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ON MEASURABLE PROJECTIONS IN BANACH SPACES

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Let E be a Banach space that is complemented in its bidual by a projection $P: E^{**} \to E$. It is shown that E has the Radon Nikodym property if and only if for every Radon probability measure λ on the unit ball K of E^{**} such that $\omega^* - \int_A x^{**} d\lambda \in E$ for every weak* Borel subset A of K, the projection P is λ-Lusin measurable and for every x^* in E^* the map x^*P satisfies the barycentric formula for λ on K.

J. J. Uhl Jr. asked the following question: Let E be a Banach space which is complemented in its bidual by a projection $P: E^{**} \to E$ which is weak* to norm universally Lusin measurable. Does E have the Radom-Nikodym property?

In [4] we showed that if E is the dual of a Banach space Y and if P is the natural projection from $E^{**} = Y^{***}$ to $Y^* = E$ then the above condition is necessary and sufficient for E to have the Radon-Nikodym property.

In [4] we also showed that for any Banach space E, if P is weak* to weak Baire-1 function then E has the Radon-Nikodym property.

Recently G. Edgar showed using an idea of Talagrand and Weizsäcker that the projection

$$L_1[0, 1]^{**} \longrightarrow L_1[0, 1]$$

is weak* to weak universally-Lusin measurable. This shows that Uhl's question does not have a positive answer in general, however if one examines the results of [4] he can see that if P is Baire-1, it is universally Lusin-measurable and for every x^* in E^* the map x^*P satisfies the barycentric formula. It turns out that a Banach space E has the Radon-Nikodym property if and only if for every Radon probability measure λ on the unit ball K of E^{**} such that $\omega^* - \int_A x^{**} d\lambda \in E$ for every ω^*-Borel subset A of K the projection P is λ-Lusin measurable and for every x^* in E^* the map x^*P satisfies the barycentric formula for λ on K.

Let us fix some terminology and conventions. All topological spaces in this paper will be completely regular. The set of all Radon probability measures on a topological space (X, τ) will be denoted by $M_+^1(X, \tau)$.

DEFINITION 1. Let (X, τ_1) and (Y, τ_2) be two topological spaces and let

$$f\colon X \longrightarrow Y \quad \text{and} \quad \mu \in M_1(X, \tau_1)$$

the map f is said μ-Lusin measurable if for every compact set K in X and for every $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset K$ such that $\mu(K \backslash K_\varepsilon) < \varepsilon$ and the restriction $f|K_\varepsilon$ of f to K_ε is continuous.

If f is μ-Lusin measurable, the image of μ denoted by $f(\mu)$ and defined by $f(\mu)(A) = \mu(f^{-1}(A))$ for every Borel subset A of (Y, τ_2) belongs to $M_+^1(Y, \tau_2)$.

DEFINITION 2. Let E be a Banach space and let (T, Σ, λ) be a probability space. A function $f\colon T \to E$, is Bochner integrable if there exists a sequence (f_n) of simple functions such that

(i) $\lim_n \|f(t) - f_n(t)\| = 0$ for λ-almost all $t \in T$ and

(ii) $\lim_n \int_T \|f(t) - f_n(t)\| d\lambda = 0$.

If f is Bochner integrable we denote by

$$\text{Bochner} - \int_A f d\lambda = \lim_n \int_A f_n d\lambda$$

for every A in Σ.

DEFINITION 3. A Banach space E is said to have the Radon-Nikodym property if for every probability space (T, Σ, λ) and every vector measure $m\colon \Sigma \to E$ such that $\|m(A)\| \leqq \lambda(A)$ for every A in Σ, there exists $f\colon T \to E$ Bochner integrable such that

$$m(A) = \text{Bochner} - \int_A f d\lambda \quad \text{for}$$

every A in Σ.

For more about the Radon-Nikodym property see [1].

If (X, τ) is a topological space, Σ the Borel subset of (X, τ) and $\lambda \in M_+^1(X, \tau)$ and $f\colon X \to (E, \|\ \|)$ which is λ-Lusin measurable and bounded then f is Bochner integrable.

If C is a w^*-compact convex subset of the dual E^* of a Banach space E and $f\colon (X, \tau) \to (C, \sigma(E^*, E))$ then f is said to be w^*-integrable with respect to $\lambda \in M_+^1(X, \tau)$ if

(i) For every $x \in E$ the map $t \to x(f(t))$ is λ-integrable.

(ii) For every $A \in \Sigma$ there exists $x_A^* \in C$ such that $x(x_A^*) = \int_A x(f(t)) d\lambda$ for every $x \in E$. The element x_A^* will be denoted by

$$x_A^* = \omega^* - \int_A f d\lambda\ .$$

Let $\mu \in M_+^1(C, \sigma(E^*, E))$ it is easy to see that the identity map $I: (C, \sigma(E^*, E)) \to (C, \sigma(E^*, E))$ is μ weak*-integrable. An affine function $h: (C, \sigma(E^*, E)) \to \boldsymbol{R}$ which is μ-Lusin measurable is said to satisfy the barycentric formula for μ on C if for every w^*-Borel subset A of C

$$h\Big(w^* - \int_A Id\mu\Big) = \int_A h \cdot Id\mu \,.$$

If $\lambda \in M_1(X, \tau)$ we denote by $\operatorname{supp} \lambda$ the support of λ.

LEMMA 4. *Let (X, τ) be a topological space and $\lambda \in M_+^1(X, \tau)$. Let C be a w^*-compact convex subset of the dual E^* of a Banach space E and f and ϕ*

$$f, \phi: (X, \tau) \longrightarrow (C, \sigma(E^*, E))$$

two λ-Lusin measurable maps such that for every Borel subset A in (X, τ),

$$\omega^* - \int_A f d\lambda = \omega^* - \int_A \phi d\lambda \,.$$

Then $f = \phi$ λ-almost everywhere.

Proof. Let K be a compact set in (X, τ) such that $\phi | K$ and $f | K$ are continuous from $(K, \tau) \to (C, \sigma(E^*, E))$ then we claim that $f = \phi$ λ-almost everywhere on K. Let $\mu = \lambda | K$, it is enough to show that

$$\phi | \operatorname{supp} \mu = f | \operatorname{supp} \mu$$

if not there exists $t_0 \in \operatorname{supp} \mu$ such that $\phi(t_0) \neq f(t_0)$. Let $x \in E$ such that $x(\phi(t_0) - f(t_0)) = 1$, the scalar map $t \to \psi(t) = x(\phi(t) - f(t))$ is continuous on K, therefore there exists a neighborhood V of t_0 in K such that

$$t \in V \Longrightarrow \psi(t) \geqq \frac{1}{2} \,.$$

Observe that $t_0 \in \operatorname{supp} \mu \Rightarrow \mu(V) > 0$ and hence

$$\int_V \psi(t) d\lambda = \int_V \psi(t) d\mu \geqq \frac{1}{2} \mu(V) > 0$$

on the other hand we have $\omega^* - \int_V f d\lambda = \omega^* - \int_V \phi d\lambda$ which in turn implies that $\int_V x(f(t)) d\lambda = \int_V x\phi(t) d\lambda$ there fore $\int_V \psi(t) d\lambda = 0$ a contradiction that finishes the proof of the claim. To finish the proof choose for every $n \geqq 1$ a compact K_n such that

(i) $f|K_n$ and $\phi|K_n$ are both continuous on K_n.

(ii) $\lambda(X\backslash K_n) \leqq 1/n$.

(iii) $K_n = H_n \cup N_n$ where $f|H_n = \phi|H_n$ and $\lambda(N_n) = 0$

Let $K = \bigcup_{n=1}^{\infty} H_n$, $M = X\backslash\bigcup_{n=1}^{\infty} K_n$ and $M = \bigcup_{n=1}^{\infty} N_n$ then $X = K \cup M \cup N$ where $\lambda(M \cup N) = 0$ and $f = \phi$ on K.

From now on, E will be a Banach space complemented in its second dual E^{**} by a projection $P: E^{**} \to E$ and K will denote the closed unit Ball of E^{**}.

THEOREM 5. *The Banach space E has the Radon-Nikodym property if and only if for every $\lambda \in M_+^1(K, \sigma(E^{**}, E^*))$ such that $\omega^* - \int_A x^{**} d\lambda \in E$ for every w^*-Borel subset A of K, the projection P is weak* to norm λ-Lusin measurable and for every x^* in E^* the map x^*P satisfies the barycentric formula for λ on K.*

Proof. Let $\lambda \in M_+^1(K, \sigma(E^{**}, E^*))$ such that

$$m(A) = \omega^* - \int_A x^{**} d\lambda \qquad \text{belongs}$$

to E for every ω^*-Borel subset A of K. It is easy to see that

$$\|m(A)\| \leqq \lambda(A) \qquad \text{for every}$$

ω^*-Borel subset A of K and therefore m is a σ-additive E-valued vector measure. If E has the Radon-Nikodym property one can find

$$f: K \longrightarrow (E, \|\quad\|)$$

λ-Bochner integrable such that for every w^*-Borel subset A of K we have

$$m(A) = \text{Bochner} - \int_A f d\lambda = \omega^* - \int_A x^{**} d\lambda .$$

Apply Lemma 4 to conclude that $f(x^{**}) = x^{**}$ λ-almost everywhere and use the fact that f is λ-Lusin measurable from $K \to (E, \|\quad\|)$ to write $K = \bigcup_{n=1}^{\infty} K_n \cup N$ where (K_n) is a sequence of disjoint norm compact subset of E and $\lambda(N) = 0$. This shows that the identity

$$I: (K, \sigma(E^{**}, E^*)) \longrightarrow (K, \|\quad\|)$$

is λ-Lusin measurable and therefore P is λ-Lusin measurable. Let x^* in E^*, we have to show that

$$x^*P\Big(\omega^* - \int_A x^{**} d\lambda\Big) = \int_A x^*P(x^{**}) d\lambda .$$

To this end observe that

$$\begin{aligned} x^*P\Big(\omega^* - \int_A x^{**}d\lambda\Big) &= x^*\Big(\omega^* - \int_A x^{**}d\lambda\Big) = x^*(m(A)) \\ &= x^*\Big(\sum_{n=1}^{\infty} m(K_n \cap A)\Big) = \sum_{n=1}^{\infty} x^*(m(K_n \cap A)) \\ &= \sum_{n=1}^{\infty} \int_{K_n \cap A} x^*(x^{**})d\lambda = \sum_{n=1}^{\infty} \int_{K_n \cap A} x^*P(x^{**})d\lambda \\ &= \int_A x^*P(x^{**})d\lambda\ . \end{aligned}$$

Conversely, let λ be in $M_+^1(K, \sigma(E^{**}, E^*))$ such that for every weak* Borel subset A of K we have

$$m(A) = \omega^* - \int_A x^{**}d\lambda \in E\ .$$

Let $x^* \in E^*$, then

$$x^*(m(A)) = x^*P(m(A)) = \int_A x^*P(x^{**})d\lambda = \int_A x^*(x^{**})d\lambda\ .$$

Therefore $\omega^* - \int_A Id\lambda = \omega^* - \int_A Pd\lambda$ where I is the identity map on K. Now apply Lemma 4 to deduce that K can be written

$$K = \bigcup_{n=1}^{\infty} K_n \cup N$$

where each K_n is w^*-compact on which $I = P$ and $\lambda(N) = 0$. This implies that for every $n \geqq 1$, K_n is norm compact and is contained in E and hence $I: (K, \sigma(E^{**}, E^*)) \to (K, \|\ \|)$ is λ-Lusin measurable. To prove now that E has the Radon-Nikodym property, let Σ be σ-algebra of all Lebesgue measurable subsets of $[0, 1]$ and let μ be the Lebesgues measure on $[0, 1]$. Consider a vector measure m: $\Sigma \to E$ such that $\|m(A)\| \leqq \mu(A)$ for every $A \in \Sigma$. By [5], there exists a map f: $[0, 1] \to K$ such that

(i) For every ω^*-Borel subset B of K, $f^{-1}(B)$ belongs to Σ.

(ii) The image measure $f(\mu)$ belongs to $M_+^1(K, \sigma(E^{**}, E^*))$.

(iii) For every $A \in \Sigma$

$$m(A) = \omega^* - \int_A fd\mu\ .$$

It follows easily that for any w^*-Borel subset B of K

$$\omega^* - \int_A x^{**}df(\mu) \in E\ .$$

Therefore $I: (K, \sigma(E^{**}, E^*)) \to (K, \|\ \|)$ is $f(\mu)$-Lusin measurable by what we did above. Consequently K can be written $K = \bigcup_{n=1}^{\infty} K_n \cup N$ where $f(\mu)(N) = \mu(f^{-1}(N)) = 0$ and K_n is norm compact subset of

E^{**}. It follows that If: $[0, 1] \to (K, \| \quad \|)$ is μ-almost separably valued. Also note that if 0 is an open set in $(K, \| \quad \|)$ then $f^{-1}(0) \in \Sigma$. This shows that the map

$$f = \text{If}: [0, 1] \to (K, \| \quad \|)$$

is μ-Lusin measurable and therefore Bochner integrable and hence

$$m(A) = \omega^* - \int_A f d\mu = \text{Bochner} - \int_A f d\mu$$

for every $A \in \Sigma$. This shows that f takes its values μ-almost everywhere in E, therefore E has the Radon-Nikodym property.

The proof of the above theorem implies the following corollary.

COROLLARY 6. *For any Banach space E the following two conditions are equivalent*:

(i) *The space E the Radon-Nikodym property.*

(ii) *For every $\lambda \in M_+^1(K, \sigma(E^{**}, E^*))$ such that $\omega^* - \int_A x^{**} d\lambda \in E$ for every w^*-Borel subset A of K, the identity*

$$(K, \sigma(E^{**}, E^*)) \to (K, \| \quad \|)$$

is λ-Lusin measurable.

If E is completed in E^{**} by a projection $P: E^{**} \to E$ then (i) *and* (ii) *are equivalent to*

(iii) *For every $\lambda \in M_+^1(K, \sigma(E^{**}, E^*))$ such that $\omega^* - \int_A x^{**} d\lambda \in E$ for every ω^*-Borel subset A of K, the projection P is λ-Lusin measurable and for every $x^* \in E^*$, the map x^*P satisfies the barycentric formula for λ on K.*

COROLLARY 7 [4]. *If E is complemented in E^{**} by a weak* to weak Baire-*1 *projection P, then E has the Radon-Nikodym property.*

Proof. If P is Baire-1, it is λ-Lusin-measurable for any $\lambda \in M_+^1(K, \sigma(E^{**}, E^*))$ and for every $x^* \in E^*$, the map x^*P is Baire-1 and therefore satisfies the barycentric formula for λ on K.

In [4] it was shown that if $P: (E^{**}, \sigma(E^{**}, E^*)) \to (E, \sigma(E^*, E))$ is Baire-1, then E is a weakly compactly generated Banach space. Using this fact we can now give the following:

Example of a Banach space having the Radon-Nikodym property and complemented in its bidual by a nonweak to weak Baire-*1 *projection.*

Let R be the Banach space constructed by Rosenthal in [2], this space bas the following properties:

(1) It is a dual space, therefore it is complemented in R^{**}.

(2) It is a closed subspace of a weakly compactly generated Banach space, therefore it has the Radon-Nikodym property [3].

(3) It is not weakly compactly generated so $P: R^{**} \to R$ is not Baire-1.

For more examples related to this paper see [4].

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