^{\perp}$ and therefore $\gamma \in Z\Gamma_2$ where Z is the center of Γ_{12} and $\Gamma_2 = \{\{R_2\}\}$. We have $R_1R_2e_1 = -\eta i\varphi e_2$ by §9.1. Hence $R_1R_2R_1e_1^{\perp} = R_1R_2e_1^{\perp} = e_2^{\perp}$. Inasmuch as $Ze_i^{\perp} = e_i^{\perp} (i = 1, 2)$ and $(R_1R_2R_1)^2 \in Z$, we see that

$$\{\gamma \in \Gamma_{12};\ \gamma\Delta_2 \subset e_1^{\perp}\} = ZR_1R_2R_1\Gamma_2\ .$$

Hence

$$e_2^{\perp} = Z\Delta_2 \cup ZR_1R_2R_1\Delta_1\ .$$

Thus $e_2^{\perp}$ is a union of $2 \# Z$ sectors with disjoint interiors. It follows at once that

$$\sphericalangle\, t_{13}p_{12}t_{31} = \frac{1}{2 \# Z} 2\pi = \frac{6-p}{2p}\pi$$

by (2.2.4).

16. The stabilizer of $\Omega(\varphi)$ in Γ.

Let ρ be order (in the multiplicative group of nonzero complex numbers) of $\bar{\eta}i\varphi^3$, and let σ be the order of $\bar{\eta}i\bar{\varphi}^3$. Let

$$r = \mathrm{order}(R_1R_2R_3)^2 \quad \text{in} \quad \mathrm{PU}(H)$$
$$s = \mathrm{order}(R_3R_2R_1)^2 \quad \text{in} \quad \mathrm{P(U)}(H) .$$

Let $\mathrm{Aut}_\Gamma \Omega$ denote the stabilizer in Γ of the region $\Omega(\varphi)$. From §9.2, we know that the order $\#\mathrm{Aut}_\Gamma \Omega$ is at most 3. The following theorem determines $\mathrm{Aut}_\Gamma\Omega$.

LEMMA 16.1. *Assume that ρ is finite (or equivalently that σ is finite)*

(i) *if $3 \nmid \rho$ then $(R_1R_2R_3)^{2\mu}R_1R_2 = J$ in Ch^2, where $3\mu + 1 \equiv 0 \mod \rho$*

(ii) *if $3 \nmid \sigma$, then $(R_3R_2R_1)^{2\nu}R_3R_2 = J^{-1}$ in Ch^2, where $3\nu + 1 \equiv 0 \mod \sigma$,*

(iii) $(R_1R_2R_3)^{2\mu}R_1R_2 = (R_2R_3R_1)^{2\mu}R_2R_3 = (R_3R_1R_2)^{2\mu}R_3R_1$
$(R_3R_2R_1)^{2\nu}R_3R_2 = (R_1R_3R_2)^{2\nu}R_1R_3 = (R_2R_1R_3)^{2\nu}R_2R_1$

(iv) $r = \mathrm{order}(\bar{\eta}i\varphi^3)^3$, $s = \mathrm{order}(\bar{\eta}i\bar{\varphi}^3)^3$.

Proof. Set $\xi = \bar{\eta}i\varphi^3$. From (9.1.1) we see that the eigenvalues $(R_1R_2R_3)^2$ are

$$-\eta^6\varphi^{-6}\ , \quad \eta^3 i\varphi^3\ , \quad \eta^3 i\varphi^3\ . \tag{16.1}$$

In addition by (9.1.3)

$$(R_1R_2R_3)^2 v_{123} = -\eta^6\varphi^{-6}v_{123}$$
$$(R_1R_2R_3)^2 e_2 = \eta^3 i\varphi^3$$
$$\langle e_2, v_{123}\rangle = 0\ .$$

This implies that $(R_1R_2R_3)^2$ fixes the point $\pi(v_{123})$ of CP^2, and each point of the complex projective line $\pi(v_{123}^\perp)$, rotates CP^2 around the point $\pi(v_{123})$ by the scalar multiple

$$\eta^3 i\varphi^3/-\eta^6\varphi^{-6} = \bar{\xi}^3 \tag{16.1}$$

stabilizes $e_2^\perp$, and rotates the complex projective line $\pi(e_2^\perp)$ around the points

$$\pi(v_{123} \cap e_2^\perp) \quad \text{and} \quad \pi(v_{123}^\perp \cap e_2^\perp) \quad \text{by}$$

scalar multiples $(\eta i\varphi^3)/(-\eta^6\varphi^{-6}) = \bar{\xi}^3$ and ξ^3 respectively.

The order of $(R_1R_2R_3)^2$ in $\mathrm{PU}(H)$ is the order of ξ^3. This proves the first part of (iv) and proof of the second part is similar.

By Lemma 12.3

$$t_{13} = \begin{cases} \pi(v_{123}^\perp \cap e_2^\perp) & \text{if} \quad \langle v_{123}, v_{123}\rangle > 0 \\ \pi(v_{123} \cap e_2^\perp) & \text{if} \quad \langle v_{123}, v_{123}\rangle < 0\ . \end{cases}$$

From §9.1 we see that $R_3R_1e_3 = -\eta i\varphi e_1$ (by symmetry from $R_1R_2e_1 = -\eta i\varphi e_2$) so that $R_3R_1e_3^{\perp} = e_1^{\perp}$. By Lemma 13.5, $R_3R_1t_{21} = t_{32}$. Hence R_3R_1 sends the quadrilateral $\boldsymbol{Q}_3 = [p_{31}t_{12}p_{23}t_{21}]$ into the $\boldsymbol{C}$-line $\pi(e_1^{\perp})$ so as to abot the quadrilateral $\boldsymbol{Q}_1 = [p_{12}t_{23}p_{31}t_{32}]$ along the geodesic line segment $[t_{32}, p_{31}]$ (cf. Figure 16.1) since $\widetilde{R_3R_1}$: $\widetilde{R_3R_1} \to (R_3R_1)^{-1}$.

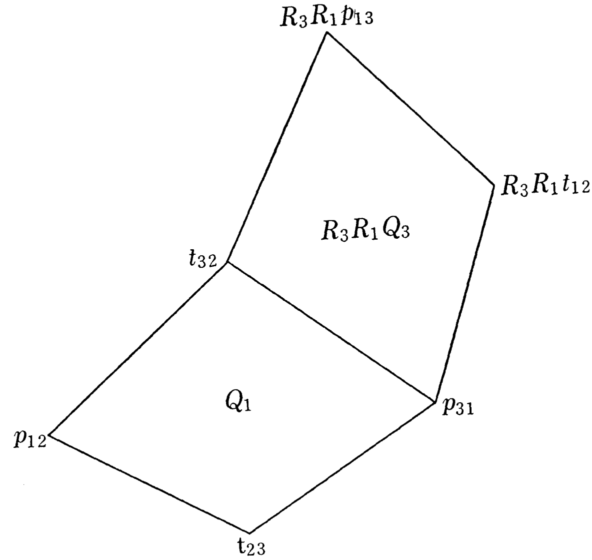


FIGURE 16.1

By symmetry one sees that image $R_3R_1 \cdot R_2R_3 \cdot R_1R_2Q_1$ is given by rotation about t_{32} through the angle $3 \sphericalangle (R_3R_1)^{-1}$, $R_1R_2 = 3(\pi/2 - \pi/p + \arg \varphi^3) = 3 \arg \xi$ in the sense from p_{12} to p_{31}, this is a rotation about the point t_{32} by the scalar multiple ξ^3 if $\langle v_{123}, v_{123}\rangle > 0$ and by the scalar multiple $\bar{\xi}^3$ if $\langle v_{123}, v_{123}\rangle < 0$.

Suppose now that $3 \nmid \rho$. Then one can choose μ so that $(R_1R_2R_3)^{2\mu}$ rotates $R_3R_1Q_3$ back into Q_1; one need only select μ so that

$$(\xi^3)^{\mu} = \bar{\xi}^1 \quad \text{if} \quad \langle v_{123}, v_{123}\rangle > 0$$
$$(\bar{\xi}^3)^{\mu} = \bar{\xi}^{-1} \quad \text{if} \quad \langle v_{123}, v_{123}\rangle < 0 \; .$$

In either case $\xi^{3\mu+1} = 1$; so select μ to satisfy $3\mu + 1 \equiv 0 (\mathrm{mod}\, \rho)$. Then we have

$$(R_3R_1R_2)^{2\mu}R_3R_1R_{31} = p_{12}$$
$$(R_3R_1R_2)^{2\mu}R_2R_1p_{23} = p_{31} \; .$$

In addition, by Lemma 13.3 (i), $R_3R_1 = J$ on I_3 sending I_3 to I_1 if $\langle v_{123}, v_{123}\rangle > 0$. In this case, $(R_3R_1R_2)^{2\mu}R_3R_1 = J$ on I_3 (by Lemma 13.3(ii)) as well as on $\pi(e_3^{\perp})$ and therefore on the ball.

It remains to consider the case $\langle v_{123}, v_{123}\rangle < 0$. In this case, the triangles Δ_{312}, Δ_{123}, Δ_{231} collapse to points and the reasoning of the proof of Lemma 13.3(i) does not apply. Instead, one argues as follows: R_3R_1 carries I_3 to I_1 and $R_3R_1 \cdot R_2R_3 \cdot R_1R_2$ carries I_1 to I_1 rotating the $\boldsymbol{C}$-line I_1 about the point t_{32} by the scalar factor $\bar{\xi}^3$. Hence by symmetry R_3R_1 restricted to I_3 is the map $\rho_1 \circ J$ where ρ_1 is the rotation of the ball about t_{32} by the scalar multiple $\bar{\xi}$. Consequently, $R_3R_1 = \rho_1 \cdot J$. By (16.1), $(R_3R_1R_2)^{2r} = \rho_1^{-1}$. Consequently $J = \rho_1^{-1}R_3R_1 = (R_3R_1R_2)^{2r}R_3R_1$. This proves (i).

(ii) follows from (i) by applying $J' : V(\bar{\varphi}) \to (\varphi)$.

(iii) follows from (i) and (ii) by symmetry. The lemma is now proved.

REMARK 1. From (9.1.6) it follows that the order in PU(H) of $R_2R_1R_2R_3$ is l.c.m.(order $-\eta i\varphi^3$, $-\eta i\bar{\varphi}^3$) = lcm(σ, ρ).

REMARK 2. If $\Omega(\varphi)$ satisfies conditions (CD1) and (CD2), then $3|\rho$ and $3|\sigma$ implies that $\#\,\mathrm{Aut}_\Gamma\,\Omega = 1$. For then the circuits around Δ_{ijk} result in the identity map. By Theorem 6.3.3(a), $\mathrm{Aut}_\Gamma\,\Omega = (1)$.

REMARK 3. More generally, in the joined $\Gamma\Omega(\varphi)$ space, if $3|\rho$ and $3|\sigma$, then $\#\,\mathrm{Aut}_\Gamma\,\Omega = 1$, provided that $|\arg\varphi^3| < \pi/2 - \pi/p$. We shall prove this in § 18.4.

REMARK 4. By Lemma 13.3(i), $J^{-1}R_1R_2$ fixes each point of I_1. Clearly $J^{-1}R_1R_2p_{12} = J^{-1}p_{12} = p_{31}$. Consequently, $J^{-1}R_1R_2$ is a rotation of $\boldsymbol{C}P^2$ about the line I_1 stabilizing the line $e_1^\perp$ and rotating p_{12} into p_{31} about t_{32}. Comparison with (16.1) shows that $J^{-1}R_1R_2$ is a rotation of $\boldsymbol{C}P^2$ about I_1 by the multiple ξ. Moreover, the matrices $J^{-1}R_1R_2$ and R_1 commute. For $JR_i = R_{i+1}J$ for any $i \bmod 3$; hence

$$J^{-1}R_1R_2R_1 = J^{-1}R_2R_1R_2 = R_1J^{-1}R_1R_2 \ .$$

17. Nonarithmetic lattices $\Gamma(\varphi)$.

17.1. *Values of φ for which $\Omega(\varphi)$ satisfies* (CD2).
Set

$$\rho = \mathrm{order}(\bar{\eta}i\varphi^3) \ , \quad \sigma = \mathrm{order}(\bar{\eta}i\bar{\varphi}^3)$$
$$r = \mathrm{order}(\bar{\eta}i\varphi^3)^3 \ , \quad s = \mathrm{order}(\bar{\eta}i\bar{\varphi}^3)^3 \ .$$

From Lemma 16.1 we know that $r = \mathrm{order}(R_1R_2R_3)^2$ in PU(H), and $s = \mathrm{order}\ (R_3R_2R_1)^2$ in PU(H). From the fact that F_{ij} is a fundamental domain for the finite group Γ_{ij}, we know that the codimension two condition (CD2) is satisfied for all 2-faces of $\Omega(\varphi)$ which contain an apex p_{ij}, $i \neq j$, $i, j \in \{1, 2, 3\}$. The only 2-faces not containing an apex are Δ_{ijk} with ijk ranging over all even permutations of (123) if $-(\pi/2 - \pi/p) < \arg \varphi^3 < 3(\pi/2 - \pi/p)$, and over all odd permutations if $-3(\pi/2 - \pi/p) < \arg \varphi^3 < \pi/2 - \pi/p$. The circuit around Δ_{ijk} is $R_iR_j \cdot R_kR_i \cdot R_jR_k \cdots \Omega$. Each such circuit places side by side around Δ_{ijk} the region bounded by two spinal surfaces whose spines lie in a common $\boldsymbol{C}$-line and which intersect in the slice I_{ijk}. By Lemma 15.1, $(\widehat{R_iR_j})^{-1}$ and $\widehat{R_jR_k}$ meet at a constant angle equal to $\pi/2 - \pi/p \pm \arg \varphi^3$ (the $(+)$ or $(-)$ corresponding to even or odd permutations). Since $(\bar{\eta} i \varphi^3)^\rho = 1$, the image of $\Omega(\varphi)$ after ρ terms of the circuit coincides with $\Omega(\varphi)$, and by definition of ρ no shorter circuit brings $\Omega(\varphi)$ back to coincidence with itself. On the other hand, by (CD2) all the interiors of the ρ images of $\Omega(\varphi)$ in the circuit must be disjoint. Hence for $|\arg \varphi^3| < 3(\pi/2 - \pi/p)$.

(i) $\rho(\pi/2 - \pi/p + \arg \varphi^3) = 2\pi$, if $\arg \varphi^3 > -(\pi/2 - \pi/p)$.

Similarly,

(ii) $\sigma(\pi/2 - \pi/p - \arg \varphi^3) = 2\pi$, if $\arg \varphi^3 < \pi/2 - \pi/p$. If $|\arg \varphi^3| \leqq \pi/2 - \pi/p$, both (i) and (ii) apply and adding we get

$$\frac{1}{2} - \frac{1}{\rho} = \frac{1}{\rho} + \frac{1}{\sigma} .$$

Setting $m = \inf(\rho, \sigma)$, this implies $m \leqq 4p/(p-2) \leqq 2m$. Thus

$$\text{(iii)} \quad \begin{aligned} &6 \leqq m \leqq 12 \quad \text{if} \quad p = 3 \\ &4 \leqq m \leqq 8 \quad \text{if} \quad p = 4 \\ &4 \leqq m \leqq 6 \quad \text{if} \quad p = 5 . \end{aligned}$$

If $3(\pi/2 - \pi/p) > \arg \varphi^3 > \pi/2 - \pi/p$, we see from (i) that

$$\rho \cdot \left(\frac{1}{2} - \frac{1}{p}\right) < 2 < \rho \cdot 4\left(\frac{1}{2} - \frac{1}{p}\right) .$$

Thus $p/(p-2) < \rho < 2p/p - 2$, yielding

$$\text{(iv)} \quad \begin{aligned} &4 \leqq \rho \leqq 5 \quad \text{for} \quad p = 3 \\ &3 \leqq \rho \leqq 3 \quad \text{for} \quad p = 4 \\ &2 \leqq \rho \leqq 3 \quad \text{for} \quad p = 5 . \end{aligned}$$

In this latter range for φ, $\Delta_{123}\Delta_{231}$, and Δ_{312} are the only 2-faces of $\Omega(\varphi)$ not containing an apex.

In order to determine all the φ for which $\Gamma(\varphi)$ is a discrete group it suffices to consider the range $0 \leqq \arg \varphi^3 < 3(\pi/2 - \pi/p)$, inasmuch as the negative values of φ are given by the symmetry $J' \colon V(\varphi) \to V(\bar{\varphi})$. We shall give the φ in two lists, one for $0 \leqq \arg \varphi^3 \leqq \pi/2 - \pi/p$, the other for $\pi/2 - \pi/p < \arg \varphi^3 < 3(\pi/2 - \pi/p)$. In case $\arg \varphi^3 = \pi/2 - \pi/p$, the geodesic triangle Δ_{321} degenerates to a point at ∞ and it can be regarded as limiting value of $\arg\varphi^3 >$ or $<$ $\pi/2 - \pi/p$; we list this case on our first table.

Set $t = (1/\pi)\arg\varphi^3$, then $t = 2/\rho - (1/2 - 1/p)$. If $0 \leqq t$ then $\inf(\rho, \sigma) = \rho$. We therefore arrange the tables according to increasing values of the integer ρ. In Table 1, $1/\rho + 1/\sigma = 1/2 - 1/p$; in Table 2, we have instead, $1/\rho + 1/\sigma = t$. The requirement that σ be an integer reduces the number

TABLE 1. $0 \leqq t \leqq 1/2 - 1/p$

p	ρ	σ	t	$\#\mathrm{Aut}_\Gamma\,\Omega$	r	s
3	6	∞	1/ 6	1	2	∞
3	7	42	5/42	3	7	14
3	8	24	1/12	3	8	8
3	9	18	1/18	1	3	6
3	10	15	1/30	3	10	5
3	12	12	0	1	4	4
4	4	∞	1/ 4	3	4	∞
4	5	20	3/20	3	5	20
4	6	12	1/12	1	2	4
4	8	8	0	3	8	8
5	4	20	1/ 5	3	4	20
5	5	10	1/10	3	5	10

TABLE 2. $1/2 - 1/p < t < 3(1/2 - 1/p)$

p	ρ	σ	t	$\#\mathrm{Aut}_\Gamma\,\Omega$	r	s
3	4	12	1/ 3	3	4	4
3	5	30	7/30	3	5	10
4	3	12	5/12	1	1	4
5	2	5	7/10	3	2	5
5	3	30	11/30	1	1	10

of cases in (iii).

REMARK. Let $p = 5$, $\varphi^3 = i$, $\psi^3 = \exp(7\pi i/10)$. We will see in §21 that $\Gamma(\varphi)$ is an arithmetic lattice despite the fact $\Omega(\varphi)$ does not satisfy (CD2). In this case $\Omega(\varphi)$ is not a fundamental domain mod $\mathrm{Aut}_\Gamma\,\Omega$; nevertheless, $\Gamma(\varphi)$ is a lattice. In fact, as we show in §21, there is an isomorphism of $\Gamma(\varphi)$ onto $\Gamma(\psi)$. Thus (CD2) is a sufficient condition for $\Gamma(\varphi)$ to be discrete but is not necessary. On the other hand, the failure of (CD2) for $\Omega(\varphi)$, $|\arg\varphi^3| < \pi/2 - \pi/p$, seems to imply the $\Gamma(\varphi)$ is not discrete. For example, (CD2) fails for $p = 5$, $t = 0$. In this case, one can compute that $\langle (R_1R_2R_3)^9e_1, e_1\rangle$ is a non-admissible value (cf 2.4.3). This implies that $\{R_1R_2R_3)^9R_1(R_1R_2R_3)^{-9}, R_1\}$ is a $\boldsymbol{C}$-reflection group which fixes a point in the ball but is infinite. It follows that $\Gamma(1)$ is not discrete.

17.2. *The field* $\boldsymbol{Q}[\mathrm{Tr\,Ad}\,\Gamma]$. In applying the test of §4 for arithmeticity of the lattice $\Gamma(\varphi)$, we must determine the field generated by $\mathrm{Tr}\,\mathrm{Ad}\,\gamma$, as γ ranges over $\Gamma(\varphi)$; we denote this by $\boldsymbol{Q}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma]$. For any subfield k of $\boldsymbol{C}$ stable under complex conjugation, we denote by $\mathrm{Re}\,k$ the subfield $k \cap \boldsymbol{R}$. We continue the notation $\rho =$ order $\bar{\eta} i \varphi^3$, $\eta = e^{\pi i/p}$. For $\arg \varphi^3 > -(\pi/2 - \pi/p)$, $\bar{\eta} i \varphi^3 = e^{2\pi i/\rho'}$ by 17.1(i).

LEMMA 17.2.1. *$\boldsymbol{Q}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma] = \mathrm{Re}\,\boldsymbol{Q}[e^{2\pi i/\rho}, 2e_{z}^{\pi i/p}]$ for $\arg \varphi^3 > -(\pi/2 - \pi/p)$ and for $\Gamma(\varphi)$ discrete.*

Proof. We identify $\mathrm{Hom}(V, V)$ with $V \otimes V^*$ in the standard way so that for any $e, f \in V$, $\alpha, \beta \in V^*$,

$$(e \otimes \alpha)(f) = \alpha(f)e$$
$$(e \otimes \alpha)(f \otimes \beta) = \alpha(f)e \otimes \beta \,.$$

Let $\{e^1, e^2, e^3\}$ denote the dual base to the base $\{e_1, e_2, e_3\}$ of our vector space $V(\varphi)$. We rewrite the generating $\boldsymbol{C}$-reflections R_1 of $\Gamma(\varphi)$:

$$R_1(x) = x + (\eta^2 - 1)\langle x_1 e_1 \rangle e_1 = \eta^2 x_1 e_1 - \eta i \bar{\varphi} x_2 e_1 - \eta i \varphi x_3 e_1$$
$$R_1 = \eta^2 e_1 \otimes e^1 - \eta i \bar{\varphi} e_1 \otimes e^2 - \eta i \varphi e_1 \otimes e^3 + e_2 \otimes e^2 + e_3 \otimes e^3$$

and

$$R_j = \sum_{k \neq j} e_k \otimes e^k + \sum_{k=1}^{3} a_{jk} e_j \otimes e^k,\ a_{ij} = \begin{cases} -\eta i \varphi^{\varepsilon(j,k)} \,, & j \neq k \\ \eta^2 \,, & j = k \end{cases}$$

where $\varepsilon(j, k) = (-1)^{k-j}$, $(j, k = 1, 2, 3)$. Then

$$\begin{aligned} \mathrm{Tr}\, R_{j_1} R_{j_2} \cdots R_{j_l} &= \sum_{q=1}^{l} \sum_{1 \leq i_1 < i_2 < \cdots < i_q < l} a_{j_{i_1} j_{i_2}} a_{j_{i_2} j_{i_3}} \cdots a_{j_{i_q} j_{i_1}} \\ &= \sum_q (\eta^2)^{q-m} (\eta i)^m (\varphi^3)^n \\ &= \sum_q (\eta^2)^q (\bar{\eta} i)^m \varphi^{3n} \end{aligned}$$

where n is the topological degree of the map of the loop $(i_1 i_2 \cdots i_p i_1)$ into the circular loop (1231) and $m - 3n$ is a nonnegative even integer $2u$. (The only terms having φ or $\bar{\varphi}$ have ηi with them.) Write $\gamma = R_{j_1} \cdots R_{j_k}$. Then $\mathrm{Tr}\,\gamma$ is a sum of terms of the form $\eta^{2p}(\bar{\eta} i)^{m-n}(\bar{\eta} i \varphi^3)^n$ with $(\bar{\eta} i)^{m-n} = (\bar{\eta} i)^{2(u+n)} = \bar{\eta}^{2(u+n)}(-1)^{u+n}$. Hence $\mathrm{Tr}\,\gamma \in \boldsymbol{Q}(\eta^2, \bar{\eta} i \varphi^3)$. But $\bar{\eta} i \varphi^3 = e^{2\pi i/\rho}$ for $\arg \varphi^3 > -(\pi/2 - \pi/p)$ and $\mathrm{Tr}\,\mathrm{Ad}\,\gamma = (\mathrm{Tr}\,\gamma)^2$ by §4. Consequently

$$\boldsymbol{Q}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma] \subset \mathrm{Re}\,\boldsymbol{Q}(e^{2\pi i/\rho}, e^{2\pi i/p}) \,.$$

In order to prove the converse inclusion, we consider $\mathrm{Tr}\,\mathrm{Ad}\,\gamma$ for $\gamma = R_2 R_1 R_2 R_3$. By (9.1.6) the eigenvalues of $R_2 R_1 R_2 R_3$ are $\eta^3 i \varphi^3$, $\eta^3 i \bar{\varphi}^3$, $-\eta^2$ and therefore

$$\begin{aligned}(\operatorname{Tr}\gamma)^2 &= |(-\eta)^2(-\eta i\varphi^3 - \eta i\bar{\varphi}^3 + 1)|^2 \\ &= 3 + 2\operatorname{Re}(\bar{\eta} i\varphi^3 + \bar{\eta} i\bar{\varphi}^3 + \varphi^6) \\ &= 3 + 2\Big(\cos\frac{2\pi}{\rho} + \cos\frac{2\pi\nu}{\sigma} + \cos 2\pi t\Big)\end{aligned}$$

where $t = \pi^{-1}\arg\varphi^3$ and ν is an integer prime to σ, the order of $\bar{\eta} i\bar{\varphi}^3$, with $\nu = 1$ for $|t| < 1/2 - 1/p$.

Additional elements of $\boldsymbol{Q}[\operatorname{Tr}\operatorname{Ad}\Gamma]$ are $\cos 2\pi/p$ and $\cos 6\pi/\rho$ arising from $\operatorname{Tr}\operatorname{Ad}R_1$ and $\operatorname{Tr}(R_1R_2R_3)^2$ respectively (cf. (9.1.1)).

Let K denote the field $\boldsymbol{Q}[e^{2\pi i/\rho}, e^{2\pi i/p}]$; K is generated by a primitive root of unity z, and its Galois group is given by automorphisms $z \to z^m$ for integers m prime to the order of z. The Galois group of K, $\operatorname{Gal} K$, is thus abelian and the subfield $\boldsymbol{Q}[\operatorname{Tr}\operatorname{Ad}\Gamma]$ is stable under all automorphisms of K. Hence

$$\cos\frac{2m\pi}{\rho} + \cos\frac{2m\nu\pi}{\sigma} + \cos 2m\pi t \in \boldsymbol{Q}[\operatorname{Tr}\operatorname{Ad}\Gamma]$$

for all m such that $z \to z^m$ is in $\operatorname{Gal} K$. Examining the cases in Tables 1 and 2 one by one, it is easy to verify that some linear combination of the foregoing elements in $\boldsymbol{Q}(\operatorname{Tr}\operatorname{Ad}\Gamma)$ yield each of $\cos 2\pi/\rho$, $\cos 2\pi n/\sigma$, and $\cos 2\pi t$ except in the case $p = 4$, $\rho = 5/12$. In the latter case one computes

$$\begin{aligned}\operatorname{Tr} R_1R_2R_1R_2R_3 &= \eta^3(\eta^3 i\bar{\varphi}^3 - \eta - 1 - \eta i\bar{\varphi}^3 + \eta i\varphi^3) \\ &= \eta^2\Big[-\Big(\frac{3}{2} + \sqrt{3}\Big) + i\Big(1 + \frac{\sqrt{3}}{2}\Big)\Big];\end{aligned}$$

thus $\boldsymbol{Q}(\operatorname{Tr}\operatorname{Ad}\Gamma)$ contains $-(\sqrt{3}/2) = \cos(2\pi n/\sigma)$ in this case too.

The field K has as a primitive generator any element $z = e^{2\pi i N/L}$ where L is the least common multiple of p and ρ and $\gcd(N, L) = 1$. Thus each element of the field K has the form $\sum_{j=0}^{l-1} a_j z^j$ with $a_j \in \boldsymbol{Q}$, and each element of the field $\operatorname{Re} K$ has the form

$$\sum_{j=0}^{l-1} a_j \cos\frac{2\pi jN}{L}.$$

As is well known, $\cos j\theta$ is a polynomial in the powers of $\cos\theta$ with integer coefficients for any integer j. Thus

$$\operatorname{Re} K = \boldsymbol{Q}\Big[\cos\frac{2\pi N}{L}\Big].$$

On the other hand, we have

$$\frac{1}{2} - \frac{1}{p} + t = \frac{2}{p}\arg\bar{\eta} i\bar{\varphi}^3 = \pi\Big(\frac{1}{2} - \frac{1}{p} - t\Big) = \Big(1 - 2\Big(\frac{1}{p} + \frac{1}{\rho}\Big)\Big).$$

Hence σ, the order of $\bar{\eta} i \bar{\varphi}^3$ is the order of $e^{2\pi i/p \cdot 2\pi i/\rho}$ if σ is even, and half that order if σ is odd. Thus if σ is even and $\gcd(p, \rho) = 1$, then $\sigma = L$ and $\exp(2\pi i\nu/\sigma)$ generates the field K. In these cases $\cos(2\pi\nu/\sigma)$ generates $\operatorname{Re} K$. In the remaining cases of Tables 1 and 2, $\exp 2\pi i t$ generates K and $\cos 2\pi t$ generates $\operatorname{Re} K$. As noted above, these generators for $\operatorname{Re} K$ are in $\operatorname{Re} \boldsymbol{Q}[\operatorname{Tr} \operatorname{Ad} \Gamma]$. Hence $\operatorname{Re} K \subset \operatorname{Re} \operatorname{Tr} \operatorname{Ad} \Gamma$. From this the lemma follows.

17.3. *Non-artithmetic lattices.*

It remains only to apply the criterion of § 4 to the groups $\Gamma(\varphi)$ listed in Tables 1 and 2 of § 17.1 to determine which of them are non-arithmetic lattices.

TABLE 3

p	ρ	t	k-gen.	#Cal k	Δ		Arith.
3	6	1/ 6	1	1			A
3	7	5/42	$\cos\frac{\pi}{21}$	6	$-\cos\frac{5\pi}{42}/\left(3\sin\frac{\pi}{3}\right)$	$\sigma_5(\Delta) = -\cos\frac{25\pi}{42}/\left(3\sin\frac{5\pi}{3}\right) < 0$	NA
3	8	1/12	$\cos\frac{\pi}{12}$	4	$-\cos\frac{\pi}{12}/\left(3\sin\frac{\pi}{3}\right)$	$\sigma_7(\Delta) = -\cos\frac{7\pi}{12}/\left(-3\sin\frac{7\pi}{3}\right) < 0$	NA
3	9	1/18	$\cos\frac{\pi}{9}$	3	$-\cos\frac{\pi}{18}/\left(3\sin\frac{\pi}{3}\right)$	$\sigma_5(\Delta) > 0,\ \sigma_{25}(\Delta) > 0$	A
3	10	1/30	$\cos\frac{\pi}{15}$	4	$-\cos\frac{\pi}{30}/\left(3\sin\frac{\pi}{3}\right)$	$\sigma_{11}(\Delta) = -\cos\frac{11\pi}{30}/\left(-3\sin\frac{11\pi}{3}\right) < 0$	NA
3	12	0	$\cos\frac{\pi}{6}$	2	$-1/3\sin\frac{\pi}{3}$	$\sigma_5(\Delta) > 0$	A
4	4	1/ 4	1	1	-1		A
4	5	3/20	$\cos\frac{\pi}{10}$	4	$-\frac{1}{2}\left(1+\frac{\cos 3\pi/20}{\sin \pi/4}\right)$	$\sigma_3(\Delta) = -\frac{1}{2}\left(1+\frac{\cos 9\pi/20}{-\sin \pi/4}\right) < 0$	NA
4	6	1/12	$\cos\frac{\pi}{6}$	2	$-\frac{1}{2}\left(1+\frac{\cos \pi/20}{\sin \pi/4}\right)$	$\sigma_5(\Delta) = -\frac{1}{2}\left(1+\frac{\cos 5\pi/12}{\sin 5\pi/4}\right) < 0$	NA
4	8	0	$\cos\frac{\pi}{4}$	2	$-\frac{1}{2}\left(1+\frac{1}{\sin \pi/4}\right)$	$\sigma_5(\Delta) > 0$	A
5	4	1/ 5	$\cos\frac{\pi}{10}$	4	$1-\frac{3}{4\sin^2 \pi/5}-\frac{\cos \pi/5}{4\sin^3 \pi/5}$	$\sigma_{11}(\Delta) = 1-2.1708+.9959 < 0$	NA
5	5	1/10	$\cos\frac{\pi}{5}$	2	$1-\frac{3}{4\sin^2 \pi/5}\ \frac{\cos \pi/10}{4\sin^3 \pi/5}$	$\sigma_3(\Delta) > 0$	A
3	4	1/ 3	$\cos\frac{\pi}{6}$	2	$-\frac{1}{2\cdot 3\sin \pi/3}$	$\sigma_5(\Delta) > 0$	A
3	5	7/30	$\cos\frac{2\pi}{15}$	4	$-\frac{\cos 7\pi/30}{3\sin \pi/3}$	$\sigma_7(\Delta) > 0,\ \sigma_{7^2}(\Delta) > 0,\ \sigma_{7^3}(\Delta) > 0$	A
4	3	5/12	$\cos\frac{\pi}{6}$	2	$-\frac{1}{2}\left(1+\frac{\cos 5\pi/12}{3\sin \pi/4}\right)$	$\sigma_5(\Delta) > 0$	A
5	2	7/10	$\cos\frac{\pi}{5}$	2	$1-\frac{3}{4\sin^2 \pi/5}+\frac{\cos 3\pi/10}{4\sin^3 \pi/5}$	$\sigma_3(\Delta) > 0$	A
5	3	11/30	$\cos\frac{2\pi}{15}$	4	$1-\frac{3}{4\sin^2 \pi/5}-\frac{\cos 11\pi/30}{4\sin^3 \pi/5}$	$\sigma_{49}(\Delta) = 1-2.17-\frac{\cos 539\pi/30}{4\sin^3 49\pi/5} > 0$	NA

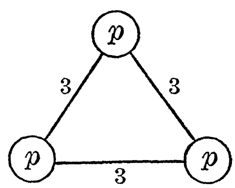
All the groups of § 17.1 are discrete. Except for the two cases $(p, \rho) = (3.6)$ and (4.4) in which $\sigma = \infty$, the regions $\Omega(\varphi)$ are compact and therefore $PU(H(\varphi))/\Gamma(\varphi)$ is compact; in the other two cases, $\Omega(\varphi)$ has three cusps at the boundary of the ball but nevertheless has finite measure. Accordingly Γ is a lattice subgroup of $PU(H)$ for all the groups listed in Tables 1 and 2.

We list the arithmeticity results in Table 3. The fourth column lists a primitive generator of the field $k = \boldsymbol{Q}[\mathrm{Ad}\,\Gamma]$, the fifth column lists the order of the Galois group of k. $\Delta = 1 - 3\alpha^2 - \alpha^3(\varphi^3 + \bar{\varphi}^3)$, the determinant of the matrix $\langle e_i, e_j\rangle$ $(i, j=1, 2, 3)$. The sixth column lists the effect of the automorphism $\sigma_n: z \to z^n$ an Δ, where z is a primitive generator of the cyclotomic field containing η^2, i, φ^3, and the value $\sigma_n(\Delta)$ for some n when $\sigma_n(\Delta) < 0$ and $\sigma_n \neq$ identity on $k = \boldsymbol{Q}[\mathrm{Tr\,Ad}\,\Gamma]$. In the last column A and NA denote arithmetic and non-arithmetic respectively.

In our situation, $\Delta = 1 - 3/(4\sin^2(\pi/p)) - \cos \pi t/(4 \sin^3(\pi/p))$ which simplifies to $-(\cos \pi t/)/(3 \sin \pi/3)$, $-(1/2)(1+(\cos \pi t)/(\sin \pi/3))$ for $p = 3, 4$ respectively.

Summing up the information contained in Table 3, we get the following result.

THEOREM 17.3. *There exists in $PU(2, 1)$ a non-arithmetic lattice generated by $\boldsymbol{C}$-reflections of order* 3, 4, *or* 5. *Up to an isometry, any such lattice with Coxeter diagram*



and $\langle e_1, e_2\rangle\langle e_2, e_3\rangle\langle e_3, e_1\rangle = e^{\pi i t}$ *is given by the seven values*

$$(p, t) = (3, 5/42), (3, 1/12), (3, 1/30)$$
$$(4, 3/20), (4, 1/12)$$
$$(5, 1/5), (5, 11/30)$$

The non co-compact lattices $\Gamma(\varphi)$ are arithmetic.

18. The space $Y(\varphi)$, $|\arg \varphi^3| < \pi/2 - \pi/p$, $(\arg \varphi)/\pi \in \boldsymbol{Q}$.

18.1. *The joined space.*

We have seen in §14.2 that for $|\arg \varphi^3| < \pi/2 - \pi/p$, the region $\Omega(\varphi)$ is bounded by the 24 3-dimensional faces

$$E_1(\Omega) = \{\tilde{R}_i^{\pm 1}, R_i, \tilde{R}_j)^{\pm 1}, (R_i R_j \tilde{R}_i)^{\pm 1}; i, j = 1, 2, 3; i \neq j\}$$

and that for each $\gamma \in \Gamma$,

$$\gamma \colon \tilde{\gamma} \longrightarrow \tilde{\gamma}^{-1} .$$

Set $\Delta(\varphi) = \{R_i^{\pm 1}, (R_i R_j)^{\pm 1}, (R_i R_j R_i)^{\pm 1}; i, j = 1, 2, 3; i \neq j\}$ and let $\Gamma(\varphi)$ denote the subgroup of Isom B generated by $\Delta(\varphi)$. Write F, Δ, Γ for $\Omega(\varphi)$, $\Delta(\varphi)$, and $\Gamma(\varphi)$ respectively. For any $\gamma \in \Delta$, set

$$e(\gamma) = \tilde{\gamma}^{-1}, \quad T(\tilde{\gamma}) = \gamma^{-1} .$$

By the proposition of §14.4, we may assert

$$\gamma(F) \cap F = e(\gamma) , \quad \text{all} \quad \gamma \in \Delta .$$

Set $\mathscr{F} - \Gamma F$. By Theorem 6.3.2(I), $\mathscr{F}$ is a connected abutted family of polyhedra. We let $\mathscr{N}$ denote the adjacency of $\mathscr{F}$ and we let $Y(\varphi)$ denote the joined $\mathscr{F}$-space. In §14.3 we have seen that the polyhedron F has 15 vertices of 4 distinct types, up to equivalence. Mod Aut F

$$\begin{array}{lll} p_{ij}(i \neq j) & \text{with} & p_{ij} = p_{ji} \\ s_{ij}(i \neq j) & \text{with} & s_{ij} = s_{ji} \\ \tilde{s}_{ij}(i \neq j) & \text{with} & \tilde{s}_{ij} = \tilde{s}_{ji} \\ t_{ij}(i \neq j) . & & \end{array}$$

We recapitulate here the results from §16. Let

$$\begin{aligned} r &= \operatorname{order}(R_1R_2R_3)^2 = \operatorname{order} \eta^3 i \bar{\varphi}^9 = \operatorname{order}(\bar{\eta} i \varphi)^3 \\ s &= \operatorname{order}(R_3R_2R_1)^2 = \operatorname{order} \eta^3 i \varphi^9 = \operatorname{order}(\bar{\eta} i \bar{\varphi}^3)^3 \\ \rho &= \operatorname{order} \bar{\eta} i \varphi^3 \\ \sigma &= \operatorname{order} \bar{\eta} i \bar{\varphi}^3 . \end{aligned}$$

These integers are finite if and only if $(2\pi)^{-1} \arg \varphi \in \boldsymbol{Q}$. In this case, $\Gamma_F = \boldsymbol{Z}_3$ if either $\rho = r$ or $\sigma = s$; equivalently 3 does not divide ρ or σ. Cyclic permutations of indices make up Aut F for all $\varphi \neq 1$. Thus for $\varphi \neq 1$, Aut $F = \boldsymbol{Z}_3$. If $\varphi = 1$, F is stable under all permutations and Aut $F = Z_6$. Set $\Gamma_\alpha = (\operatorname{Aut} \Gamma)\Gamma$. Then modulo Γ_α, the vertices of $E_1(F)$ are

(18.1.0) $E_4(F)/\Gamma_\alpha$: p_{12}, s_{12}, t_{32}, t_{23}

modulo Γ_α, the remaining $(4-k)$-faces $E_k(F)$ are

$E_3(F)/\Gamma_\alpha$: $p_{12}t_{32}$, p_{12}, t_{23}, $p_{12}s_{12}$(via s_{12}^*), $p_{12}s_{12}$(via s_{21}^*)

(18.1.1) $s_{12}t_{32}$, $s_{31}t_{23}$,

$\tilde{s}_{13}s_{12}$, $\tilde{s}_{21}s_{31}$

$E_2(F)/\Gamma_\alpha$: $e_1^\perp \cap F$, Δ_{123}, Δ_{321}

(18.1.2) $\tilde{R}_1 \cap \tilde{R}_1 R_2$, $\tilde{R}_2 \cap \tilde{R}_1 R_2$, $\tilde{R}_1 \cap R_1\tilde{R}_2R_1$, $\tilde{R}_2 \cap R_1\tilde{R}_2R_1$, $R_1\tilde{R}_2 \cap R_1\tilde{R}_2R_1$, $\tilde{R}_2R_1 \cap R_1\tilde{R}_2R_1$

(18.1.3) $E_1(F)/\Gamma_\alpha$: $\tilde{R}_1$, $\tilde{R}_1R_2$, $\tilde{R}_2R_1$, $R_1\tilde{R}_2R_1$.

18.2. *Stabilizers.*

Up to an automorphism of Γ, the stabilizers of all faces in the polyhedral decomposition $\mathscr{F}$ of the space $Y(\varphi)$ is given by the stabilizers of the faces $\eta(e, F)$ where η denotes the canonical map of $F \times \mathscr{F}$ onto $Y(\varphi)$ (cf. §6.2) and e ranges over the faces in the list (18.1). As above, $\Gamma(e, F)$ denotes the stabilizer of the face $\eta(e, F)$.

(18.2.1) $\Gamma_{(p_{12}, F)}$ is the subgroup Γ_{12} of Aut $B(=\mathrm{PU}(H))$ generated by R_1 and R_2; it has order $24(p/(6-p))^2$.

Proof. Let $\gamma \in \Gamma_{(p_{12},F)}$. Then $\gamma p_{12} = p_{12}$ and γF is p_{12}-connected to F in $\mathscr{F}$, and $\gamma F \in \mathscr{F}_{p_{12},F}$. By Remark 2 of §6.6, $\mathscr{F}_{p_{12},F} = \Gamma_{12}F_{p_{12}}$. Since F lies in a fundamental domain for the action of Γ_{12} on B, Γ_{12} operates simply transitively on $\mathscr{F}_{p_{12},F}$ and $\gamma \in \Gamma_{12}$. The order of Γ_{12} is given in §2.

PROPOSITION 18.2.3.

(i) $\Gamma_{(s_{12},F)} = \{\{R_3R_1R_2R_1, R_1^{-1}R_3R_1R_2\}\} = \{\{R_3R_2R_1R_2, R_2^{-1}R_3R_2R_1\}\}$
$\Gamma_{(\tilde{s}_{12},F)} = \{\{R_1R_2R_1R_3, R_2R_1R_3R_1^{-1}\}\} = \{\{R_2R_1R_2R_3, R_1R_2R_3R_2^{-1}\}\}$
and these stabilizers are abelian.

(ii) (a) *If* $3 \nmid \rho\sigma$, *then* $\Gamma_{(s_{12},F)} = \{\{(R_3R_1R_2)^2, (R_3R_2R_1)^3\}\} = \boldsymbol{Z}_r \times \boldsymbol{Z}_s$.

(b) *If* $3 \mid \rho\sigma$, *then* $\{\{(R_3R_1R_1)^2, (R_3R_2R_1)\}^2\}\}$ *is a subgroup of index* 3 *in* $\Gamma_{(s_{12},F)}$.

(iii) $\Gamma_{(s_{12},F)}\backslash G(s_{12}, F)$ *has* 6 *elements represented by*

$$1,\ R_1^{-1},\ R_2^{-1},\ R_3,\ R_3R_1,\ R_3R_2\ .$$

(iv) $\Gamma_{(s_{12},F)}\backslash\mathscr{F}_{s_{12},F}$ *has* $\begin{cases} 2 \text{ elements represented by } F, R_3F \\ \qquad\qquad\qquad\qquad \text{if } \Gamma_F \neq \{1\} \\ 6 \text{ elements} \qquad\qquad\quad \text{if } \Gamma_F = 1\ . \end{cases}$

Proof. Set $a = R_3R_1R_2R_1$, $b = R_1^{-1}R_3R_1R_2$, $c = b^{-1}a$. Then $c = R_2^{-1}R_1^{-1}R_3^{-1}(R_1R_3R_1)R_2R_1 = R_2^{-1}R_1^{-1}R_3^{-1}R_3R_1R_3R_2R_1 = R_2^{-1}R_3R_2R_1$. Set $\Gamma^1 = \{\{a, b\}\}$. Then $\Gamma^1 = \{\{a, c\}\} = \{\{R_3R_2R_1R_2, R_2^{-1}R_3R_2R_1\}\}$.

We shall prove (i) with the help of Proposition 6.6. By §14, the 3-faces of F containing s_{12} are

$$\tilde{R}_1,\ \tilde{R}_2,\ \tilde{R}_3^{-1},\ R_1\tilde{R}_2,\ \tilde{R}_2R_1,\ (R_3\tilde{R}_1)^{-1},\ (R_3\tilde{R}_2)^{-1},\ R_1\tilde{R}_2R_1\ .$$

Thus the adjacent cells $\mathscr{N}(F_{s_{12}})$ are of the form $(\gamma F)_{s_{12}}$ with γ in the subset of Δ denoted

$$F[s_{12}]\colon R_1^{-1},\ R_2^{-1},\ R_3,\ (R_1R_2)^{-1},\ (R_2R_1)^{-1},\ R_3R_1,\ R_3R_2,\ (R_1R_2R_1)^{-1}\ .$$

We first verify

$$\mathscr{N}(\mathscr{N}(F_{s_{12}})) \subset \Gamma^1(\mathscr{N}(F_{s_{12}}) \cup F_{s_{12}})\ . \tag{1}$$

In verifying this, we make use of the evident fact: for any $\gamma \in \Gamma$, $F \in \mathscr{F}$, $e \in E_k(\gamma F)$

$$\mathscr{N}((\gamma F)_e) = \gamma \mathscr{N}(F_{\gamma^{-1}e})\ .$$

In order to prove (1), we must show for each $\gamma \in F[s_{12}]$ and $\delta \in F[\gamma^{-1}s_{12}]$ that

$$\gamma\delta \in \Gamma^1(F[s_{12}] \cup 1)\ . \tag{1'}$$

For $\gamma = R_1^{-1}$, we find $\gamma^{-1}s_{12} = R_1s_{12} = \tilde{s}_{31}$. From §11, we find the 3-faces of F containing $\tilde{s}_{31}$ and accordingly we get for

$$F[\tilde{s}_{31}]\colon R_3,\ R_1,\ R_3R_1,\ R_1R_3,\ R_1R_3R_1,\ R_2^{-1},\ (R_1R_2)^{-1},\ (R_3R_2)^{-1}$$

and $R_1^{-1}F[\tilde{s}_{31}] = R_1^{-1}R_3,\ 1,\ R_1^{-1}R_3R_1,\ R_3,\ R_3R_1, (R_2R_1)^{-1},\ (R_1R_2R_1)^{-1}, (R_3R_2R_1)^{-1}$. We have $R_1^{-1}R_3R_1R_2 = b \in \Gamma^1$. Thus

$$R_1^{-1}R_3 = b(R_1R_2)^{-1} \subset \Gamma^1F[s_{12}]\ .$$

Similarly $R_1^{-1}R_3R_1 = (R_1^{-1}R_3R_1R_2)R_2^{-1} \subset \Gamma^1F[s_{12}]$, and

$$(R_3R_2R_1)^{-1} = (R_2^{-1}R_3R_2R_1)^{-1}R_2^{-1} \subset \Gamma^1F[s_{12}]\ .$$

Consequently $R_1^{-1}F[\tilde{s}_{31}] \subset \Gamma^1(F[s_{12}] \cup 1)$. The verification of (1′) for $\gamma = R_2^{-1}$ and R_3 is similar. For $\gamma = (R_1R_2)^{-1}$, we find $(R_1R_2)s_{12} = s_{23}$, and

$$F[s_{23}]\colon R_2^{-1},\ R_3^{-1},\ R_1,\ (R_2R_3)^{-1},\ (R_3R_2)^{-1},\ R_1R_2,\ R_1R_3,\ (R_2R_3R_2)^{-1}$$

$$(R_1R_2)^{-1}F[s_{23}]\colon (R_1R_2R_1)^{-1},\ (R_3R_1R_2)^{-1},\ R_2^{-1},\ (R_2R_3R_1R_2)^{-1}(R_3R_2R_1R_2)^{-1}\ ,$$
$$1,\ R_2^{-1}R_3,\ (R_2R_3R_2R_1R_2)^{-1}$$

$$R_2^{-1}R_3 = R_2^{-1}R_3R_2R_1(R_2R_1)^{-1} \in \Gamma^1F[s_{12}]\ .$$

Similarly, $(R_2R_3R_2R_1R_2)^{-1} = a^{-1}R_2^{-1} \in \Gamma^1F[s_{12}]$. Verification of (1′) for $\gamma = (R_2R_1)^{-1},\ R_3R_1,\ R_3R_1$, and $(R_1R_2R_1)^{-1}$ is similar. For example, $R_1R_2R_1s_{12} = \tilde{s}_{12}$ and we have for

$$F[\tilde{s}_{12}]\colon R_1,\ R_2,\ R_1R_2,\ R_2R_1,\ R_2R_1R_2,\ R_3^{-1},\ R_2$$

$$(R_1R_2R_1)^{-1}F(\tilde{s}_{12})\colon (R_2R_1)^{-1},\ (R_1R_2)^{-1},\ R_1^{-1},\ R_2^{-1},\ 1,\ (R_3R_1R_2R_1)^{-1}\ , \quad \text{and}$$

$$(R_2R_3R_1R_2R_1)^{-1} = a^{-1}R_2^{-1},\ (R_1R_3R_1R_2R_1)^{-1} = a^{-1}R_1^{-1}$$

all in $\Gamma^1(F[s_{12}] \cup 1)$, in verification of (1′).

Next we compute $S = (\{1\} \cup F[s_{12}])T_F \cap \Gamma_{(s_{12},F)}$. Let J denote the automorphism of B arising from the cyclic permutation $(1, 2, 3) \to (2, 3, 1)$. Then $\mathrm{Aut}\, F = \{\{J\}\}$. From §16 we know that if 3 does not divide ρ,

$$J = (R_1R_2R_3)^{2\mu}R_1R_2 = (R_2R_3R_1)^{2\mu}R_2R_3 = (R_3R_1R_2)^{2\nu}R_3R_1$$

where $3\mu + 1 \equiv 0(\mathrm{mod}\, \rho)$; if 3 does not divide σ

$$J^{-1} = (R_3R_2R_1)^{2\nu}R_3R_2 = (R_2R_1R_3)^{2\nu}R_2R_1 = (R_1R_3R_2)^{2\nu}R_1R_3$$

where $3\nu + 1 \equiv 0(\mathrm{mod}\, \sigma)$; if $3|\rho$ and $3|\sigma$ then $\Gamma_F = (1)$.

From the relation of §13:

$$R_is_{ij} = \tilde{s}_{ik},\ R_k\tilde{s}_{ij} = s_{ij}(i \neq j)(s_{ij} = s_{ji}) .$$

These permit us to find

$$R_1s_{12} = \tilde{s}_{13},\ R_2s_{12} = \tilde{s}_{32},\ R_3\tilde{s}_{21} = s_{12}$$
$$R_1R_2s_{12} = s_{23},\ R_2R_1s_{12} = s_{31},\ R_3R_1s_{31} = s_{12},\ R_3R_2s_{23} = s_{12}R_1R_2R_1s_{12} = \tilde{s}_{21} .$$

Consequently if

[A] $\quad \Gamma_F = (1),\ S = \{\gamma \in \{1\} \cup F[s_{12}];\ \gamma s_{12} = s_{12}\} = (1)$

[B] $\quad \Gamma_F = \{\{J\}\},\ S = \{1,\ (R_1R_2)^{-1}J,\ (R_2R_1)^{-1}J^{-1},\ R_3R_1J^{-1},\ R_3R_2J\}$.

If $3 \nmid \rho$, $(R_1R_2)^{-1}J = R_2^{-1}R_1^{-1}(R_1R_2R_3)^{2\mu}R_1R_2 = (R_3R_1R_2)^{2\mu} \in \Gamma^1$. If $3 \nmid \sigma$, $(R_1R_2)^{-1}J = (R_1R_2)^{-1}(R_3R_2)^{-1}(R_3R_2R_1)^{-2\nu} = (R_3R_2R_1R_2)^{-1}(R_3R_2R_1)^{-2\nu} \in \Gamma^1$. Similarly, we find in Case [B] that $S \subset \Gamma^1$. Hence $\Gamma^1 = \Gamma_{(s_{12},F)}$. By Proposition 6.6. $\Gamma_{(s_{12},F)} = \Gamma^1S = \Gamma^1$. The result for $\Gamma_{(\tilde{s}_{12},F)}$ follows from the isomorphism of $\Gamma(\varphi)$ to $\Gamma(\bar{\varphi})$ provided by complex conjugation which sends $s_{ij}(\varphi)$ to $\tilde{s}_{ij}(\bar{\varphi})$ and $R_i(\varphi)$ to $R_i^{-1}(\bar{\varphi})$.

Proof of (ii). $ac = (R_3R_2R_1)^2$ and $bc = R_1^{-1}R_3R_1R_3R_2R_1 = R_1^{-1}R_1R_3R_1R_2R_1 = a$. Therefore $a^3 = ab \cdot ac \in \Gamma^1$. Let E denote the subgroup $\{\{ab,\ ac\}\}$. Then Γ^1/E has order dividing 3. By §10 the matrix $R_3R_1R_2R_1$ is diagonalizable with eigenvalues $(\eta^3i\bar{\varphi}^3,\ \eta^3i\varphi^3,\ -\eta^2)$ so that its order in $\mathrm{PU}(H)$ is the order of the diagonal matrix $d = (-\eta i\bar{\varphi}^3,\ -\eta i\varphi^3)$. On the other hand, by §10, the order $(R_3R_1R_2)^2$ in $\mathrm{PU}(H)$ is the order of $\eta^3i\bar{\varphi}^9 = (-\eta i\bar{\varphi}^3)^3$ and the order of $(R_3R_2R_1)^2 = (-\eta i\varphi^3)^3$. If 3 does not divide the order of d, then a^3 has the same order as a and $a^3 \in E$ implies that $a \in E$. Therefore $E = \Gamma^1$, and $\Gamma_{(s_{12},F)} = \{(R_3R_1R_2)^2,\ (R_3R_2R_1)^2\}$. The intersection of the cyclic groups $\{(R_3R_1R_2)^2 \cap \{(R_3R_2R_1)^2\} = (1)$ for any element in this intersection fixes each point of the orthogonal $\boldsymbol{C}$-lines containing Δ_{312} and Δ_{321} respectively and therefore fixes all the points of B. Thus $\Gamma_{(s_{12},F)} = \boldsymbol{Z}_r \times \boldsymbol{Z}_s$ in Case 3 does not divide the order of d, or equivalently the order of $\bar{d} = (\bar{\eta}i\varphi^3,\ \bar{\eta}i\bar{\varphi}^3)$. This proves (iia).

If 3 divides the order of d, then 3 divides the order of a so that a is not in E. Hence E is of index 3 in $\Gamma_{(s_{12},F)}$, proving (iib).

Proof of (iii). Write Γ^1 for $\Gamma_{(s_{12},F)}$, $\mathscr{F}^1$ for $\mathscr{F}_{s_{12},F}$. We know from the proof of (i) that

$$G(s_{12}, F) = \Gamma^1(\{1\} \cup F[s_{12}])F .$$

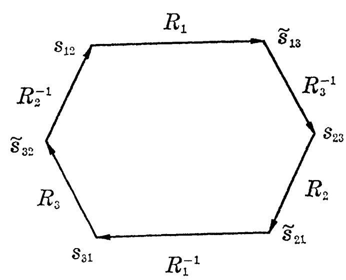
Clearly

$$\begin{aligned}(R_3R_1R_2R_1)^{-1}R_3R_1 &= (R_2R_1)^{-1}\\ (R_3R_2R_1R_2)^{-1}R_3R_2 &= (R_1R_2)^{-1}\\ (R_3R_2R_1R_2)(R_1R_2R_1)^{-1} &= R_3 .\end{aligned}$$

Consequently, $G(s_{12}, F) = \Gamma^1\{1, R_1^{-1}, R_2^{-1}, R_3, R_3R_1, R_3R_2\}$. For any two distinct elements γ_1, γ_2 of the six elements $Q = \{1, R_1^{-1}, R_2^{-1}, R_3, R_3R_1, R_3R_2\}$ one can order them so that $\gamma_1\gamma_2^{-1}$ is either in Q or is one of $\{R_1^{-1}R_2,\ R_3R_1^2R_3R_2^2R_3R_1R_2R_3R_2R_1R_3R_1R_2^{-1}R_3^{-1}\}$. From §13 we have the diagrams

$$\begin{tikzcd}[column sep=small] & (s_{23}, \tilde{s}_{21}) \arrow[dr, "R_2R_3"] & \\ (s_{12}, \tilde{s}_{13}) \arrow[ur, "R_1R_2"] & & (s_{31}, \tilde{s}_{32}) \arrow[ll, "R_3R_1"'] \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & (\tilde{s}_{32}, s_{12}) \arrow[dl, "R_2R_1"'] \arrow[dr, "R_3R_2"] & \\ (\tilde{s}_{21}, s_{31}) \arrow[rr, "R_1R_3"'] & & (\tilde{s}_{13}, s_{23}) \end{tikzcd}$$

and also



From these relations it is easy to verify that for any $\gamma_1, \gamma_2 \in Q$, $\gamma_1\gamma_2^{-1}s_{12} \neq s_{12}$. For example

$$R_3R_1R_2^{-1}(s_{12}) = R_3R_1(s_{23}) \neq s_{12}$$

since $(R_3R_1)^{-1}s_{12} = s_{31} \neq s_{23}$. Hence $G(s_{12}, F)$ has exactly six $\Gamma_{(s_{12},F)}$ orbits.

Proof of (iv). The set $\Gamma_{(s_{12},F)}\backslash\mathscr{F}_{s_{12},F}$ is the space of double cosets $\Gamma_{(s_{12},F)}\backslash G(s_{12}, F)/\Gamma_F$. It remains therefore to find which pairs of elements g_1 and g_2 of Q are in the same double coset; this is the case if and only if

$$g_2 J^{\pm 1} g_1 \in \Gamma_{(s_{12}, F)} \quad (= \Gamma^1) .$$

Clearly if $\Gamma_F = \{1\}$, then $\mathscr{F}_{s_{12}, F}$ is represented by the six elements of $G(s_{12}, F)$. If $\Gamma_F \neq \{1\}$, then either $3 \nmid \rho$ or $3 \nmid \sigma$.

Case 1. $3 \nmid \rho$. Then

$$\begin{aligned}
R_3R_1 \cdot J^{-1} \cdot 1 &= R_3R_1[(R_3R_1R_2)^{2\mu}R_3R_1]^{-1} \in \Gamma^1 \\
R_3R_2 \cdot J \cdot 1 &= R_3R_2[(R_1R_2R_3)^{2\mu}R_1R_2] = R_3(R_1R_2R_3)^{2\mu}R_2R_1R_2 \\
&= (R_3R_1R_2)^{2\mu}R_3R_2R_1R_2 \in \Gamma^1 \\
R_3JR_1 &= R_3(R_1R_2R_3)^{2\mu}R_1R_2R_1 = (R_3R_1R_2)^{2\mu}R_3R_1R_2R_1 \in \Gamma^1 \\
R_3J^{-1}R_2 &= R_3[(R_2R_3R_1)^{2\mu}R_2R_3]^{-1}R_2 = R_3[R_2(R_3R_1R_2)^{2\mu}R_3]^{-1}R_2 \\
&= (R_3R_1R_2)^{2\mu} \in \Gamma^1 .
\end{aligned}$$

Case 2. $3 \nmid \sigma$

$$\begin{aligned}
R_3R_1 \cdot J^{-1} \cdot 1 &= R_3(R_2R_1R_3)^{2\nu}R_1R_2R_1 = (R_3R_2R_1)^{2\nu}R_3R_1R_2R_1 \in \Gamma^1 \\
R_3R_2J \cdot 1 &= R_3R_2[(R_3R_2R_1)^{2\nu}R_3R_2]^{-1} \in \Gamma^1 \\
R_3JR_1 &= R_3[(R_1R_3R_2)^{2\nu}R_1R_3]^{-1}R_1 = [R_1(R_3R_2R_1)^{2\nu}]^{-1}R_1 \in \Gamma^1 \\
R_3JR_2 &= R_3[(R_2R_1R_3)^{2\nu}R_2R_1]R_2 = (R_3R_2R_1)^{2\nu}R_3R_2R_1R_2 \in \Gamma^1 .
\end{aligned}$$

Thus if either $3 \nmid \rho$ or $3 \nmid \sigma$, the space of double cosets has at most 2 distinct elements represented by 1 and R_3. It is easily verified that $R_3J^{\pm 1}s_{12} \neq s_{12}$. Proof of (iv) is now complete.

PROPOSITION 18.2.4. *Let* $\rho = \text{order}\, \bar{\eta} i \bar{\varphi}^3$, $\sigma = \text{order}\, \bar{\eta} i \bar{\eta}^3$, $r = \text{order}(R_1R_2R_3)^2$, $s = \text{order}(R_3R_2R_1)^2$.

(i) *If* $3 \nmid \rho\sigma$, *then* $\Gamma_{(t_{13}, F)} = \{\{R_2, (R_1R_2R_3)^2\}\} = \boldsymbol{Z}_p \times \boldsymbol{Z}_r$ *and* $\Gamma_{(t_{31}, F)} = \{\{R_2, (R_3R_2R_1)^2\}\} = \boldsymbol{Z}_p \times \boldsymbol{Z}_s$, *and* $r = \rho$, $\sigma = s$.

(ii) (a) *If* $3 | \sigma$ *but* $3 \nmid \rho$, *then*

$$\Gamma_{(t_{13}, F)} = \{\{R_2, (R_1R_2R_3)^2\}\} = \boldsymbol{Z}_p \times \boldsymbol{Z}_r , \quad r = \rho$$

and

$$\Gamma_{(t_{31}, F)} = \{\{R_2, (R_3R_1R_2)^{2\mu}R_3R_1R_2R_1\}\} = \boldsymbol{Z}_p \times \boldsymbol{Z}_{3s}, \sigma = 3s$$

where $3\mu + 1 \equiv 0 (\text{mod}\, r)$.

(b) *If* $3 | \rho$ *but* $3 \nmid \sigma$, *then*

$$\Gamma_{(t_{31}, F)} = \{\{R_2, (R_3R_2R_1)^2\}\} = \boldsymbol{Z}_p \times \boldsymbol{Z}_s , \quad s = \sigma$$

and

$$\Gamma_{(t_{13}, F)} = \{\{R_2, (R_2R_1R_3)^{2\nu}R_2R_1R_2R_3\}\} = \boldsymbol{Z}_p \times \boldsymbol{Z}_{3r} , \quad \rho = 3r$$

where $3v + 1 \equiv 0 (\mathrm{mod}\, s)$.

(iii) *If* $3|\rho$ *and* $3|\sigma$, *then*

$$\Gamma_{(t_{13}, F)} = \{\{R_2, (R_1R_2R_3)^2\}\} = \boldsymbol{Z}_p \times \boldsymbol{Z}_r \ , \quad \rho = 3r$$
$$\Gamma_{(t_{31}, F)} = \{\{R_2, (R_3R_2R_1)^2\}\} = \boldsymbol{Z}_p \times \boldsymbol{Z}_s \ , \quad \sigma = 3s \ .$$

(iv) $\Gamma_{(t_{13}, F)} \backslash \mathscr{F}_{t_{13}, F}$ *has 3 elements represented by* F, $(R_2R_3)^{-1}F$, $(R_1R_2)F$ *if* $\Gamma_F = \{1\}$, *and has only one element if* $\Gamma_F \neq \{1\}$.

Proof. From §13, we have $R_iR_jt_{kj} = t_{ik}$ for any distinct i, j, k. Thus we get the diagrams

$$\begin{tikzcd} & t_{13} \arrow[dr, "R_2R_3"] & \\ t_{32} \arrow[ur, "R_1R_2"] & & t_{21} \arrow[ll, "R_3R_1"'] \end{tikzcd} \qquad \begin{tikzcd} & t_{31} \arrow[dl, "R_2R_1"'] \arrow[dr, "R_3R_2"] & \\ t_{23} \arrow[rr, "R_1R_3"'] & & t_{12} \end{tikzcd}$$

Let J be the element in $\mathrm{Aut}\, F$ permuting (1, 2, 3) into (2, 3, 1). From §16 we know that if $3 \nmid \rho$,

$$J = (R_1R_2R_3)^{2\mu}R_1R_2 = (R_2R_3R_1)^{2\mu}R_2R_3 = (R_3R_1R_2)^{2\mu}R_3R_1$$

where $3\mu + 1 \equiv 0(\mathrm{mod}\, \rho)$.

Similarly, if $3 \nmid \sigma$,

$$J^{-1} = (R_3R_2R_1)^{2\nu}R_3R_2 = (R_2R_1R_3)^{2\nu}R_2R_1 = (R_3R_1R_2)^{2\nu}R_3R_1$$

where $3\nu + 1 \equiv 0(\mathrm{mod}\, \sigma)$.

For every product of the form $\gamma = (R_1R_2)(R_2R_3)(R_3R_1) \cdots$, $t_{13} \in \gamma F$; and thus every such γ which fixes t_{13} is in $\Gamma_{(t_{13}, F)}$. A similar assertion holds for any permutation of (1, 2, 3).

If 3 does not divide ρ, we infer from $JR_2R_1(t_{31}) = J(t_{23}) = t_{31}$ that $JR_2R_1 \in \Gamma_{(t_{12}, F)}$. Moreover, $(JR_2R_1)^3 = ((R_3R_1R_2)^{2\mu}R_3R_1R_2R_1)^3 \cdot = (R_3R_1R_2)^{6\mu}$ $(R_3R_1R_2R_1)^3$ since by Proposition 18.2.3(i), $R_3R_1R_2R_1$ commutes with $(R_3R_1R_2)^2$. As before, set $a = R_3R_1R_2R_1$, $b = R_1^{-1}R_3R_1R_2$, $c = R_2^{-1}R_3R_2R_1$. Then $a^3 = a^2 \cdot a = a^2bc = ab \cdot ac = (R_3R_1R_2)^2(R_3R_2R_1)^2$. Therefore

$$(R_3R_1R_2)^{6\mu}(R_3R_1R_2R_1)^3 = (R_3R_1R_2)^{-2} \cdot (R_3R_1R_2)^2(R_3R_2R_1)^2 = (R_3R_2R_1)^2 \ .$$

Similarly, if $3 \nmid \sigma$, then we infer from $J^{-1}R_2R_3(t_{13}) = J^{-1}(t_{21}) = t_{13}$ that $J^{-1}R_2R_3 \in \Gamma_{(t_{13}, F)}$. Moreover,

$$(J^{-1}R_2R_3)^3 = (R_2R_1R_3)^{2\nu}R_2R_1R_2R_3 \cdot J^{-2} = (R_2R_1R_3)^{6\nu}(R_2R_1R_2R_3)^3$$

since $R_2R_1R_2R_3$ commutes with $(R_2R_1R_3)^2$ by the cited result.

As before

$$(R_2R_1R_3)^{6\nu}(R_2R_1R_2R_3)^3 = (R_2R_1R_3)^{-2\nu} \cdot (R_2R_1R_3)^2(R_1R_2R_3)^2 = (R_1R_2R_3)^2 \ .$$

If $3 \nmid \rho$, then $\{\{JR_1R_2\}\} = \{\{(JR_1R_2)^3\}\} = \{\{(R_3R_2R_1)^2\}\}$, and similarly, if $3 \nmid \sigma$, then $\{\{J^{-1}R_2R_3\}\} = \{\{(R_1R_2R_3)^2\}\}$. Set

$$\Gamma^1 = \begin{cases} \{\{R_2, (R_2R_1R_3)^{2\nu}R_2R_1R_2R_3\}\} & \text{if } 3 \nmid \sigma \\ \{\{R_2, (R_1R_2R_3)^2\}\} & \text{if } 3 \mid \sigma \end{cases}$$

$$\tilde{\Gamma}^1 = \begin{cases} \{\{R_2, (R_3R_1R_2)^{2\mu}R_3R_1R_2R_1\}\} & \text{if } 3 \nmid \rho \\ \{\{R_2, (R_3R_2R_1)^2\}\} & \text{if } 3 \mid \rho . \end{cases}$$

We have $\Gamma^1 \subset \Gamma_{(t_{13}, F)}$ and $\hat{\Gamma}^1 \subset \Gamma_{(t_{31}, F)}$. By Proposition 18.2.3(i), $(R_2R_1R_3)^{2\nu}R_2R_1R_2R_3$ fixes the point $\tilde{s}_{12}$ as well as t_{13}, hence each point of the $\boldsymbol{C}$-line through Δ_{123}, and hence commutes with R_2. Each element in $\{\{R_2\}\} \cap \{\{(R_2R_1R_3)^{2\nu}R_2R_1R_2R_3\}\}$ fixes each point of two $\boldsymbol{C}$-lines and therefore is the identity. Hence the order of Γ^1 is $p\rho$. Similarly, the order of $\tilde{\Gamma}^1 = p\sigma$. Thus assertions (i), (ii), (iii) of the proposition is equivalent to: $\Gamma^1 = \Gamma_{(t_{13}, F)}$ and $\tilde{\Gamma}^1 = \Gamma_{(t_{31}, F)}$. Making use of the isomorphism $\Gamma(\varphi) \to \Gamma(\bar{\varphi})$ given by complex conjugation, which sends $t_{31}(\varphi)$ to $t_{13}(\bar{\varphi})$, it suffices to prove only that $\Gamma^1 = \Gamma_{(t_{13}, F)}$.

To prove this equality, we apply Proposition 6.6, arguing as in the proof of the preceding proposition. By Lemma 13.4 the 3-faces of F containing t_{13} are

$$\tilde{R}_2,\ \tilde{R}_2^{-1},\ \tilde{R}_2R_3,\ (R_1\tilde{R}_3)^{-1},\ R_2\tilde{R}_3R_2,\ (R_2\tilde{R}_1R_3)^{-1} .$$

Thus the cells in $\mathscr{F}_{t_{13}, F}$ adjacent to $F_{t_{13}}$ are of the form γF with γ in the set

$$F[t_{13}]\colon R_2^{-1},\ R_2,\ (R_2R_3)^{-1},\ R_1R_2,\ (R_2R_3R_2)^{-1},\ R_2R_1R_2 .$$

First we verify

$$(1) \qquad \mathscr{N}(\mathscr{N}(F_{t_{13}})) \subset \Gamma^1(\mathscr{N}(F_{t_{13}}) \cup F_{t_{13}}) .$$

We must show that for each $\gamma \in F[t_{13}]$ and $\delta \in F[\gamma^{-1}t_{13}]$

$$(1') \qquad \gamma\delta \in \Gamma^1(F[t_{13}] \cup 1) .$$

For $\gamma = R_2^{\pm 1}$, we have $\gamma \in \Gamma^1$ and the assertion is obvious. If $\gamma = (R_2R_3)^{-1}$, $\gamma^{-1}t_{13} = R_2R_3t_{13} = t_{21}$; we have for

$$\begin{aligned} &F[t_{21}]\colon R_3^{-1},\ R_3,\ (R_3R_1)^{-1},\ R_2R_3,\ (R_3R_1R_3)^{-1},\ R_3R_2R_3 \\ &(R_2R_3)^{-1}F[t_{21}]\colon R_3^{-1}R_2^{-1}R_3^{-1},\ R_3^{-1}R_2^{-1}R_3,\ R_3^{-1}R_2^{-1}R_1^{-1}R_3^{-1},\ 1,\ (R_3R_1R_3R_2R_3)^{-1}, \\ &\qquad R_2 \\ &(R_3R_2R_3)^{-1} \in F[t_{13}],\ R_3^{-1}R_2^{-1}R_3 = R_2R_3^{-1}R_2^{-1} = R_2(R_2R_3)^{-1} \in \Gamma^1F[t_{13}] , \\ &\qquad R_3^{-1}R_2^{-1}R_1^{-1}R_3^{-1} = (R_1R_2R_3)^{-2}R_1R_2 \in \Gamma^1F[t_{13}] , \end{aligned}$$

and

$$(R_3R_1R_3R_2R_3)^{-1} = (R_3R_1R_2R_3R_2)^{-1} = R_2^{-1}(R_1R_2R_3)^{-2}R_1R_2R_3 \cdot R_3^{-1} \in \Gamma^1R_1R_2 .$$

Thus $(R_2R_3)^{-1}F[t_{21}]\subset\Gamma^1(F[t_{13}]\cup 1)$. In a similar way we verify the remaining claims in (1′). For example,

$$\begin{aligned}R_2R_1R_2F[(R_2R_1R_2)^{-1}t_{13}] &= R_2R_1R_2F[t_{32}]\\ &= R_2R_1R_2\{R_1^{-1},\, R_1,\, (R_1R_2)^{-1},\, R_3R_1\,, (R_1R_2R_1)^{-1},\, R_1R_3R_1\}\\ &= \{R_1R_2,\, R_2R_1R_2R_1,\, R_2,\, R_2R_1R_2R_3R_1,\, 1,\, R_2R_1R_2R_1R_3R_1\}\,.\end{aligned}$$

We have $R_2R_1R_2R_3R_1 = R_2(R_1R_2R_3)^2(R_2R_3)^{-1}$ and $R_2R_1R_2R_1R_3R_1 = R_2(R_1R_2R_3)^2(R_2R_3)^{-1}R_3$, and $R_3^{-1}R_2^{-1}R_3 = R_2R_3^{-1}R_2^{-1}$. Therefore

$$R_2R_1R_2R_1R_3R_1 = R_2(R_1R_2R_3)^2R_2R_3^{-1}R_2^{-1} = R_2^2(R_1R_2R_3)^2(R_2R_3)^{-1}\,.$$

In this way we see that $R_2R_1R_2F[(R_2R_1R_2)^{-1}t_{13}]\subset\Gamma^1(F[t_{13}]\cup 1)$. So much for the proof of (1).

From Proposition 6.6, it follows that

$$\Gamma_{(t_{13},F)} = \Gamma^1 S$$

where $S=\{\gamma\in(F[t_{13}]\cup\{1\})\Gamma_F;\ \gamma t_{13}=t_{13}\}$. Set $T^1=F[t_{13}]\cup\{1\}$. For any subset $S^1\subset T^1$ such that $\Gamma^1T^1=\Gamma^1S^1$, we have $\Gamma^1(S^1\Gamma_F\cap\Gamma_{(t_{13},F)})=\Gamma_{(t_{13},F)}$ by Remark 1 following Proposition 6.6. The images of t_{13} under $F[t_{13}]\Gamma_F$ are seen from

$$R_2t_{13}=t_{13},\ (R_2R_3)^{-1}t_{21}=t_{13},\ (R_1R_2)t_{32}=t_{13},\ R_1R_2R_1t_{32}=t_{13}\,.$$

Inasmuch as $\Gamma^1(R_2R_3)^{-1}$ contains $(R_2R_3R_2)^{-1}$, we can ignore the contribution of $(R_2R_3R_2)^{-1}$. That is, set $S^1=\{1,\,(R_2R_3)^{-1},\,R_1R_2\}$. Then

$$(*)\qquad \begin{aligned}\Gamma^1S^1 &= \Gamma^1(\{1\}\cup F[t_{13}])\,,\\ t_{13} &= (R_2R_3)^{-1}Jt_{13} = (R_1R_2)J^{i1}t_{13}\,.\end{aligned}$$

Consequently

$$(**)\qquad \Gamma_{(t_{13},F)} = \begin{cases}\Gamma^1\{1,\,(R_2R_3)^{-1}J,\,(R_1R_2)J^{-1}\} & \text{if}\quad J\in\Gamma_F\\ \Gamma^1 \quad \text{if}\quad J\notin\Gamma_F\,.\end{cases}$$

All the assertions of the proposition follow from (**) and (*).

REMARK 18.2.5. The number of polyhedra in the polyhedral space $Y(t_{13},\mathscr{F})$ surrounding t_{13} is

$$\#\mathscr{F}_{t_{13},F} = p\rho\,.$$

Similarly,

$$\#\mathscr{F}_{t_{31},F} = p\sigma\,.$$

From Proposition 18.2.3, we have

$$\#\mathscr{F}_{s_{ij},F} = \#\mathscr{F}_{\tilde{s}_{ij},F} = 2\rho\sigma\,.$$

PROPOSITION 18.2.6. *The stabilizer in Γ of $e \in E_k(F)(k = 1, 2, 3)$ is given by the following lists*

$\eta(e, F)$	$\Gamma_{(e,F)}$	*order*
$p_{12}t_{32}$ *or* $p_{12}t_{23}$	$\{\{R_1\}\}$	p
$p_{12}s_{12}$(*via* s_{12}^* *or* s_{21}^*)	$\{1\}$	1
$s_{12}t_{32}$	$\begin{cases}\{\{J^{-1}R_1R_2\}\} \\ \{\{(R_3R_1R_2)^2\}\}\end{cases}$	$\begin{cases}\rho \textit{ if } J \in \Gamma \\ r \textit{ if } J \notin \Gamma\end{cases}$
$s_{31}t_{23}$	$\begin{cases}\{\{JR_1R_3\}\} \\ \{\{(R_2R_1R_3)^2\}\}\end{cases}$	$\begin{cases}\sigma \textit{ if } J \in \Gamma \\ s \textit{ if } J \notin \Gamma\end{cases}$
$\tilde{s}_{13}s_{12}$	$\begin{cases}\{\{J^{-1}R_1R_2,\ R_3R_1R_2\}\} \\ \{\{R_3R_1R_2\}\}\end{cases}$	$\begin{cases}2\rho \textit{ if } J \in \Gamma \\ 2r \textit{ if } J \notin \Gamma\end{cases}$
$\tilde{s}_{21}s_{31}$	$\{\{R_2R_1R_3\}\}$	$2s$
$e_1^{\perp} \cap F$	$\{\{R_1\}\}$	p
$\varDelta_{123}$	$\begin{cases}J^{-1}R_1R_2 \\ \{\{(R_1R_2R_3)^2\}\}\end{cases}$	ρ if $J \in \Gamma$ r if $J \notin \Gamma$
$\varDelta_{321}$	$\begin{cases}JR_1R_3 \\ \{\{(R_3R_2R_1)^2\}\}\end{cases}$	$\begin{cases}\sigma \text{ if } J \in \Gamma \\ s \text{ if } J \notin \Gamma\end{cases}$
$\tilde{R}_1 \cap \widetilde{R_1R_2}$, $\tilde{R}_2 \cap \widetilde{R_1R_2}$, $\tilde{R}_1 \cap R_1\tilde{R}_2R_1$, $\tilde{R}_2 \cap R_1\tilde{R}_2R_1$		1
$\widetilde{R_1R_2} \cap R_1\tilde{R}_2R_1$, $\widetilde{R_2R_1} \cap R_1\tilde{R}_2R_1$		1
$\tilde{R}_1$, $\widetilde{R_1R_2}$, $\widetilde{R_2R_1}$, $R_1\tilde{R}_2R_1$		1

Proof. One verifies that $R_3R_1R_2\tilde{s}_{13} = s_{12}$ and $R_3R_1R_2s_{12} = \tilde{s}_{13}$. Thus the stabilizer of the 1-face $\tilde{s}_{13}s_{12}$ contains the cyclic group $\{\{R_3R_1R_2\}\}$. Similar for $\tilde{s}_{21}s_{31}$. By reasoning as in the preceding propositions, one proves the asserted results for $\tilde{s}_{13}s_{12}$ and $\tilde{s}_{21}s_{31}$. For $\tilde{R}_1 \cap \widetilde{R_1R_2}$, the stabilizer leaves fixed each of its vertices p_{12}, $\tilde{s}_{32}$, s_{12} and its stabilizer is in the intersection of the stabilizer of its vertices; hence the stabilizer fixes each point of the smallest geodesic subspace containing the faces.

18.3. *Riemannian manifold structure on Y.*

Assume that $|\arg \varphi^3| < \pi/2 - \pi/p$ and that $(1/\pi) \arg \varphi^3 \in \boldsymbol{Q}$. Then the space $Y(\varphi)$ has been defined as well as the canonical map π of Y onto $\boldsymbol{C}h^2$. At any point of Y which is not in the Γ orbit of a point in one of the six $\varDelta_{ijh}$, the map π is a local homeomorphism. At faces lying in some $\varDelta_{ijh}$, the neighborhood of the faces are given by the results in §18.2. The hypotheses of Proposition 6.4.1 are satisfied by the abutted family $\Gamma\Omega$. Indeed one can give Y a complex analytic structure so that $\pi: Y \to \boldsymbol{C}h^2$ is holomorphic (cf. Proposition 6.4.2). Thus we conclude (cf. §19 for details).

LEMMA 18.3. *For* $|\arg \varphi^3| < \pi/2 - \pi/p$ and $(1/\pi)\arg \varphi^3 \in \boldsymbol{Q}$, *the space* $Y(\varphi)$ *has the structure of a Riemannian manifold such that* $\pi: Y \to \boldsymbol{C}h^2$ *is differentiable.*

The infinitesimal Riemannian metric in $\boldsymbol{C}h^2$ pulls back to a metric on Y which may be degenerate at points at which π is not a local homeomorphism. However, it is not difficult to define a Riemannian metric on $Y(\varphi)$ that is preserved by Γ. For example, let Γ_0 be a torsion-free normal subgroup of Γ of finite index — such a subgroup exists in any finitely generated metric group by a result of Selberg. Then choose a Riemannian metric on the manifold $\Gamma_0\backslash Y$. Averaging over the finite group Γ/Γ_0, we can assume that the metric d on $\Gamma_0\backslash Y$ is Γ-stable. The pull-back of d to Y gives a Γ-stable metric on Y.

18.4. $\mathrm{Aut}_\Gamma\, \Omega$, $|\arg \varphi^3| < \pi/2 - \pi/p$, $3\,|\,\gcd(\rho, \sigma)$.

We take up the question of the order of $\mathrm{Aut}_\Gamma\, \Omega$ that was mentioned in Remark 3 of §16.

LEMMA 18.5. *Assume* $|\arg \varphi^3| < \pi/2 - \pi/p$. *Let* $\rho = \mathrm{order}\, \bar{\eta} i \varphi^3$, $\sigma = \mathrm{order}\, \bar{\eta} i \bar{\varphi}^3$ *and assume that* $3\,|\,\gcd(\rho, \sigma)$. *Then* $\# \mathrm{Aut}_\Gamma\, \Omega = 1$.

Proof. Let Y denote the manifold $Y(\varphi)$, let Y^* denote the simply connected covering space of Y, and let Γ^* denote the lift of Γ to Y. Then $\Gamma^*/\pi_1(Y) = \Gamma$ and Γ^* operates discontinuously as a group of isometries on the Riemannian manifold Y^*. Let Ω^* denote a lift of the polyhedron Ω; inasmuch as Ω is a topological cell (by Lemma 3.3.2) Ω^* is a cell mapping homeomorphically onto Ω. The space Y^* is the joined Γ^*-space of the abutted family of polyhedron $\Gamma^*\Omega^*$. The group Γ^* operates discontinuously on Y^* and satisfies both (CD1) and (CD2) of Theorem 6.3.3. It follows that $\mathrm{Aut}_{\Gamma^*}\, \Omega^*$ is generated by R_2^*, the set of words corresponding to shortest circuits in Y^* around 2-faces of Ω^*. From this in turn it follows that $\mathrm{Aut}_\Gamma\, \Omega$ is generated by $\mathscr{R}_2$, the set of words corresponding to shortest circuits in Y around 2-faces of Ω. Inasmuch as the circuits around 2-faces of Ω which do not contain an apex correspond to trivial words, we need only consider the circuits around Δ_{ijh} for permutations (ijh) of (123). By symmetry, it suffices to consider only circuits around Δ_{123}.

Given such a circuit Ω, R_1R_2, $R_1R_2 \cdot R_3R_1\Omega$, $R_1R_2 \cdot R_3R_1 \cdot R_2R_3\Omega$, $\cdots$ with $R_1R_2 \cdot R_3R_1 \cdot \cdots \Omega = \Omega$, then set $\gamma = R_1R_2 \cdot R_3R_1 \cdots$. We have $\gamma \in \mathrm{Aut}_\Gamma\, \Omega$. We can assume that $\varphi \neq 1$, otherwise $3\,|\,\rho$ and $3\,|\,\sigma$, imply $p = 3$ and $\Omega(\varphi)$ satisfies (CD_2); in that case the lemma is known (cf. Remark 2 of §16). Hence $\mathrm{Aut}_\Gamma\, \Omega = \boldsymbol{Z}_3$ and $\gamma = 1$, J, or J^{-1}. Thus γ sends into Δ_{123} either Δ_{123}, Δ_{312}, or Δ_{231}. Accordingly, γ has the form

$$(R_1R_2R_3)^{2m},\ (R_1R_2R_3)^{2m}R_1R_2,\ (R_1R_2R_3)^{2m}R_1R_2R_3R_1\ .$$

We wish to prove $\gamma = 1$. Hence we need only dismiss the possibilities $\gamma = (R_1R_2R_3)^{2m}R_1R_2$ or $(R_1R_2R_3)^{2m}R_1R_2R_3R_1$.

Consider the canonical map $\pi: Y \to Ch^2$. The circuit in Ch^2 corresponding to γ places side by side $3m+1$(resp. $3m+2$) images of Ω bounded by spinal surfaces meeting in a common slice containing Δ_{123}. Each of the $3m+1$(resp. $3m+2$) angles formed equals $\sphericalangle(R_1\hat{R}_2)^{-1}$, $R_2R_3 = \arg \bar{\eta} i\varphi^3$ (cf. §15). Hence $(\bar{\eta} i\varphi^3)^{3m+\varepsilon} = 1$ $\varepsilon = 1$ or 2. Consequently, $\rho | 3m+\varepsilon(\varepsilon = 1, 2)$. This contradicts $3|\rho$. It follows that $\mathrm{Aut}_\Gamma \Omega = 1$.

19. Complex analytic structure on $Y(\varphi)$, $|\arg \varphi^3| < \pi/2 - \pi/p$, $\pi^{-1} \arg \varphi \in \boldsymbol{Q}$.

We continue the notation of §18, writing $\Gamma = \Gamma(\varphi)$, $\Delta = \Delta(\varphi)$, $F = \Omega(\varphi)$, $Y = Y(\varphi)$, $\rho = \text{order } \bar{\eta} i\varphi^3$, $\sigma = \text{order } \bar{\eta} i\bar{\varphi}^3$. Assume that ρ and σ are finite. Then

$$\rho\left(\frac{\pi}{2} - \frac{\pi}{p} + \arg \varphi^3\right) = 2m\pi,\ \sigma\left(\frac{\pi}{2}\frac{\pi}{p} - \arg \varphi^3\right) = 2n\sigma$$

where $\gcd(m, \rho) = 1 = \gcd(n, \sigma)$. Set

$$\xi = \exp(2\pi i/\rho)\ ,\quad \xi = \exp(2\pi i/\sigma)\ .$$

Then $\xi^m = \bar{\eta} i\varphi^3$, $\zeta^n = \bar{\eta} i\bar{\varphi}^3$.

The canonical map π of Y onto the ball B is clearly a homeomorphism in the neighborhood of any point p of thecell $\eta(\gamma, F)$ of Y if p does not lie on $\eta(\gamma, \Delta_{ijk})(i, j, k$ any permutation of 1, 2, 3); in the neighborhood of such p, one chooses as coordinates the pull-back of a standard coordinate system on the ball B.

We next describe the choice of a coordinate system in the neighborhood of a point in $\eta(\gamma, \Delta_{ijk})$. By symmetry, we can take $\gamma = 1$ and $p \in \eta(1, \Delta_{312})$. We shall cover a neighborhood of $\eta(1, \Delta_{312})$ in Y by pull-backs of three balls in B centered at the vertices of Δ_{312}.

We first consider the case $p = s_{12}$. Recall that (cf. (12.1))$s_{12} = I_1 \cap I_2'$ and that the C-lines I_1, $I_2^\perp$ are orthogonal at s_{12} (cf. Remark 2 following Lemma 12.3). Therefore we can choose a standard nonhomogeneous coordinate system centered at s_{12} with $u = 0$ on I_2' and $v = 0$ on I_1. Since u and v are unique up to a scalar factor of modulus one, (each line has a unique holomorphic structure induced from C^3!) we can make the choice unique by the additional conditions $u(\tilde{s}_{13}) > 0$, $v(\tilde{s}_{32}) > 0$. Let α be a positive number such that the ball $B_\alpha(s_{12})$ with center at s_{12} and radius d meet no face of F other than those containing s_{12}. Let C_{12} denote the subset of $B_\alpha \times C \times C$ defined by

$$C_{12} = \{(x, \tilde{u}, \tilde{v});\ \tilde{u}^m = u(x),\ \tilde{v}^n = v(x)\}\ .$$

Let $\psi: C_{12} \to B_\alpha(s_{12})$ denote the projection on the first factor. We can define an operation of $\Gamma_{(s_{12},F)}$ on C_{12} so that ψ is a $\Gamma_{(s_{12},F)}$ map. We see this as follows.

By (9.1.6) the eigenvalue of $R_2R_1R_2R_3$ corresponding to its fixed point s in B is $-\eta^2$ and on standard nonhomogeneous coordinates centered at s its eigenvalues are $(\eta^3 i\varphi^3/-\eta^2,\ \eta^3 i\bar{\varphi}^3(-\eta^2))$; that is, $(\bar{\zeta}^n, \bar{\xi}^m)$. The same is true for $R_3R_2R_1R_2$ (for it has the same characteristic polynomial) which fixes s_{12}. Hence

$$u(R_3R_2R_1R_2x) = \bar{\zeta}^n u(x)$$
$$v(R_3R_2R_1R_2x) = \bar{\xi}^m v(x)$$

for all $x \in B$. Similarly by (9.1.1) and (9.1.2)

$$u((R_3R_1R_2)^2x) = u(x),\ v((R_3R_1R_2)^2x) = \bar{\xi}^m v(x)$$
$$u((R_3R_2R_1)^2x) = \bar{\zeta}^n u(x),\ v((R_3R_2R_1)^2x) = v(x)$$

for all $x \in B$.

Thus we define for all $y = (x, \tilde{u}, \tilde{v}) \in C_{12}$,

$$\tilde{u}(R_3R_2R_1R_2y) = \bar{\zeta}\tilde{u}(y)\ ,\quad \tilde{v}(R_3R_2R_1R_2y) = \bar{\xi}\tilde{v}(y)$$
$$\tilde{u}((R_3R_1R_2)^2y) = \tilde{u}(y)\ ,\quad \tilde{v}((R_3R_1R_2)^2y) = \bar{\xi}\tilde{v}(y)$$
$$\tilde{u}((R_3R_2R_1)^2y) = \bar{\zeta}\tilde{u}(y)\ ,\quad \tilde{v}((R_3R_2R_1)^2y) = \tilde{v}(y)\ .$$

Holomorphically, take $C_{12} \approx \{(\tilde{u}, \tilde{v}) \in \boldsymbol{C}^2;\ |\tilde{u}|^{2m} + |\tilde{v}|^{2n} < r^2\}$. It is clear now that $\psi: C_{12} \to B_\alpha(s_{12})$ is a $\Gamma_{(s_{12},F)}$ map and also holomorphic. The canonical map $\pi: y_{s_{12},F} \to B$ branches over $B_r(s_{12})$ in exactly the same way that C_{12} does. Hence the projection $\psi: C_{12} \to B$ factors through $Y_{s_{12},F}$ and there is a unique $\Gamma_{(s_{12},F)}$ homeomorphism $f_{12}: C_{12} \to Y_{s_{12},F} \cap \pi^{-1}(B_\alpha)$ such that $\pi \circ f_{12} = \psi$, $\tilde{u} > 0$ on $f_{12}^{-1}(\eta([s_{12}, \tilde{s}_{13}], F))$, and $\tilde{v} > 0$ on $f_{12}^{-1}((\eta[s_{12}, \tilde{s}_{13}], F))$. We sometimes denote f_{12} as $f_{s_{12}}$.

In exactly the same way, one defines a space C_{ij}(resp. $C_{\tilde{i}\tilde{j}}$) for each vertex s_{ij}(resp. $s_{\tilde{i}\tilde{j}}$), $i \neq j$, $i, j = 1, 2, 3$; this yields six space. Given any $\gamma \in G(s_{12}, F)$ (cf. § 6.6 for definition), $\bar{\gamma}^{-1}s_{12} \in \mathscr{S} = \{s_{ij}, s_{\tilde{i}\tilde{j}};\ i \neq j,\ i, j = 1, 2, 3\}$, and for each $s \in \mathscr{S}$ we get a commutative diagram

$$\begin{array}{ccc}
C_s & \xrightarrow{\ \gamma\ } & C_{12} \\
{\scriptstyle f_s}\downarrow\approx & & \approx\downarrow{\scriptstyle f_{12}} \\
\pi^{-1}(B) \cap Y_{s,F} & \xrightarrow{\ \gamma\ } & Y_{s_{12},F} \cap \pi^{-1}(B_\alpha) \\
\downarrow & & \downarrow \\
B_\alpha(s) & \xrightarrow{\ \gamma\ } & B_\alpha(s_{12})\ .
\end{array}$$

We next prove that:

$$\gamma: C_s \longrightarrow C_{12} \quad \text{is holomorphic for all } \gamma \in G(s_{12}, F) \text{ where } s = \gamma^{-1}s_{12}\ .$$

Inasmuch as f_s is a $\Gamma_{(s,F)}$ map, it suffices to prove that γ is holomorphic for a set of representatives of $\Gamma_{(s_{12},F)}G(s_{12},F)$; by Proposition 18.2.3(iii), $\{1, R_1^{-1}, R_2^{-1}, R_3, R_3R_1, R_3R_2\}$ is a set of representatives. We shall give the argument only for the cases $\gamma = R_3$, $\gamma = R_1^{-1}$ and γ R_3R_1, the other cases following by symmetry.

Returning to Figure 14.1, we see that

$$R_3[\tilde{s}_{21}, s_{23}] = [s_{12}, \tilde{s}_{13}] . \tag{19.3}$$

Thus $R_3I_2 = I_1$, and in the $\boldsymbol{C}$-line I_1 the geodesic triangle $R_3\Delta_{123}$ abuts Δ_{312} along the geodesic line segment $[s_{12}, \tilde{s}_{13}]$. Similarly

$$R_3[\tilde{s}_{21}, s_{31}] = [s_{12}, \tilde{s}_{32}] , \quad R_3I_1' = I_2'$$

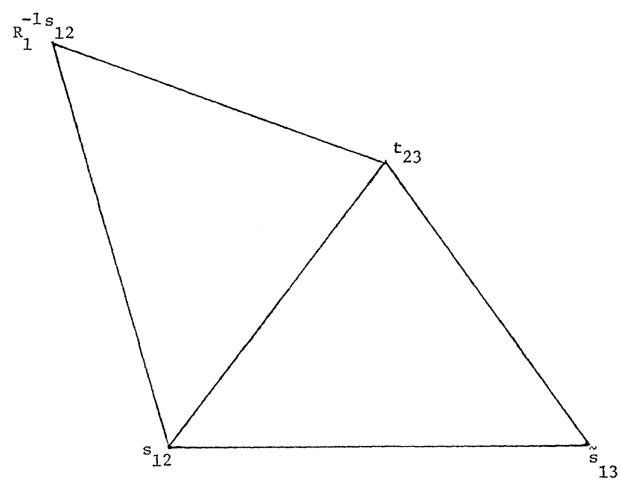
and the geodesic triangle $R_3\ \Delta_{213}$ abuts Δ_{321} along the geodesic line segment $[s_{12}, \tilde{s}_{32}]$. Inasmuch as R_3 is an isometry of the ball, it carries any standard nonhomogeneous coordinate on the $\boldsymbol{C}$-line I_2 (resp. I_1') centered at $\tilde{s}_{21}$ to a standard nonhomogeneous coordinate on the $\boldsymbol{C}$-line I_1 (resp. I_2') centered at s_{12}. We next compute the transformation of the coordinates of $C_{\tilde{2}\tilde{1}}$ induced by R_3.

By definition $C_{\tilde{2}\tilde{1}}$ has coordinates $\tilde{u}$, $\tilde{v}$; $\tilde{u}^m$(resp. $\tilde{v}^n$) is the unique standard nonhomogeneous coordinates on I_2 (resp. I_1') centered at $\tilde{s}_{21}$ with $\tilde{u} > 0$ on $f_{21}^{-1}(\gamma([\tilde{s}_{21}, s_{23}], F))$(resp. $\tilde{v} > 0$) on $f_{\tilde{2}\tilde{1}}^{-1}(\gamma([\tilde{s}_{21}, s_{31}], F))$. By (19.3), R_3 sends the defining data of the $\tilde{u}$, $\tilde{v}$ coordinates of $C_{\tilde{2}\tilde{1}}$ to the defining data for the $\tilde{u}$, $\tilde{v}$ coordinates of C_{12}. Consequently, the map $R_3\,C_{\tilde{2}\tilde{1}} \to C_{12}$ is the map $(\tilde{u}, \tilde{v}) \to (\tilde{u}, \tilde{v})$. Thus R_3 is holomorphic.

Consider next the map R_1^{-1}: $C_{\tilde{1}\tilde{3}} \to C_{12}$. We have (cf. (13.1))

$$R_1^{-1}[\tilde{s}_{13}, t_{32}] = [s_{12}, t_{32}]$$
$$R_1^{-1}[\tilde{s}_{13}, s_{23}] = [s_{12}, \tilde{s}_{32}] .$$

The second relation implies that R_1^{-1} sends the $\tilde{v}$ coordinates of $C_{\tilde{1}\tilde{3}}$ to the $\tilde{v}$ coordinates of C_{12}, by the same argument that was used above. The first relation yields the diagram



This shows that the image $R_1^{-1}[\tilde{s}_{13}, s_{23}]$ is the geodesic line segment from s_{12} to $R_1^{-1}s_{12}$, which forms with $[s_{12}, \tilde{s}_{13}]$ an angle equal to $2 \sphericalangle t_{32}s_{12}\tilde{s}_{13} = \pi/2 - \pi/p - \arg \varphi^3$ by Lemma 15.2. Thus transformation of coordinates induced by $R_1^{-1}: C_{\tilde{13}} \to C_{12}$ is

$$(\tilde{u}, \tilde{v}) \longrightarrow (\zeta\tilde{u}, \tilde{v}) ;$$

it too is holomorphic.

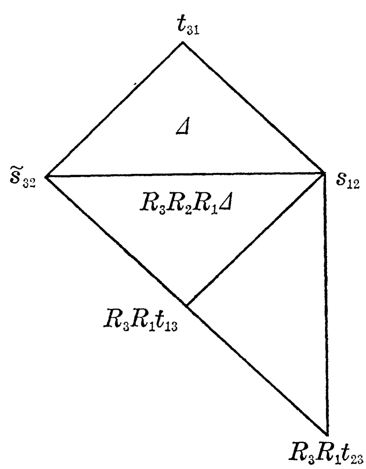
Finally, consider the map $R_3R_1: C_s \to C_2$ where $s = (R_3R_1)^{-1}s_{12} = s_{31}$ by (13.1)

$$R_3R_1[s_{31}, \tilde{s}_{32}] = [s_{12}, \tilde{s}_{13}]$$
$$R_3R_1[s_{31}, t_{23}] = [s_{12}, R_3t_{23}] .$$

The first relation implies by the argument used above that R_3R_1 transforms the $\tilde{u}$ coordinate of C_{31} to the $\tilde{u}$ coodinate of C_{12}. The implication of the second relation becomes clear upon observing that

$$R_3t_{23} = R_3(R_2R_1t_{31}) \quad \text{by Lemma 13.3(i)}$$
$$R_3R_1t_{23} = (R_3R_2R_1)t_{31}$$

and $R_3R_2R_1s_{12}, \tilde{s}_{32}] = [\tilde{s}_{32}, s_{12}]$. Consequently $R_3R_2R_1$ rotates Δ_{321} about the midpoint of $[s_{12}, \tilde{s}_{32}]$ $180°$ and the image of Δ_{213} under R_3R_1 is given (cf. the diagram)



by rotating $[s_{12}, \tilde{s}_{32}]$ towards the direction of $\operatorname{Im} u < 0$ through an angle $2 \sphericalangle t_{31}s_{12}\tilde{s}_{32}$; that is $-(\pi/2 - \pi/p + \arg \varphi^3)$ by Lemma 15.2. Thus the transformation of coordinates induced by $R_3R_1: C_{31} \to C_{12}$ is

$$(\tilde{u}, \tilde{v}) \longrightarrow (\tilde{u}, \bar{\xi}\tilde{v})$$

which is holomorphic.

Analogously, we can consider balls $B_\beta(t_{32})$ centered at t_{32} of radius β and meeting no edge of Δ_{312} other than those containing t_{32}. We can select α and β so that

$$\Delta_{312} \subset B_\alpha(s_{12}) \cup B_\alpha(\tilde{s}_{13}) \cup B_\beta(t_{32}) .$$

Define for any integer i mod 3,

$$C_i = \{(x, \tilde{w});\ x \in B_\beta(t_i),\ \tilde{w} \in C;\ \tilde{w}^m = w(x)\}$$

where $t_i = t_{i-1,i+1}$, w is the unique standard nonhomogeneous coordinates on I_i centered at t_i and positive on the geodesic lin segment $[t_i s_{ii+1}]$. Then define the homeomorphism

$$f_i\colon C_i \xrightarrow{\approx} Y_{t_i,F} \cap \pi^{-1}(B_\beta(t_i))$$

with properties analogous to (19.1). By the same type of reasoning as above, one proves:

(19.4) *For all $\gamma \in G(t_i, F)$ the map induced by γ from C_j to C_i is holomorphic, where $\gamma^{-1}t_i = t_j (i = 1, 2, 3)$.*

The coordinate system on y that we have selected above has the property:

(19.5) *For any two overlapping coordinate neighborhoods, the coordinates are biholomorphically related.*

The proof is quite simple. For by choice of coordinates, the intersection of two coordinate neighborhoods either

(i) projects by π biholomorphically onto a neighborhood in the ball; or

(ii) has one of the forms

(a) $Y_{s,F} \cap Y_{s',F} \cap \pi^{-1}(B_\alpha(s) \cap B_\alpha(\tilde{s}))$, $s, s' \in \Delta_{ijk}$

(b) $Y_{s,F} \cap Y_{t,F} \cap \pi^{-1}(B_\alpha(s) \cap B_\beta(t))$, $s, t \in \Delta_{ijk}$.

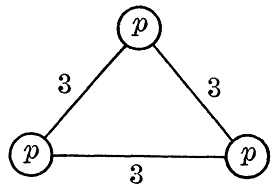
In Case (i), assertion (19.5) is obvious. In Case ii(a) the two coordinate systems are related by $R_i R_j R_k$ which belongs to $G(s, F)$. In case $s = s_{12}$, this is assured by (19.2); for any other s, the analogue of (19.2) is valid by symmetry. Thus it remains only to consider Case (iib). By symmetry, we may take $s = s_{12}$, $t = t_{32}$. In the overlap (iib), the coordinate function $\tilde{v}$ of C_{12} has no branch locus on $B_\beta(t)$. Moreover, the $\tilde{u}^m$ of C_{12} and $\tilde{w}^m$ of C_1 are related by the fractional linear transformation of the Poincaré disc I_1 which relates the two standard nonhomogeneous coordinates centered at s_{12} and t_{32} respectively. Hence

$$\tilde{u} = \sqrt[m]{\frac{a\tilde{w}^m + b}{b\tilde{w}^m + \bar{a}}}$$

where neither numerator nor denominator vanish on $B_\alpha(s_{12}) \cap B_\beta(t)$. From this assertion (19.5) follows.

From (19.2) and (19.4) we can conclude that the group Γ acts holomorphically on Y. We summarize our conclusions in the following theorem.

THEOREM 19.1. *Let φ be any complex number of modulus* 1 *with* $|\arg \varphi^3| < \pi/2 - \pi/p$ *and* $(1/\pi) \arg \varphi$ *a rational number. Let* $\Gamma(\varphi)$ *denote the group generated by C-reflections with Coxeter diagram*



$p = 3, 4, 5$

and phase shift $\varphi_{12} = \varphi_{23} = \varphi_{31} = \varphi$. *Let* $Y(\varphi)$ *denote the joined* $\Gamma(\varphi)$*-space on which* $\Gamma(\varphi)$ *operates discontinuously* (*cf.* §6.5), *and* $\pi: Y(\varphi) \to Ch^2$ *the canonical* $\Gamma(\varphi)$*-map of* $Y(\varphi)$ *to the ball. Then* $Y(\varphi)$ *has the structure of a complex analytic manifold satisfying*

(1) π *is holomorphic.*

(2) *Each* $\gamma \in \Gamma(\varphi)$ *acts biholomorphically on* $Y(\varphi)$.

REMARK 1. If $\pi/2 - \pi/p < \arg \varphi^3 < 3(\pi/2 - \pi/p)$ and $(1/\pi) \arg \varphi \in \boldsymbol{Q}$, the abutted smooth family of polyhedra $\Gamma\Omega(\varphi)$ satisfies conditions BR of § 6.4 but not hypothesis (2) of Proposition 6.4.2. For $\arg \varphi^3 > \pi/2 - \pi/p$, $\Delta_{123} \cap \Delta_{231} = \tilde{s}_{21} = s_{31}$ and $\Delta_{123} \cap \Delta_{312} = s_{23} = \tilde{s}_{13}$, the C-reflections $(R_1R_2R_3)^2$, $(R_2R_3R_1)^2$ may not be an admissible pair (cf. § 2). The subgroup of $\Gamma(\varphi)$ that fixes the point s_{31} may not be finite. Thus hypothesis (3) of Proposition 6.4.2 is generally violated.

REMARK 2. The proof of Theorem 19.1 is so much longer than the proof of Proposition 6.4.2 because it proves the added assertion: Γ operates holomorphically on Y.

REMARK 3. By a well-known theorem of Selberg (cf. [9]) any finitely generated matrix group Γ has a subgroup of finite index Γ_0 that has no elements of finite order. It follows readily that if $\gamma p = p$, $\gamma \in \Gamma_0$ and $p \in B$, then $\gamma = 1$. *A fortiori*, Γ_0 operates freely on Y. Therefore Y/Γ_0 is a complex analytic manifold and compact for $|\arg \varphi^3| < \pi/2 - \pi/p$ and $(1/\pi) \arg \varphi^3 \in \boldsymbol{Q}$. In order to prove that Y/Γ_0 is an algebraic manifold, it suffices by Kodaira's fundamental theorem to construct a Kaehler metric on Y/Γ_0 admitting a positive line bundle. 3uch a metric can be constructed with the help of the

Bergman kernel function for the domain $|u|^{2m} + |v|^{2n} < 1$ in C^2. The canonical line bundle turns out to be positive (cf. [8]). In [8], it is shown that if $m = 1$, the Kaehler metric can be selected so as to have everywhere negative sectional curvature.

20. Presentation for Γ.

Let φ be a complex number with $|\varphi| = 1$, $\varphi^N = 1$, and $|\arg \varphi^3| < \pi/2 - \pi/p$. Let $\Gamma = \Gamma(\varphi)$, $Y = Y(\varphi)$, and $\pi\colon Y \to \boldsymbol{Ch}^2$ the canonical F map. Let $Y^\natural$ denote the simply connected covering space of Y, let $\Gamma^\natural$ denote the lift of Γ to $Y^\natural$, and let $\sigma\colon Y^\natural \to Y$ denote the covering map. Thus for each $\gamma^\natural \in \Gamma^\natural$ we have the commutative diagram

$$\begin{tikzcd} Y^\natural \arrow[r, dash, "\gamma^\natural"] \arrow[d, "\sigma"'] & Y^\natural \arrow[d, "\sigma"] \\ Y \arrow[r, "\gamma"] & Y \end{tikzcd} .$$

The cell decomposition $Y = \Gamma\Omega$ lifts to a cell decomposition $Y^\natural = \Gamma^\natural\Omega^\natural$ with $\sigma\colon \Omega^\natural \to \Omega$ a homeomorphism. By the result in § 18.3, Y has a Riemannian matrix which is preserved by Γ. The pull-back of this metric to $Y^\natural$ gives a metric preserved by $\Gamma^\natural$. The hypotheses of Theorem 6.3.2 are satisfied by $(\Omega^\natural, \Gamma^\natural, Y^\natural)$.

THEOREM 20.1. *Let φ be a complex number of modulus* 1 *with* $|\arg \varphi| < \pi/2 - \pi/p$. *Let $\eta = \exp(\pi i/p)$. Set*

$$\begin{aligned} \rho &= \text{order } \bar{\eta} i \varphi^3, & \sigma &= \text{order } \bar{\eta} i \bar{\varphi}^3 \\ r &= \rho \text{ if } 3 \nmid \rho, & s &= \sigma \text{ if } 3 \nmid \sigma \\ &\quad \rho/3 \text{ if } 3 \mid \rho & &= \sigma/3 \text{ if } 3 \mid \sigma . \end{aligned}$$

Choose μ, ν so that

$$\begin{aligned} 3\mu + 1 &\equiv 0 \bmod \rho & &\text{if } 3 \nmid \rho \\ 3\nu + 1 &\equiv 0 \bmod \sigma & &\text{if } 3 \nmid \sigma . \end{aligned}$$

Let $\mathscr{F}$ denote the free group with generators $\{R_i \; i = 1, 2, 3\}$. Let $\mathscr{R}'$ denote the normal subgroup of $\mathscr{F}$ generated by the words

$$\{R_i^p,\ R_i R_j R_i R_j^{-1} R_i^{-1} R_j^{-1},\ (R_1 R_2 R_3)^{2r},\ (R_3 R_2 R_1)^{2s},\ i, j = 1, 2, 3\} .$$

Set

$$\begin{aligned} \rho_i &= (R_i R_{i+1} R_{i+2})^{2\mu} R_i R_{i+1} \text{ if } 3 \nmid \rho \\ \sigma_i &= (R_i R_{i-1} R_{i-2})^{2\nu} R_i R_{i-1} \text{ if } 3 \nmid \sigma . \end{aligned}$$

Let $\mathscr{R}''$ denote the normal subgroup of $\mathscr{F}$ generated by the words

$$\begin{array}{ll}
\{\rho_1 = \rho_2, \sigma_1 = \sigma_2\} & \text{if } 3 \nmid \rho\sigma,\ |\arg \varphi^3| < \dfrac{\pi}{2} - \dfrac{\pi}{p} \\
\{\rho_1 = \rho_2\} & \text{if } 3 \nmid \rho, 3 | \sigma,\ |\arg \varphi^3| < \dfrac{\pi}{2} - \dfrac{\pi}{p} \\
\{\sigma_1 = \sigma_2\} & \text{if } 3 | \rho, 3 \nmid \sigma,\ |\arg \varphi^3| < \dfrac{\pi}{2} - \dfrac{\pi}{p} \\
\{1\} & \text{if } 3 | \rho \text{ and } 3 | \sigma,\ |\arg \varphi| < \dfrac{\pi}{2} - \dfrac{\pi}{p} \\
\{\rho_1 = \rho_2\} & \text{if } 3 \nmid \rho,\ \dfrac{\pi}{2} - \dfrac{\pi}{p} < \arg \varphi^3 \\
\{\sigma_1 = \sigma_2\} & \text{if } 3 | \rho,\ \arg \varphi^3 < -\left(\dfrac{\pi}{2} - \dfrac{\pi}{p}\right) .
\end{array}$$

Let Γ denote the action of $\Gamma(\varphi)$ on $\mathbf{C}h^2$.
(i) *If, in addition $|\arg \varphi^3| < \pi/2 - \pi/p$, then*

$$\Gamma^{\natural} = \mathscr{F} | \mathscr{R}' \cdot \mathscr{R}'' .$$

(ii) *If $|\arg \varphi| < \pi/2 - \pi/p$ and $\Omega(\varphi)$ satisfies* (CD1) *and* (CD2) (cf. § 6.3), *then*

$$\Gamma = \mathscr{F} | \mathscr{R}' \mathscr{R}'' .$$

Proof. We first observe that the relations $\mathscr{R}'$ and $\mathscr{R}''$ are symmetric in 1, 2, 3. $\mathscr{R}'$ is symmetric since $R_i^{-1}(R_iR_jR_k)^2R_i = (R_jR_kR_i)^2$ for any i, j, k. As for $\mathscr{R}''$, given $\rho_1 = \rho_2$, we have on the one hand

$$R_2^{-1}\rho_1 = (R_1R_2R_3)^{2\rho}R_2^{-1}R_1R_2 \quad \text{(cf. Proof of Lemma 14.1 (i))} .$$

From $R_1R_2R_1 = R_2R_1R_2$, we get $R_2^{-1}R_1R_2 = R_1R_2R_1^{-1}$. Hence $R_2^{-1}\rho_1 = \rho_1R_1^{-1}$. On the other hand

$$R_2^{-1}\rho_1 = R_2^{-1}\rho_2 = (R_3R_1R_2)^2R_3 = \rho_3R_1^{-1} .$$

Hence $\rho_1R_1^{-1} = \rho_3R_1^{-1}$ and $\rho_1 = \rho_3$. Similarly $\sigma_1 = \sigma_2$ implies $\sigma_2 = \sigma_3$. The relations $\mathscr{R}' \cup \mathscr{R}''$ coincide with the relations obtained around the codimension-2 circuits of the region $\Omega(\varphi)$ by (10.1.2) and the results in § 17.1 and § 18.4. Theorem 20.1 now follows directly from Theorem 6.3.2.

REMARK 1. From the relations $\mathscr{R}'$ one can infer

$$\rho_3\rho_2\rho_1 = 1 \qquad \sigma_1\sigma_2\sigma_3 = 1 . \tag{20.2}$$

Proof.

$$\begin{aligned}\rho_3\rho_2\rho_1 &= (R_3R_1R_2)^{2\mu+1}R_2^{-1}\cdot(R_2R_3R_1)^{2\mu+1}R_1^{-1}(R_1R_2R_3)^{2\mu+1}R_3^{-1}\\ &= (R_3R_1R_2)^{2\mu+1}(R_3R_1R_2)^{2\mu+1}R_2^{-1}R_1^{-1}(R_1R_2R_3)^{2\mu+1}R_3^{-1}\\ &= (R_3R_1R_2)^{4\mu+2+2\mu} = (R_3R_1R_2)^{2(3\mu+1)} = 1\ .\end{aligned}$$

The isomorphism $J'V(\varphi)\to V(\bar\varphi)$ yields $\sigma_1\sigma_2\sigma_3 = 1$. The added relation $\rho_1 = \rho_2$ therefore implies $\rho_1^3=1$. The independence of $\mathscr{R}''$ from $\mathscr{R}'$ is assured by the existence of the 3-fold branched Γ-cover $Y^\#$ of § 6.5. Similarly $\sigma_1 = \sigma_2$ implies that $\sigma_1^3 = 1$. The group $\mathscr{F}/\mathscr{R}'$ is the group operating on the lift of Γ to the simply connected covering space of $Y^\#$ (which is not a manifold!).

REMARK 2. For those φ for which $\Omega(\varphi)$ satisfies condition (CD1) and (CD2) (cf. § 17.1) Theorem 20 gives a presentation for the image of the lattice subgroup $\Gamma(\varphi)$ in PU (H) – or equivalently, for $\Gamma(\varphi)/Z$, where Z is the set of scalar multiplies of the identity matrix in the matrix group $\Gamma(\varphi)$. In general, one may possibly have a nontrivial extension

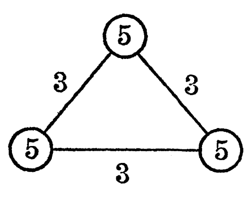
$$1\longrightarrow N\longrightarrow \Gamma\longrightarrow \Gamma\longrightarrow 1$$

where $N=\pi_1(Y)$, the fundamental group of the space $Y(\varphi)$. I do not know whether there exist any values of φ with $N(\varphi)\neq\{1\}$; equivalently, can $Y(\varphi)$ fail to be simply connected?

21. Some examples of isomorphisms among $\Gamma(\varphi)$.

Let $c_i = (R_iR_{i+1}R_{i+2})^2$, regarded as an element of PU (H) for any integer i mod 3.

LEMMA 21.1. *Let Γ denote the group $\Gamma(\varphi)$ action on $\boldsymbol{C}h^2$ with diagram*



and $\varphi^3=\sqrt{-1}$. Then

(i) *Γ is an arithmetic lattice in* PU (H).

(ii) *Γ is generated by $\{c_1, c_2, c_3\}$.*

Proof of (i). Let $k=\boldsymbol{Q}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma]$. Then $k=\boldsymbol{Q}[\cos 2\pi/5]$ by Lemma 17.2.1. Direct calculation shows that ${}^\sigma\!\Delta>0$ for $1\neq\sigma\in\mathrm{Gal}\,k$, Δ denoting the determinant of the matrix $\langle e_i, e_j\rangle$ $(i, j = 1, 2, 3)$. Hence Γ is an arithmetic lattice (cf. § 4).

Proof of (ii). By Lemma 16.1, the order of the transformation c_j equals the order of $\bar{\eta} i \varphi^3$, which is the order of $-\eta^5$. Thus c_j is of order 5 for $j = 1, 2, 3$. Inasmuch as $\arg \varphi^3 > -(\pi/2 - \pi/5)$, we have $\langle v_{123}, v_{123} \rangle > 0$ (by (9.1.4)′) and $I_{i+1} = v^{\perp}_{i,i+1,i+2} \cap V^-$ is not empty. By Lemma 13.3 (ii), c_i fixes each point of I_{i+1} and is thus a C-reflection. Moreover, by Lemma 12.3

$$I_i \cap I_{i+1} = t_{i,i+1} \qquad (i \text{ integer mod } 3)$$

and thus each of I_1, I_2, I_3 meets the other two. Thus c_1, c_2 fix the point t_{23}. Direct calculation shows that for $\varphi^3 = i$,

$$\text{(21.1)} \qquad \frac{\langle v_{123}, v_{231} \rangle}{|v_{123}|\,|v_{231}|} = -\alpha\varphi \frac{(1 - \eta - 2\bar{\eta}^2)}{1 + \eta + \bar{\eta}^2} = -\alpha\varphi\bar{\eta}^2$$

$$\text{(21.2)} \qquad (R_3R_1R_2)^{-2} t_{23} = p_{12} \,.$$

The first equation implies that $\{\{c_1, c_2\}\} \approx F_{12}$. The second equation states that the stabilizer $\Gamma_{p_{12}}$ of the point p_{12} contains $\{c_3^{-1}c_1c_3, c_3^{-1}c_2c_3\}$. Since Γ is a discrete group, $\Gamma_{p_{12}}$ is a finite group. Since ⑤$\overset{3}{\text{———}}$⑤ is a maximal subgroup of PU (2) generated by C-reflections, it follows that $\{\{c_3^{-1}c_1c_3, c_3^{-1}c_2c_3\}\} = \Gamma_{12}$. Hence $\{\{c_1, c_2, c_3\}\} \supset \{R_1, R_2\}$. By symmetry $\{\{c_1, c_2, c_3\}\} \supset \{R_1, R_2, R_3\}$. This implies (ii).

EXAMPLE 1. Equation (21.1) has an interesting interpretation. $(\varphi\bar{\eta}^2)^3 = \varphi^3\bar{\eta}^6 = -\bar{\eta} i$. Thus there is an isomorphism A of Γ to $\Gamma(\psi)$ with $\arg \psi^3 = -7\pi/10$ given by

$$\text{A:} \qquad c_i(\varphi) \longrightarrow R_i^2(\psi) \qquad (i = 1, 2, 3)\,.$$

The reason for not sending c_1 to R_1 is that c_1 rotates about its fixed point set via $\bar{\eta} i \varphi^3$ by (9.1.1), i.e., by $-\bar{\eta}$. Thus c_1^3 should map to $R_1(\psi)$. The isomorphism A is induced by the automorphism of C^3

$$v_{i,i+1,i+2} \longrightarrow e_i \qquad (i \text{ integer mod } 3)\,.$$

An alternative description is to say that A is induced by the isometry of the Ch^2 formed from $V(\varphi)$ to the Ch^2 formed from $V(\psi)$ which takes

$$t_{23} \longrightarrow p_{12},\ t_{31} \longrightarrow p_{23},\ t_{12} \longrightarrow p_{23}\,.$$

If we compose A with $J'\colon V(\psi) \to V(\psi)$, we get that $J' \circ A$ induces

$$t_{12} \longrightarrow p_{12},\ t_{23} \longrightarrow p_{23},\ t_{31} \longrightarrow p_{31}\,.$$

This example provides us with a fundamental domain for the arithmetic lattice $\Gamma(\varphi)$. For $\Omega(\varphi)$ does *not* satisfy condition (CD2) and is therefore *not* a fundamental domain $\bmod \operatorname{Aut}_\Gamma \Omega$. However, $\Omega(\psi)$ is a fundamental domain $\bmod \operatorname{Aut}_\Gamma \Omega(\psi)$ $(=\{1, J, J^2\})$.

The next example will provide a geometric isomorphism between two arithmetic lattices generated by $\boldsymbol{C}$-reflections having different Coxeter diagrams.

EXAMPLE 2. Let Γ' denote the group generated by reflections with diagram

$$\begin{array}{ccccc} \textcircled{5} & \overset{3}{\text{———}} & \textcircled{5} & \overset{3}{\text{———}} & \textcircled{5} \\ e_1' & & e_2' & & e_3' \end{array} .$$

Inasmuch as there are no closed loops in the diagram, the phase shifts can be arbitrary. Here $\{\{R_1', R_2'\}\}$ and $\{\{R_2', R_3'\}\}$ are isomorphic groups of Γ' where as $R_1'R_3' = R_3'R_1'$. It is easily verified that Γ' is discrete by the arithmeticity test of § 4. Let $\varphi = \exp \pi i/6$ as in Example 1. Then there is an isomorphism of Γ' to $\Gamma(\varphi)$ given by

$$\mathrm{A}': \qquad \begin{aligned} R_1' &\longrightarrow R_1(\varphi) \\ R_2' &\longrightarrow R_2(\varphi) \\ R_3' &\longrightarrow (R_3R_1R_2)^6(\varphi) . \end{aligned}$$

Let $p_{ij}' = e_i^\perp \cap e_j^\perp$ $(i, j = 1, 2, 3, i \neq j)$. The isomorphism A$'$ is induced by the isometry of $\boldsymbol{C}h^2$ which takes

$$p_{13}' \longrightarrow t_{32}(\varphi),\ p_{12}' \longrightarrow p_{12}(\varphi),\ p_{23}' \longrightarrow t_{31}(\varphi) .$$

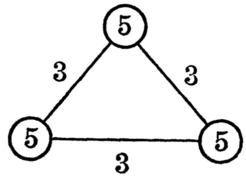
It should be noted that the image of A$'$ is all of $\Gamma(\varphi)$ since it contains $(R_3R_1R_2)^2$, $(R_2R_3R_1)^2 = R_2(R_3R_1R_2)^2R_2^{-1}$, and $(R_1R_2R_3)^2$.

22. Nonstandard homomorphisms.

Given a nonarithmetic lattice Γ in PU(2, 1), there is a field automorphism σ of $\boldsymbol{C}$ such that ${}^\sigma\Gamma$ is not a bounded set of matrices and σ is nontrivial on the field $\boldsymbol{Q}[\operatorname{Tr} \operatorname{Ad} \Gamma]$ (cf. Lemma 4.1). Such a monomorphism σ of Γ cannot be extended to a rational representation of PU(2, 1) $-$ for such a representation ρ would satisfy $\operatorname{Tr} \operatorname{Ad} \rho(\gamma) = \operatorname{Tr} \operatorname{Ad} \gamma$ for all $\gamma \in \Gamma$. This is in sharp contrast to the remarkable "super-rigidity" theorem proved by Margulis for semi-simple groups of $\boldsymbol{R}$-rank > 1.

There is however, another kind of violation of super-rigidity: There is a homomorphism ρ of $\Gamma(\varphi_1)$ onto $\Gamma(\varphi_2)$ which is not a composition of a rational homomorphism and a field automorphism; this will follow once we show that $\operatorname{Ker} \rho$ is infinite. Moreover, in our example $\Gamma(\varphi_1)$ and $\Gamma(\varphi_2)$ are arithmetic lattices.

Let $\Gamma_1 = \Gamma(\varphi_1)$ and $\Gamma_2 = \Gamma(\varphi_2)$ be the groups generated by $\boldsymbol{C}$-reflections with Coxeter diagrams



with $\arg \varphi_1^3 = -\pi/10$, $\arg \varphi_2^3 = 7\pi/10$. By Theorem 20.1, we know that the relations in the presentation for Γ_1 and Γ_2 are $\{R_i^5 = 1, R_iR_jR_i = R_jR_iR_j\ i, j = 1, 2, 3\}$ and in addition for Γ_1

$$\begin{cases}(R_1R_2R_3)^{20} = 1,\ (R_3R_2R_1)^{10} = 1, \\ (R_1R_2R_3)^6R_1R_2 = (R_2R_3R_1)^6R_2R_3 = ((R_3R_2R_1)^6R_3R_2)^{-1}\end{cases}$$

and for Γ_2

$$\begin{cases}(R_1R_2R_3)^4 = 1 \\ (R_1R_2R_3)^2R_1R_2 = (R_2R_3R_1)^2R_2R_3\ .\end{cases}$$

Moreover, in Γ_2 we have $((R_3R_2R_1)^6R_3R_2)^{-1} = (R_1R_2R_3)^6R_1R_2$, by Lemma 16.1 and $(R_3R_2R_1)^{10} = 1$ by (9.1.2). Thus the map ρ: $R_i(\varphi_1) \to R_i(\varphi_2)$ $(1 = 1, 2, 3)$ is a homomorphism. Its kernel is a normal subgroup N of $\Gamma(\varphi_1)$ containing $(R_1R_2R_3)^4$, $(R_2R_2R_1)^4$, $(R_3R_1R_2)^4$, whose common fixed point set is $I_1 \cap I_2 \cap I_3$ which is empty. Since every finite subgroup of Γ_1 fixes a point in the ball, N is not finite. Consequently ρ is not a composition of the stated type. Consulting Table 3 of § 17.3, we see that both $\Gamma(\varphi_1)$ and $\Gamma(\varphi_2)$ are arithmetic lattices. Both these lattices are cocompact.

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Received August 3, 1979. Supported in part by N. S. F. Grant #MCS76-10435.

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