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CONDITIONS FOR BEING AN FGC DOMAIN

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A domain R is said to be FGC if every finitely generated R-module decomposes into a direct sum of cyclic submodules. The main result is: if R is a domain with quotient field Q, then R is FGC if and only if all of the following three conditions are satisfied: (1) R is Bezout, (2) Q/R is an injective R-module, and (3) there does not exist a continuous embedding of βN into spec R relative to the patch topology of spec R. This result is also true if (3) is replaced: (3′) every nonzero element of R is an element of only finitely many maximal ideals of R. Using entire functions, there exists an example of a domain satisfying (1) and (2), but not satisfying (3). Also presented are some partial results towards generalizing the main result to commutative rings.

Introduction. All *rings* will be commutative with identity. R will always denote a ring. N will denote the set of all positive integers. Giving N the discrete topology, βN will denote the Stone-Cech compactification of N. Use spec R to denote the set of all prime ideals of R and m spec R to denote the set of all maximal ideals of R. For $a \in R$, use $V(a)$ for $\{P \in \operatorname{spec} R\colon a \in P\}$ and $D(a)$ for $\{P \in \operatorname{spec} R\colon a \notin P\} = \operatorname{spec} R - V(a)$. The *patch topology* of spec R is the topology which has $\{V(a)\}_{a \in R} \cup \{D(b)\}_{b \in R}$ as a subbasis of open sets. R is a *valuation ring* if the set of all the ideals of R forms a chain with respect to set inclusion (possibly R has zero-divisors). R is a *Bezout ring* if every finitely generated ideal of R is cyclic. We shall find it convenient to also use the following nonstandard notation: if $r \in R$, then $m\operatorname{spec}(r) = \{M \in m\operatorname{spec} R\colon r \in M\} = V(r) \cap m\operatorname{spec} R$. The main reference for this paper is [3], which includes characterizations of FGC rings and discussions of βN and the patch topology of spec R (although no homological algebra).

We next discuss the historical development of this subject to motivate the results. An R-module A is *linearly compact* if every family of cosets of submodules of A, that has the finite intersection property, has a nonempty intersection. A ring R is *maximal* if R is a linearly compact R-module. A ring R is *almost maximal* if R/I is a maximal ring for all nonzero proper ideals I of R. In 1952 I. Kaplansky [5] proved that if R is a valuation domain, then R is FGC if and only if R is almost maximal. In 1959 E. Matlis [6] proved that if R is a valuation domain with quotient field Q, then R is FGC if and only if Q/R is an injective R-module. One can restate these results as follows. If R is domain with only one maximal ideal, then R is FGC if and only if R is a valuation domain and R is almost maximal. If R is a domain with only one maximal ideal and Q is the quotient field of R, then R is FGC if and only if R is a valuation domain and Q/R is an injective R-module.

The I. Kaplansky result has already been generalized to domains, and in fact to commutative rings [3, Theorem 9.4 and Main Theorem 9.1]. For example, if R is a domain, then R is FGC if and only if R is a Bezout domain and R is almost maximal. One may view this paper as a generalization of the E. Matlis result. For several years we had conjectured the obvious generalization: if R is a domain with quotient field Q, then R is FGC if and only if R is a Bezout domain and Q/R is an injective R-module. After several years of unsuccessful attempts at proving this conjecture, a counterexample was discovered. We want to thank P. Eakin and W. Heinzer for the conversation which led to this example. This example quickly led to the main result.

The first section gives the proof of the main theorem. The second section gives the example which indicates why condition (3) or (3′) is necessary for the main theorems. The third section gives some partial results towards generalizing from domains to commutative rings.

1. The domain case. Of importance in this section is the following definition, first introduced by E. Matlis. A domain R is *h-local* if every nonzero prime ideal of R is a subset of only one maximal ideal of R and every nonzero element of R is an element of only finitely many maximal ideals of R. A major portion of the proofs given here really amounts to verifying the h-local conditions. See [3] and [2] for a general discussion of h-local domains. We shall also need the following of E. Matlis [7, Theorem 3.3]: if R is an h-local domain and A is an R-module, then $\operatorname{inj}\dim_R A = \sup\{\operatorname{inj}\dim_{R_M} A_M : M \in m \operatorname{spec} R\}$. To make the proof of the main results more readable, it will be broken up into a series of lemmas.

LEMMA 1.1. *If $x \in \beta N$, then there exists a continuous embedding $j: \beta N \to \beta N$ such that $x \notin j(\beta N)$.*

Proof. Let E be the set of all positive even integers. Then $\{V(E), V(N-E)\}$ is a partition of βN with both $V(E)$ and $V(N-E)$ homeomorphic to βN. Thus the homeomorphism βN onto $V(E)$ or βN onto $V(N-E)$ is the required embedding. $\square$

LEMMA 1.2. *If R is a valuation ring, then there does not exist a continuous embedding of βN into* $\operatorname{spec} R$ *relative to the patch* *topology of* spec R.

Proof. Let all topological statements for spec R be relative to the patch topology. R is a valuation ring implies spec R is a chain and the topology of spec R is the order topology, i.e., a subbasis of open sets is $\{I \in \operatorname{spec} R: I \subsetneqq P\}_{P \in \operatorname{spec} R} \cup \{I \in \operatorname{spec} R: P \subsetneqq I\}_{P \in \operatorname{spec} R}$. The order topology of a chain cannot have a subspace homeomorphic to βN.

This paragraph just verifies the last statement. Suppose $i: \beta N \to \operatorname{spec} R$ is a continuous embedding. Let $P_1 = \cap\, i(\beta N)$. Then $P_1 \in i(\beta N)$. By Lemma 1.1, there exists a continuous embedding $i_2: \beta N \to i(\beta N)$ such that $P_1 \notin i_2(\beta N)$. Let $P_2 = \cap\, i_2(\beta N)$. Then $P_2 \in i_2(\beta N)$ and $P_1 \neq P_2$. In particular $P_1 \subsetneqq P_2$ and $P_1, P_2 \in i(\beta N)$. Inductively one proceeds to get $P_1, P_2, \cdots, P_n, \cdots \in i(\beta N)$ with $P_n \subsetneqq P_{n+1}$ for all $n \in N$. Let $C = \{P_n\}_{n \in N} \cup \{\bigcup_{n \in N} P_n\}$. Then C is a closed subset of $i(\beta N)$. By [3, Theorem 7.6] every infinite closed subset of βN has a subset homeomorphic to βN. Thus there exists a continuous embedding of βN into C. But then $|N| = |C| \geqq |\beta N| = 2^c$, which is a contradiction. □

LEMMA 1.3. *Suppose R is a Bezout domain such that there exists a continuous embedding of βN into* spec R *relative to the patch topology of* spec R. *Then there exists a nonzero element of R which is an element of infinitely many maximal ideals of R.*

Proof. Let all topological statements for spec R be relative to the patch topology. Suppose $i: \beta N \to \operatorname{spec} R$ is a continuous embedding. By Lemma 1.1, we may assume $\{0\} \notin i(\beta N)$. By [3, Lemma 6.2], a basis of open sets of spec R is $\{D(a) \cap V(b): a, b \in R\}$ since R is Bezout. Thus every closed subset of spec R is of the form $\bigcap_{j \in I} (V(a_j) \cup D(b_j))$ for some $a_j, b_j \in R$. Since βN is compact, $i(\beta N)$ is a closed subset of spec R. Thus there exist $a, b \in R$ such that $i(\beta N) \subset V(a) \cup D(b)$ and $\{0\} \notin V(a) \cup D(b)$. In particular $a \neq 0$ and $b = 0$, so $D(b) = D(0) = \varnothing$. Thus $i(\beta N) \subset V(a)$. We claim that $|m \operatorname{spec}(a)| = \infty$, i.e., a is the required nonzero element of R which is an element of infinitely many maximal ideals of R. For suppose not, and so suppose $m \operatorname{spec}(a) = \{M_1, \cdots, M_n\}$. Since $i(\beta N) \subset V(a)$, we have $a \in P$ for all $P \in i(\beta N)$. Thus for some k with $1 \leqq k \leqq n$. we have $|\{P \in i(\beta N): P \subset M_k\}| = \infty$. Let $L(M_k) = \{P \in \operatorname{spec} R: P \subset M_k\}$. We claim that $L(M_k)$ is a closed subset of spect R. Because if $J \in \operatorname{spec} R - L(M_k)$, then there exists $x \in J - M_k$ and so $D(x)$ is a closed subset of spec R with $J \notin D(x)$ and $D(x) \supset L(M_k)$. This verifies that $L(M_k)$ is a closed subset of spec R. Therefore $\{P \in i(\beta N): P \subset M_k\} = L(M_k) \cap i(\beta N)$ is an infinite closed subset of $i(\beta N)$. By [3, Theorem

7.6], there exists a continuous embedding of βN into $L(M_k) \cap i(\beta N)$. But $L(M_k)$ is homeomorphic to spec R_{M_k} and R_{M_k} is a valuation domain. This contradicts Lemma 1.2. $\square$

LEMMA 1.4. *Suppose R is a Bezout domain, Q is the quotient field of R, and Q/R is an injective R-module. Then there does not exist a nonzero prime ideal of R which is a subset of two maximal ideals of R.*

Proof. Suppose the result is not true, and so suppose $P \in \operatorname{spec} R$ and $M_1, M_2 \in m \operatorname{spec} R$ with $\{0\} \neq P \subset M_1 \cap M_2$ and $M_1 \neq M_2$. Then $PR_P \subset R_{M_1} \cap R_{M_2}$.

We first claim that if A is an R-module such that $Q \supset A \supset R$ with $A_{M_1} \neq Q$ or $A_{M_2} \neq Q$, then there exists $x_0 \in R - \{0\}$ such that $x_0(A/R)_{M_i} \cong \{0\}$ for both $i = 1, 2$. To verify this, we without loss of generality suppose $A_{M_1} \neq Q$. Since R_{M_1} is a valuation domain, there exists $r \in R - \{0\}$ such that $(1/r)R_{M_1} \supset A_{M_1}$. Then $PrA \subset PrA_{M_1} \subset PR_{M_1} \subset PR_P \subset R_{M_1} \cap R_{M_2}$. In particular, there exists $x_0 \in R - \{0\}$ such that $x_0 A \subset R_{M_1} \cap R_{M_2}$, and this is the required x_0

Let us use K for Q/R. We claim that if $K = K_1 \oplus K_2$ for R-submodules K_1 and K_2 of K, then for some $i \in \{1, 2\}$ we have $(K_i)_{Mj} \cong \{0\}$ for both $j = 1, 2$. To verify this, suppose $K = K_1 \oplus K_2$. Then $Q/R_{M_1} \cong K_{M_1} \cong (K_1)_{M_1} \oplus (K_2)_{M_1}$. But R_{M_1} is a valuation domain and so Q/R_{M_1} is indecomposable as an R_{M_1}-module. Hence $(K_i)_{M_1} \cong \{0\}$ for some $i \in \{1, 2\}$. Without loss of generality we suppose that $(K_1)_{M_1} \cong \{0\}$. There exists an R-submodule A of Q such that $K_1 = A/R$. Then $A_{M_1} \neq Q$ and so by the last paragraph there exists an $x_0 \in R - \{0\}$ such that $x_0(K_1)_{M_2} \cong \{0\}$. As above $K_{M_2} \cong (K_1)_{M_2} \oplus (K_2)_{M_2}$ and K_{M_2} is indecomposable as an R_{M_2}-module. $x_0(K_1)_{M_2} \cong \{0\}$ implies $(K_2)_{M_2} \cong K_{M_2}$. Thus we must have $(K_1)_{M_2} \cong \{0\}$, verifying the claim.

Choose $m_1 \in M_1 - M_2$ and $m_2 \in M_2 - M_1$. Since R is a Bezout domain, there exist $m, d \in R$ such that $Rm = Rm_1 + Rm_2$ and $Rd = Rm_1 \cap Rm_2$. Define $x = (m/d) + R$, $y_1 = (m_1/d) + R$, and $y_2 = (m_2/d) + R$. Then x, y_1, and y_2 are nonzero elements of K, and $Rx = Ry_1 \oplus Ry_2 \subset K$. Since $Q/R = K$ is an injective R-module, there exist R-submodules K_i of K such that $Ry_i \subset K_i$ and $K = K_1 \oplus K_2$. But $R_{M_2}y_1 \ncong \{0\}$ and $R_{M_1}y_2 \ncong \{0\}$, so $(K_1)_{M_2} \ncong \{0\}$ and $(K_2)_{M_1} \ncong \{0\}$. This contradicts the last paragraph. $\square$

LEMMA 1.5. *Let R be a ring and suppose Y is a subset of $m \operatorname{spec} R$ with at least two elements. Then there exist $y, z \in R$ satisfying $V(y) \cap Y \neq \varnothing$, $V(z) \cap Y \neq \varnothing$, and $(V(y) \cap Y) \cap (V(z) \cap Y) = \varnothing$.*

Proof. Suppose this is not true. Define $P = \{r \in R: V(r) \cap Y \neq \varnothing\}$. From the assumption that the lemma is not true and $V(a + b) \supset V(a) \cap V(b)$, we infer that $a, b \in P$ implies $a + b \in P$. Also $a \in P$ and $r \in R$ implies $ar \in P$, $0 \in P$, and $1 \notin P$. Thus P is a proper ideal of R. Since Y has at least two elements, there exists $M \in Y$ such that $M \not\subset P$. Choose $x \in M - P$. Then $V(x) \cap Y \neq \varnothing$, so $x \in P$. This contradicts $x \notin P$. □

LEMMA 1.6. *Let R be a Bezout ring and suppose Y is an infinite subset of* $m \operatorname{spec} R$. *Then there exists $\{x_n\}_{n \in N} \subset R$ such that $\{V(x_n) \cap Y\}_{n \in N}$ is a family of distinct nonempty pairwise disjoint sets.*

Proof. If $x \in R$ with $0 < |V(x) \cap Y| < \infty$ and $M \in V(x) \cap Y$, then there exists $x' \in R$ such that $V(x') \cap Y = \{M\}$. Define $F = \{M \in Y$: there exists $x \in R$ with $V(x) \cap Y = \{M\}\}$. If F is an infinite set, then the desired conclusion of the lemma follows. So we will assume that F is a finite set. It follows that if $x \in R$ with $V(x) \cap (Y - F) \neq \varnothing$, then $|V(x) \cap Y| = |V(x) \cap (Y - F)| = \infty$.

Let $Y_0 = Y - F$. We recursively define x'_n, x''_n and Y_n for $n \in N$ such that $x'_n, x''_n \in R$, $V(x'_n) \cap Y_{n-1} \neq \varnothing$, $Y_n = V(x''_n) \cap Y_{n-1}$ is an infinite set, and $(V(x'_n) \cap Y_{n-1}) \cap (V(x''_n) \cap Y_{n-1}) = \varnothing$. For $n = 1$, by Lemma 1.5, there exist $x'_1, x''_1 \in R$ such that $V(x'_1) \cap Y_0 \neq \varnothing$, $V(x''_1) \cap Y_0 \neq \varnothing$, and $(V(x'_1) \cap Y_0) \cap (V(x''_1) \cap Y_0) = \varnothing$. Let $Y_1 = V(x''_1) \cap Y_0$. By the first paragraph Y_1 is an infinite set. Suppose x'_{n-1}, x''_{n-1} and Y_{n-1} have been defined as required. By Lemma 1.5, there exist $x'_n, x''_n \in R$ such that $V(x'_n) \cap Y_{n-1} \neq \varnothing$, $V(x''_n) \cap Y_{n-1} \neq \varnothing$, and $(V(x'_n) \cap Y_{n-1}) \cap (V(x''_n) \cap Y_{n-1}) = \varnothing$. Let $Y_n = V(x''_n) \cap Y_{n-1}$. Since R is a Bezout ring, there exists $g_n \in R$ such that $Rg_n = \sum_{i=1}^{n} Rx''_i$. Thus $Y_n = V(x''_n) \cap Y_{n-1} = V(x''_n) \cap \cdots \cap V(x''_1) \cap Y_0 = V(g_n) \cap Y_0$, which is an infinite set by the first paragraph. This completes the construction of the x'_n, x''_n and Y_n.

It follows that $\{V(x'_n) \cap (Y - F)\}_{n \in N}$ is a family of distinct nonempty pairwise disjoint sets. Choose $P_n \in V(x'_n) \cap (Y - F)$. Since F is a finite set, there exists $y_n \in P_n$ such that $y_n \notin M$ for all $M \in F$. Since R is a Bezout ring, there exists $x_n \in R$ such that $Rx_n = Rx'_n + Ry_n$. Then $\{x_n\}_{n \in N}$ is the required set. □

THEOREM 1.7. *Let R be a domain with quotient field Q. Then R is an* FGC *domain if and only if all of the following three conditions are satisfied:*

(1) *R is a Bezout domain,*

(2) *Q/R is an injective R-module, and*

(3) *there does not exist a continuous embedding of βN into* $\operatorname{spec} R$ *relative to the patch topology of* $\operatorname{spec} R$.

Proof. Suppose R is an FGC domain. By [3, Theorems 9.4 and 2.9] R is an almost maximal Bezout domain, and so R is h-local and locally almost maximal. For $M \in m\,\mathrm{spec}\,R$, R_M is an almost maximal valuation domain, and so Q/R_M is an injective R_M-module by [6]. By [7, Theorem 3.3] $\mathrm{inj\,dim}_R\, Q/R = \sup\{\mathrm{inj\,dim}_{R_M}(Q/R)_M \colon M \in m\,\mathrm{spec}\,R\} = \sup\{\mathrm{inj\,dim}_{R_M}\, Q/R_M \colon M \in m\,\mathrm{spec}\,R\} = 0$, and so Q/R is an injective R-module. Since R is h-local, there does not exist a nonzero element of R which is an element of infinitely many maximal ideals of R. Using Lemma 1.3, we infer that condition (3) is satisfied. This verifies that if R is an FGC domain, then the three conditions are satisfied.

Conversely, we suppose the three conditions are satisfied, and we must prove that R is an FGC domain. For the first step, we show that there does not exist a nonzero element of R which is an element of infinitely many maximal ideals of R. Suppose this is not true, i.e., suppose there exists $r_0 \in R - \{0\}$ such that $|m\,\mathrm{spec}\,(r_0)| = \infty$. By Lemma 1.6 there exists $\{x_n\}_{n \in N} \subset R$ such that $\{V(x_n) \cap m\,\mathrm{spec}\,(r_0)\}_{n \in N}$ is a family of distinct nonempty pairwise disjoint sets. R is a Bezout ring implies there exists $y_n \in R$ such that $Ry_n = Rx_n + Rx_0$, and so $V(y_n) \cap m\,\mathrm{spec}\,R = V(x_n) \cap m\,\mathrm{spec}\,(r_0)$.

Choose $M_n \in V(y_n) \cap m\,\mathrm{spec}\,R$. For $r \in R$ define $S_r = \{n \in N \colon r \in M_n\}$, and define $\mathscr{S} = \{S_r \in \mathscr{P}(N) \colon r \in R\}$, where $\mathscr{P}(N)$ is the set of all subsets of N. We claim that $\mathscr{S} = \mathscr{P}(N)$. Because suppose $B \subset N$. Define $I = \sum_{n \in N} R(r_0/y_n)$. Note that $y_n | r_0$ in R, so I is an ideal of R. Define $f \colon I \to Q/R$ by $f(\sum_{i=1}^{k} r_i(r_0/y_i)) = \sum_{i \in \{1,\cdots,k\}-B} r_i((1/y_i) + R)$, for $r_i \in R$. It can be shown that f is a well-defined R-homomorphism. By condition (2), Q/R is an injective R-module, so there exists $q \in Q - \{0\}$ such that $f(r) = r(q + R)$ for all $r \in I$. Since R is a Bezout domain, there exists $t \in R$ such that $Rt = R \cap Rr_0 q$. It follows that $B = S_t \in \mathscr{S}$, and hence $\mathscr{S} = \mathscr{P}(N)$ are claimed.

Define $i_0 \colon N \to \mathrm{spec}\,R$ by $i_0(n) = M_n$ for $n \in N$. Giving N the discrete topology, i_0 is continuous relative to the patch topology of $\mathrm{spec}\,R$. But $\mathrm{spec}\,R$ is compact Hausdorff relative to the patch topology [3, Theorem 6.4], so by the universal property of βN there exist a continuous $i \colon \beta N \to \mathrm{spec}\,R$ such that $i | N = i_0$. We claim that i is a one-to-one function. Suppose $z_1, z_2 \in \beta N$ and $z_1 \neq z_2$. There exists $A \subset N$ such that $z_1 \in V(A)$ and $z_2 \in V(N - A)$. By the last paragraph, there exists $r_1, r_2 \in R$ such that $S_{r_1} = A$ and $S_{r_2} = N - A$. Then $i(z_1) \in V(r_1)$, $i(z_2) \in V(r_2)$, $V(r_1) \cap V(r_2) \cap i_0(N) = \varnothing$, and so $i(z_1) \neq i(z_2)$. This verifies that i is a one-to-one function. This contradicts condition (3), and so we have shown that there does not exist a nonzero element of R which is an element of infinitely many maximal ideals of R. Condition (2), Lemma 1.4, and this last statement implies that R is an h-local domain.

Condition (2) means that $\operatorname{inj\,dim}_R Q/R = 0$. By [7, Theorem 3.3], R is h-local implies $0 = \operatorname{inj\,dim}_R Q/R = \sup\{\operatorname{inj\,dim}_{R_M}(Q/R)_M\colon M \in m\operatorname{spec} R\}$. Thus for all $M \in m\operatorname{spec} R$, $(Q/R)_M \cong Q/R_M$ is an injective R_M-module. By the local case [6], R_M is an FGC domain, and hence R_M is almost maximal. Thus R is h-local, locally almost maximal, and Bezout. By [3, Theorem 2.9 and Theorem 9.4] R is FGC. $\square$

THEOREM 1.8. *Let R be a domain with quotient field Q. Then R is an* FGC *domain if and only if all of the following three conditions are satisfied*:

(1) *R is a Bezout domain,*

(2) *Q/R is an injective R-module, and*

(3′) *every nonzero element of R is an element of only finitely many maximal ideals of R.*

Proof. If R is an FGC domain, then (1) and (2) follow from Theorem 1.7 and (3′) follows from [3, Theorem 9.4 and Theorem 2.9] or from the previous proof. Conversely, suppose (1), (2) and (3′) are satisfied. By Lemma 1.3, condition (3) of Theorem 1.7 is satisfied, and so by Theorem 1.7 R is FGC. $\square$

We briefly comment on some alternative forms of the last two theorems. In [1, Theorem 2.3], for R a valuation domain with quotient field Q, there are eleven equivalent conditions for R to be FGC (including R is almost maximal and Q/R is an injective R-module). One of these is the condition that every R-homomorphic image of Q is an injective R-module. Looking at the proofs given above it should be clear that both Theorems 1.7 and 1.8 are true if condition (2) is replaced by:

(2′) every R-homomorphic image of Q is an injective R-module.

Another of the equivalent conditions in [1, Theorem 2.3] is that H is a maximal ring, where H is the completion of R in the R-topology (or equivalently $H = \operatorname{Hom}_R(Q/R, Q/R)$). However the condition that H is a maximal ring cannot replace condition (2) in Theorems 1.7 or 1.8 since by [1, Theorem 4.9], if H is a maximal ring then R has only finitely many maximal ideals, and of course there do exist FGC domains with infinitely many maximal ideals (for example, the ring of integers).

2. An example. We wish to illustrate that the main Theorems 1.7 and 1.8 require conditions (3) or (3′), since there is an example of a domain satisfying conditions (1) and (2), but not (3) or (3′). The example deals with the ring of entire functions. For the necessary algebraic facts about entire functions, the reader is referred to the exercises on p. 146–148 of R. Gilmer's text [4].

EXAMPLE 2.1. There exists a Bezout domain R with quotient field Q such that Q/R is an injective R-module, yet R is not an FGC domain.

Proof. Let E be the set of all entire functions. With the standard operations E is a Bezout domain. Let Q be the quotient field of E, and so Q is the field of meromorphic functions. Let $S = \{f \in E: f(n) \neq 0 \text{ for all } n \in N\}$. Then S is a multiplicatively closed subset of E. Define $R = E_S$. Then R is a Bezout domain with quotient field Q and R conists of all the meromorphic functions that have no poles on the set N. We claim that this is the required R.

For $n \in N$ define $P_n = \{f \in E: f(n) = 0\}$. Then P_n is a maximal ideal of E. For $n \in N$ define $M_n = RP_n$. Then M_n is a maximal ideal of R for all $n \in N$. In order to show that Q/R is an injective R-module, we suppose I is a nonzero ideal of R and $g: I \to Q/R$ is a R-homomorphism. We must show that there exists $q \in Q$ such that $g(f) = f(q + R)$ for all $f \in I$.

For each $n \in N$ choose $f_n \in I$ such that $R_{M_n} f_n = R_{M_n} I$. This is possible since R_{M_n} is a discrete rank one valuation domain. Choose $q_n \in Q$ such that $g(f_n) = q_n + R$. By possibly replacing q_n by $1 + q_n$, we may assume that $R_{M_n} q_n \supset R_{M_n} f_n$. Suppose the Laurent series expansion of q_n/f_n about $z = n$ is $\sum_{i=r_n}^{\infty} a_{n,i}(z - n)^i$ for some nonpositive integer r_n and $a_{n,i}$ are complex numbers. By the Mittag-Leffler theorem, there exists $q \in Q - \{0\}$ such that for all $n \in N$, the Laurent series expansion of q at $z = n$ has as its principal part $\sum_{i=r_n}^{-1} a_{n,i}(z-n)^i$, and q has no other poles. We claim that this is the required q, i.e., $g(f) = f(q + R)$ for all $f \in I$. Because if $f \in I$ and $g(f) = q' + R$ then the principal part of the Laurent series expansion about $z = n$ of fq is the same as for q', and so $fq - q' \in R$. This completes the proof that Q/R is an injective R-module.

Let $f \in R$ be defined by $f(z) = \sin(\pi z)$ for any complex number z. Then $f \in M_n$ for all $n \in N$ and so there exists a nonzero element of R which is an element of infinitely many maximal ideals of R. By definition, R is not h-local. Thus by [3, Theorem 9.4 and Theorem 2.9] R is not FGC. □

The example given above satisfies conditions (1) and (2) of Theorem 1.8, but not condition (3′). It follows from Theorems 1.7 and 1.8 that for this R there exists a continuous embedding of βN into $\operatorname{spec} R$ relative to the patch topology. The reader may check that for this example, βN is homeomorphic to $m \operatorname{spec} R$, relative to either the patch topology or the Zariski topology of spec R.

For those readers familiar with the group of divisibility of a domain, we suspect that the group of divisibility of R in the above example is order isomorphic to Z^N, where Z is the set of integers and Z^N has the product ordering.

COROLLARY 2.2. *Suppose R is a Bezout domain with quotient field Q such that Q/R is an injective R-module. Then examples show that R may be h-local or R may not be h-local. If R is h-local, then R is an* FGC *domain. If R is not h-local, then there exists a nonzero element of R which is an element of infinitely many maximal ideals of R, and there exists a continuous embedding of βN into* spec R *relative to the patch topology of* spec R.

Proof. The ring of integers is an h-local Bezout domain R with quotient field Q such that Q/R is an injective R-module. The Example 2.1 is a non-h-local Bezout domain R with quotient field Q such that Q/R is an injective R-module. If R is h-local then by Theorem 1.8 R is FGC. If R is not h-local, then R is not FGC by [3, Theorem 9.4 and Theorem 2.9], and so by Theorems 1.7 and 1.8 the last statement of the corollary is true. □

3. Partial results for commutative rings. We wish to comment on attempts to generalize Theorems 1.7 and 1.8 to commutative rings, with most of the results being of a negative nature. By [3, Main Theorem 9.1], if R is a direct sum of the ring of integers with itself, then R is an FGC ring, and clearly there exists a nonzero element of R which is an element of infinitely many maximal ideals of R. Thus condition (3′) of Theorem 1.8 needs to be changed in order to hope to generalize this theorem. Also one needs something to replace "quotient field" in condition (2) of Theorems 1.7 and 1.8.

Recall that by our convention rings are commutative with identity. If R is a ring, then, a *regular element* of R is a nonzero nonzero-divisor of R. The *total ring of quotients* of R is R_S where S is the set of all regular elements of R. Just as for domains, if R is a ring and Q is its total ring of quotients, then there is a standard embedding of R into Q and via this embedding one considers R as an R-submodule of Q. Hence one can consider the R-module Q/R.

We first consider the obvious two choices for conditions (2) and (3′), namely use "total ring of quotients of R" for Q and use "regular element" for nonzero element of R.

EXAMPLE 3.1. There exists a ring R with total ring of quotients

Q such that

(1) R is a Bezout ring,

(2) Q/R is an injective R-module, and

(3′) every regular element of R is an element of only finitely many maximal ideals of R,

and yet R is not an FGC ring.

Proof. Let R be a countably infinite product of fields. Then every regular element of R is a unit of R, so $Q/R \cong \{0\}$. Then R satisfies conditions (1), (2), and (3′). Since R has infinitely many minimal prime ideals, R is not FGC by [3, Theorem 8.5]. □

A *Boolean ring* is a ring R with $x^2 = x$ for all $x \in R$. We wish to thank R. Weigand for pointing out that a countably infinite Boolean ring has the properties described in the next example.

EXAMPLE 3.2. There exists a ring R with total ring of quotients Q such that

(1) R is a Bezout ring,

(2) Q/R is an injective R-module, and

(3) there does not exist a continuous embedding of βN into $\operatorname{spec} R$ relative to the patch topology of $\operatorname{spec} R$,

and yet R is not an FGC ring.

Proof. Let $R = \{A \subset N: |A| < \infty \text{ or } |N - A| < \infty\}$. Make R into a Boolen ring by the usual definition of addition and multiplication: $A + B = (A \cup B) - (A \cap B)$ and $AB = A \cap B$ for $A, B \in R$. The only regular element of R is the identity, N, so $Q/R \cong \{0\}$. Since R is a countable set and $|\beta N| = 2^c$, it follows that R satisfies conditions (1), (2), and (3). For $n \in N$ define $P_n = \{A \in R: n \notin A\}$. Define $P_\infty = \{A \in R: |A| < \infty\}$. Then $\operatorname{spec} R = \{P_n\}_{n \in N \cup \{\infty\}}$, and all prime ideals of R are minimal prime ideals of R. By [3, Theorem 8.5], an FGC ring has only finitely many minimal ideals, so R is not FGC. □

The above two examples show that either of the set of three conditions is not sufficient to imply that R is an FGC ring. The following shows that these conditions are not necessary.

EXAMPLE 3.3. There exists a ring R with total ring of quotients Q such that R is an FGC ring and Q/R is not an injective R-module.

Proof. Let Z be the additive group of integers, and let Z^2 be ordered lexicographically. Let R_0 be the long power series ring relative to the field of complex numbers and the group Z^2 (see [3,

§11]). Then R_0 is a maximal valuation domain [3, Proposition 11.5] and so R_0 is an FGC domain [3, Main Theorem 9.1]. Let P_0 be the nonzero nonmaximal prime ideal of R_0. Define $R = R_0/P_0^2$. Then R is maximal valuation ring, and so R is an FGC ring. If $P = P_0/P_0^2$, then P is the minimal prime ideal of R. Letting Q be the total ring of quotients of R, there exists a surjective R-homomorphism $f: P \to Q/R$. Since $P(Q/R) \cong \{0\}$, it follows that there does not exist an element $x \in Q/R$ such that $f(p) = px$ for all $p \in P$. Thus by Baer's criterion, Q/R is not an injective R-module. □

The above three examples suggest that the "total ring of quotients" is not the appropriate choice for replacing "quotient field" if one wants to generalize the earlier theorems. Another choice is to use the "injective envelope of R" for the "quotient field."

EXAMPLE 3.4. There exists a ring R with E an injective envelope of R such that

(1) R is a Bezout ring,

(2) E/R is an injective R-module, and

(3) there does not exist a continuous embedding of βN into spec R relative to the patch topology of spec R,

and yet R is not an FGC ring.

Proof. Use the same R as in the proof of Example 3.2. If I is an ideal of R and there exists $A \in I$ with $|A| = \infty$, then I is a cyclic ideal of R. If I is an ideal of R and $A \in I$ implies $|A| < \infty$, then there exists a subset S_I of N such that $I = \{A \subset S_I : |A| < \infty\}$.

Let E be the set of all subsets of N. One makes E an R-module with the standard operations (same as used for R). One checks that E is an essential extension of R. That E is an injective R-module is checked by using Baer's criterion and considering the two different types of ideals of R described above. Thus E is an injective envelope of R. Again one checks that E/R is an injective R-module by using Baer's criterion and considering the two different types of ideals of R. Thus R has the required properties. □

Of course the R in the last proof also satisfies the condition: (3′) every regular element of R is an element of only finitely many maximal ideals of R. Thus if one uses the "injective envelope of R" to replace the "quotient field of R", then either of the set of three conditions is not sufficient to imply that R is an FGC ring. The following example shows that these conditions are not necessary.

EXAMPLE 3.5. There exists a ring R with E an injective envelope of R is an FGC ring and E/R is not an injective R-module.

Proof. Use the same R as in the proof of Example 3.3, with Q being the total ring of quotients of R. One easily checks that Q is an essential extension of R, and using Baer's criterion, one can check that Q is an injective R-module. Thus Q is an injective of R. Taking $E = Q$, one gets the desired conclusion. □

As a summary, if one wants to generalize Theorems 1.7 or 1.8 to commutative rings, then neither "the total ring of quotients of R" cr "an injective envelope of R" is an appropriate choice for "the quotient field of R."

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