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TENSOR PRODUCTS FOR $SL_2(\mathscr{K})$, I COMPLEMENTARY SERIES AND THE SPECIAL REPRESENTATION

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We obtain the decomposition of the tensor product of two irreducible unitary representations of SL_2 of a local field in the case when at least one of them is the special representation or in the complementary series. This is done by considering the analytic continuation of the unitary principal series.

1. Introduction and notation. Martin ([3]) has studied the tensor product of a principal series representation of $SL_2(\mathscr{K})$ with any irreducible unitary representation. His work makes extensive use of the Mackey machinery. In particular, an application of Mackey's Tensor Product Theorem ([2], pp. 128–133) shows that the tensor product of two principal series representations is unitarily isomorphic to a representation induced from the diagonal subgroup. This idea was originated by Williams ([7]) and used by Martin. It is also worked out in detail for $SL_2(\boldsymbol{R})$ in [4].

In this paper we shall view the complementary series and the special representation as analytic continuations of the class one principal series, and study tensor products with them in this light. We obtain the decomposition of the tensor product of any unitary irreducible representation with a complementary series or special representation.

Let $\mathscr{K}$ be a local field of odd residual characteristic, with ring of integers $\mathscr{O}$ and prime ideal $\mathscr{P} = (\pi)$; let q be the order of $\mathscr{O}/\mathscr{P}$. Let Φ be an additive character of $\mathscr{K}$ which is trivial on $\mathscr{O}$ but not on $\mathscr{P}^{-1}$.

Let $G = SL_2(\mathscr{K})$, let A be the diagonal subgroup, let N (resp. V) be the subgroup of upper (resp. lower) triangular unipotent matrices, and let $K = SL_2(\mathscr{O})$, a maximal compact subgroup of G.

For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$, $x \in \mathscr{K}$, let $x \cdot g = (ax + c)/(bx + d)$. If σ is a quasicharacter of $\mathscr{K}^\times$, define a representation T_σ of G on $L^2(\mathscr{K})$ by $T_\sigma(g)f(x) = \sigma(bx + d)|bx + d|^{-1}f(x \cdot g)$. This is called a (non-unitary) principal series representation. In particular, if $\sigma = |\cdot|^{-s}$, we write $T_\sigma = T_s$. The representations T_s with $s \in \boldsymbol{R}$ are unitary, the "class one principal series".

Following Sally ([5]) we give another realization of T_σ. Let C_π be the kernel of the norm map of the quadratic extension $\mathscr{K}(\sqrt{\pi})$ over $\mathscr{K}$. We take σ, a quasicharacter of $\mathscr{K}^\times$, and extend it to $\mathscr{K}(\sqrt{\pi})^\times$. We define a map $E_\sigma: L^2(C_\pi) \to L^2(\mathscr{K})$ by

$$E_\sigma f(x) = 1/\sqrt{2}\,|1 - \sqrt{\pi}\,x|^{-1}\sigma(1 - \sqrt{\pi}\,x)f((1 + \sqrt{\pi}\,x)/(1 - \sqrt{\pi}\,x))$$
$$E_\sigma^{-1} f(t) = \sqrt{2}\,|(1 + t)/2|^{-1}\sigma((1 + t)/2)f((t - 1)/(\sqrt{\pi}(t + 1)))$$

(cf. [5], 2.20, 2.21).

If σ is unitary, then E_σ is a unitary isomorphism, and in any case E_σ gives an isomorphism between the realization of T_σ on $L^2(\mathscr{K})$ and another realization on $L^2(C_\pi)$ (see [5], 2.22).

2. Tensor products of principal series representations. Let σ, σ' be quasicharacters of $\mathscr{K}^\times$; following Sally ([5]), we write $\sigma = \sigma^*|\cdot|^{-s}$, $\sigma' = \sigma'^*|\cdot|^{-s'}$, with σ^*, σ'^* characters of $\mathscr{O}^\times$. We assume that $s, s' \in i\boldsymbol{R} \cup [-1, 0)$ and that if s (resp. s') $\in [-1, 0)$ then σ^* (resp. σ'^*) is trivial. Though we will not say so from now on, we will interpret $s \in i\boldsymbol{R}$ to mean $-\pi/\ln q \leqq \operatorname{Im} s \leqq \pi/\ln q$.

As remarked above, if $s, s' \in i\boldsymbol{R}$, then $T_\sigma \otimes T_{\sigma'} \approx \operatorname{Ind}_A^G(\sigma(\sigma')^{-1})$. This is proved in [3], where it is also shown that this last representation depends only on the value of $\sigma(\sigma')^{-1}$ at $-\mathrm{Id}$. We write the equivalence explicitly.

We define:

$$\begin{aligned} L = L_\sigma &: L^2(\mathscr{K} \times \mathscr{K}) \longrightarrow L^2(\mathscr{K} \times \mathscr{K}) \\ T = T_\sigma &: L^2(\mathscr{K} \times \mathscr{K}) \longrightarrow L^2(\mathscr{K} \times \mathscr{K}) \\ \hat{} &: L^2(\mathscr{K} \times \mathscr{K}) \longrightarrow L^2(\mathscr{K} \times \mathscr{K}) \\ S = S_{s,s'} &: L^2(\mathscr{K} \times \mathscr{K}) \longrightarrow L^2(\mathscr{K} \times \mathscr{K}) \end{aligned}$$

as follows:

$$\begin{aligned} Lf(x, y) &= |x|^{-1}\sigma(x)f(y + 1/x, y) \\ Tf(x, y) &= |x - y|^{-1}\sigma(x - y)f(1/(x - y), y) \\ \hat{}f(x, y) &= \int_{\mathscr{K}} \Phi(xz)f(z, y)dz \\ Sf(x, y) &= |x|^{(s-s')/2}f(x, y) \end{aligned} \tag{2.1}$$

(we understand $\sigma(0) = 0$; $\hat{}$ is the partial Fourier transform).

All these maps are unitary isomorphisms. Note $L = T^{-1}$.

For any quasicharacter η of A, we realize $\operatorname{Ind}_A^G \eta$ on $L^2(\mathscr{K} \times \mathscr{K}) \approx L^2(N \times V) \approx L^2(A\backslash G)$. For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$, the action is given as follows:

$$g \cdot f(x, y) = \eta(by + d)^{-1}f((by + d)((by + d)x + b), yg)\ . \tag{2.2}$$

The transform of this action by $(\hat{})^{-1}$ is given as follows:

$$\begin{aligned} &(\hat{})g(\hat{})^{-1}f(x, y) \\ &\quad = \eta(by + d)^{-1}|by + d|^{-2}\Phi(-bx/(by + d))f(x/by + d)^2, yg)\ . \end{aligned} \tag{2.3}$$

Now $T_\sigma \otimes T_{\sigma'}$ acts on $L^2(C_\pi) \otimes L^2(C_\pi) \approx L^2(C_\pi \times C_\pi)$ or on

$L^2(\mathscr{K}\times\mathscr{K})$, and $E = E_\sigma \otimes E_{\sigma'}$ gives the isomorphism between these two spaces. The map $S\circ\hat{}\circ L\circ E\colon L^2(C_\pi \times C_\pi) \to L^2(\mathscr{K} \times \mathscr{K})$ gives a unitary isomorphism from $T_\sigma \otimes T_{\sigma'}$ to $\mathrm{Ind}_A^G(\sigma^*\sigma'^*)^{-1})$, with the action (2.2), where $\eta = \sigma^*(\sigma'^*)^{-1}$.

3. Extension to complementary series. Now suppose we allow one or both of s, s' to be in $[-1, 0)$. The maps defined by the formulae (2.1) need no longer be unitary. And the action of G on $L^2(C_\pi \times C_\pi)$ or $L^2(\mathscr{K} \times \mathscr{K})$ is not unitary. We recall that G *does* act unitarily with respect to a different inner product (see [5], pp. 429–431, where it is worked out for $s, s' \in (0, 1]$, but the results are similar, or [1], p. 169); if s or s' equals -1, then we must take a quotient space. If either s or $s' \in (-1, 0)$, let $L^2(\mathscr{K} \times \mathscr{K})^u$ be the Hilbert space on which G acts unitarily (i.e., completion with respect to the unitary inner product of, say, the Schwartz functions $\mathscr{S} = \mathscr{S}(\mathscr{K} \times \mathscr{K})$). Similarly, let $L^2(C_\pi \times C_\pi)^u$ be the Hilbert space of functions on $C_\pi \times C_\pi$ on which G acts unitarily. We see from [5], (3.7), that the characters of $C_\pi \times C_\pi$ form an orthogonal basis for this space. Note too that every element of $L^2(C_\pi \times C_\pi)^u$ is also in $L^2(C_\pi \times C_\pi)$ and the identity map gives a continuous injection

$$J\colon L^2(C_\pi \times C_\pi)^u \longrightarrow L^2(C_\pi \times C_\pi)$$

(see [5], (3.7) and (3.9), noting however that $s > 0$ there).

We now consider the map defined by

$$F = S\circ\hat{}\circ L\circ E\circ J\ .$$

This is a map from some subspace of $L^2(C_\pi \times C_\pi)^u$ to some subspace of $L^2(\mathscr{K} \times \mathscr{K})$. It is a G-map from the unitary representation $T_\sigma \otimes T_{\sigma'}$ on $L^2(C_\pi \times C_\pi)^u$ to the unitary representation $\mathrm{Ind}_A^G(\sigma^*(\sigma'^*)^{-1})$ realized on $L^2(\mathscr{K} \times \mathscr{K})$ by (2.2). We shall show that F is a closed map and study its domain and range so that we can apply Schur's Lemma.

Let $\mathscr{S}_0 \subset \mathscr{S}(\mathscr{K} \times \mathscr{K})$ be the subset which vanish on $\{0\} \times \mathscr{K}$.

LEMMA 3.1. *For $s, s' \in i\boldsymbol{R} \cup [-1, 0)$,*

$$(S\circ\hat{}\,)(\mathscr{S}_0 \cap \mathit{domain}(S\circ\hat{}\,))\ \textit{is dense in}\ L^2(\mathscr{K} \times \mathscr{K})\ .$$

Proof. Clearly domain$(S) \supseteq \mathscr{S}_0$, so $S_0 \cap (\mathrm{domain}(S\circ\hat{}\,)) \supseteq S_0 \cap (\hat{}(\mathscr{S}_0))$. Thus

$$(S\circ\hat{}\,)(\mathscr{S}_0 \cap \mathrm{domain}(S\circ\hat{}\,)) \supseteq \left\{f \in \mathscr{S}_0\colon \int f(x, y)\,|x|^{(s'-s)/2}dx = 0,\ \forall y\right\}\ .$$

Since $\mathscr{S}_0$ is dense in $L^2(\mathscr{K} \times \mathscr{K})$, the fact that the above set is dense is a consequence of the fact that $|x|^{(s'-s)/2}$ is not square-integrable on $\mathscr{K}$. $\square$

Trivial computations show that (at least formally):

$$L_\sigma^* = T_{\bar{\sigma}^{-1}} \qquad T_\sigma^* = L_{\bar{\sigma}^{-1}}$$
$$S_{s,s'}^* = S_{\bar{s},\bar{s}'} \qquad S_{s,s'}^{-1} = S_{s',s} \tag{3.1}$$
$$E_\sigma^* = (E_{\bar{\sigma}^{-1}})^{-1}\,.$$

We also define two more spaces of functions. Let $\mathscr{S}_d \subset \mathscr{S}(\mathscr{K} \times \mathscr{K})$ be the subset of those functions which vanish on the diagonal. Let $\mathscr{P} \subset L^2(C_\pi \times C_\pi)$ be the set of finite linear combinations of characters of $C_\pi \times C_\pi$ which vanish on the diagonal and on $\{-1\} \times C_\pi$ and $C_\pi \times \{-1\}$.

It is easy to verify from the definitions (2.1) that

$$E(\mathscr{P}) = \mathscr{S}_d \quad L(\mathscr{S}_d) = \mathscr{S}_0 \quad T(\mathscr{S}_0) = \mathscr{S}_d\,. \tag{3.2}$$

These relations hold for *any* choice of σ, σ' (specifically, they remain true when σ is replaced by $\bar{\sigma}^{-1}$, as suggested by (3.1)).

From now on we assume

$$\operatorname{Re} s \geqq \operatorname{Re} s' \tag{3.3}$$

PROPOSITION 3.2. *F is a closed, injective map.*

Proof. What we mean by this statement is that F is "closable", i.e. its closure is a function, and moreover that it is an injective function. If the closure were not a function, then there would exist $f_n \in L^2(C_\pi \times C_\pi)^u$ such that $f_n \to 0$ and $Ff_n \to f \neq 0$. Since J is continuous, $Jf_n \to 0$. Since $((S^*)^{-1}\circ\hat{\ })(\mathscr{S}_0 \cap \operatorname{domain}((S^*)^{-1}\circ\hat{\ }))$ is dense in $L^2(\mathscr{K} \times \mathscr{K})$, there exists $\phi \in \mathscr{S}_0 \cap \operatorname{domain}((S^*)^{-1}\circ\hat{\ })$ such that $\langle (S^*)^{-1}\circ\hat{\ }(\phi), f\rangle = c \neq 0$.

Now

$$\begin{aligned}\langle (S^*)^{-1}\circ\hat{\ }(\phi), Ff_n\rangle &= \langle (S^*)^{-1}\circ\hat{\ }(\phi), S\circ\hat{\ }\circ L\circ E\circ Jf_n\rangle = \langle \phi, L\circ E\circ Jf_n\rangle \\ &= \langle T_{\bar{\sigma}^{-1}}\phi, E\circ Jf_n\rangle\,.\end{aligned}$$

Note $T_{\bar{\sigma}^{-1}}\phi = \psi \in \mathscr{S}_d$; we claim $E^*\psi \in L^2(C_\pi \times C_\pi)$. Indeed, since $\operatorname{supp}\psi$ is bounded, $\bar{\sigma}^{-1}((1+t)/2)\bar{\sigma}'^{-1}((1+t')/2)$ is bounded on $\operatorname{supp}(E^*\psi)$, so $E^*\psi = (E_{\bar{\sigma}^{-1}} \otimes E_{\bar{\sigma}'^{-1}})^{-1}\psi \in L^2(c_\pi \times c_\pi)$. Letting $\theta = E^*\psi$, we have that $\langle ((S^*)^{-1}\circ\hat{\ })\phi, Ff_n\rangle = \langle \theta, Jf_n\rangle$, and $0 \neq c = \lim\langle ((S^*)^{-1}\circ\hat{\ })\phi, Ff_n\rangle = \lim\langle \theta, Jf_n\rangle$, which is impossible, since $Jf_n \to 0$. This contradiction shows F is closed.

Next we show that the closure of F is injective. It suffices to show that the image of F^* is dense in domain(F), or, since J is injective, that image$((S\circ\hat{}\circ L\circ E)^*)$ is dense.

In the case where $\mathrm{Re}(s-s')<1$, image$(S^*)\supset\mathscr{S}$, so image$(\hat{}\circ S^*)\supset\mathscr{S}_0$. Thus, by (3.1), (3.2), image$((S\circ\hat{}\circ L\circ E)^*)\supseteqq(L\circ E)^*(\mathscr{S}_0)=\mathscr{P}$, which is dense in $L^2(C_\pi\times C_\pi)$.

By (3.3), the other possibility occurs when $s'=-1$, $s\in i\boldsymbol{R}$. In this case, σ is a unitary character and L and T are unitary maps. Now $E^*=(E_{\bar\sigma^{-1}})^{-1}\otimes(E_{\sigma'^{-1}})^{-1}$; we write $E_{\sigma'^{-1}}=E_1^{-1}\circ M$, where 1 means the trivial character and M is the multiplication operator on $L^2(\mathscr{K})$ given by $Mf(x)=\sigma'^{-1}(2/(1-\sqrt{\pi}x))f(x)=|(1-\sqrt{\pi}x)/2|\cdot f(x)$. Then $E^*=[(E_{\bar\sigma^{-1}})^{-1}\otimes E_1^{-1}]\circ[1\otimes M]$, and since the left-hand operator is unitary, it suffices to prove the image of $(1\otimes M)\circ L^*\circ\hat{}\circ S^*$ is dense in $L^2(\mathscr{K}\times\mathscr{K})$. Note that the operator $1\otimes M$ maps $\mathscr{S}$ onto itself.

It would suffice to show we can approximate any element of $\mathscr{S}$ by an element of the image. In fact, it suffices to approximate a function of the form $f(x,y)=h(x)\phi(y)$, where h is a Schwartz function and ϕ is the characteristic function of a set $y_0+\mathscr{P}^n$, where n is large enough that $h(x)$ and $|1-\sqrt{\pi}x|$ are constant on cosets of $\mathscr{P}^n$.

We would like to find h' close to h in $L^2(\mathscr{K})$ and such that $h'\otimes\phi$ is in the image of $L^*\circ\hat{}\circ S^*$. The image of S^* contains $\mathscr{S}_0$, so h' should satisfy

$$\int(L^*)^{-1}(h'\otimes\phi)(x,y)dx=0;\ \text{equivalently,}$$

$$\int|x|^{-1}\bar\sigma^{-1}(x)h'(y+1/x)dx=0\,,\quad\text{for all}\quad y\in y_0+\mathscr{P}^n;\ \text{i.e.}$$

$$\int|x|^{-1}\bar\sigma(x)h'(y+x)dx=0\,,\quad\text{for all}\quad y\in y_0+\mathscr{P}^n;\ \text{i.e.}$$

$$\int|x|^{-1}\bar\sigma(x)h'(y_0+x)dx=0\quad\text{using the condition on } n)\,.$$

Since $|x|^{-1}\bar\sigma(x)$ is not square-integrable on $\mathscr{K}$, it is possible to find h' satisfying this last condition and arbitrarily close to h. Rather than $h'\otimes\phi$, we let $f'(x,y)=h'(x+y_0-y)\phi(y)$. This f' is close to f and is also in $(S\circ\hat{}\circ L)^*(\mathscr{S}_0)$.

By the choice of n, we know that M just multiplies ϕ by a constant $K\geqq 1$, so f is approximated by $f'=(1\otimes M)((1/K)f')$, which is in the image of $(1\otimes M)\circ L^*\circ\hat{}\circ S^*$, as desired. □

Now we consider the image of F.

PROPOSITION 3.3. *Image(F) is dense in $L^2(\mathscr{K}\times\mathscr{K})$.*

Proof. By Lemma 3.1, it suffices to show that $\mathscr{S}_0$ is contained in the image of $L\circ E\circ J$, which is clear from (3.2). □

Now we turn to the study of the domain of F. Of course, if $s, s' \in i\boldsymbol{R}$, it is all of $L^2(C_\pi \times C_\pi)$.

PROPOSITION 3.4. *F is defined on a G-invariant subspace, which is dense if s or $s' \in i\boldsymbol{R}$ or if $s, s' \in (-1, 0)$ and $s + s' \geqq -1$.*

If, on the other hand, $s, s' \in (-1, 0)$ and $s + s' < -1$, then the orthogonal complement of domain(F) is a subspace on which G acts as the (irreducible) complementary series representation $T_{s+s'+1}$.

Proof. By (3.2), (3.3), we see that domain$(F) \supseteqq \mathscr{P}$. Since $\hat{}$ is a unitary isomorphism, to show that domain(F) contains the G-span of this set, it suffices to show that the action (2.3) of G on $L^2(\mathscr{K} \times \mathscr{K})$ takes $\mathscr{S}_0$ into $L^2(\mathscr{K} \times \mathscr{K})$, when we let $\eta = \sigma(\sigma')^{-1}$. In light of (3.3), this is obvious.

The span of the G-translates of $\mathscr{P}$ (even the C_π-translates) contains the space of all locally constant functions which vanish on the diagonal in $C_\pi \times C_\pi$. Let us call this space $\mathscr{P}_d$. So domain$(F) \supseteqq \mathscr{P}_d$.

Now let $\chi \in \hat{C}_\pi$ and consider

$$L^2(C_\pi \times C_\pi)_\chi = \{f \in L^2(C_\pi \times C_\pi)\colon f(\alpha t, \alpha t') = \chi(\alpha) f(t, t'),\ \forall \alpha, t, t' \in C_\pi\} .$$

The space $L^2(C_\pi \times C_\pi)_\chi$ is is isomorphic to $L^2(C_\pi)$ under the mapping $f \to f_1$, where $f_1(t) = f(1, t)$.

The projection $(\mathscr{P}_d)_\chi$ of $\mathscr{P}_d$ on $L^2(C_\pi \times C_\pi)_\chi \approx L^2(C_\pi)$ is the space of finite linear combinations ϕ of characters of C_π such that $\phi(1) = 0$. We index the characters of C_π as $\psi_{\pm i}$, $0 \leqq i$, as in [5], p. 420. Then an orthogonal basis of $L^2(C_\pi \times C_\pi)_\chi$ is given by $\phi_{\pm i} = \chi\psi_{\mp i}^{-1} \otimes \psi_{\pm i}$, and

$$(\mathscr{P}_d)_\chi = \left\{\phi = \sum_{\text{finite}} a_{\pm i}\phi_{\pm i}\colon \sum a_{\pm i} = 0\right\} .$$

We wish to determine the closure of $(\mathscr{P}_d)_\chi$ with respect to the norm induced by the norm $\|\cdot\|^u$ in $L^2(C_\pi \times C_\pi)^u$. If $s \in i\boldsymbol{R}$, let $\|\cdot\|_s$ be the ordinary norm on $L^2(C_\pi)$; if $s \in (-1, 0)$, let $\|\cdot\|_s$ be defined as in [5], (3.8). Then

$$\|\phi_{\pm i}\|^u = \|\chi \cdot \psi_{\pm i}^{-1}\|_s \cdot \|\psi_{\pm i}\|_{s'} . \tag{3.4}$$

We note that if s and/or $s' \in (-1, 0)$ and if i is large enough so that cond$(\psi_{\pm i}^{-1}) \supsetneqq$ cond(χ), then cond$(\chi\psi_{\pm i}^{-1}) =$ cond$(\psi_{\pm i})$, and

$$\|\chi\psi_{\pm i}^{-1}\|_s = q^{-hs/2} \quad \|\psi_{\pm i}\|_{s'} = q^{-hs'/2} . \tag{3.5}$$

Here h is the number such that $\psi_{\pm i}$ is trivial on $C_\pi^{(h)}$, nontrivial on $C_\pi^{(h-1)}$, cf. [5], (2.17), (3.13); note that for each $h \geqq 1$, there are $2q^h(1 - 1/q)$ such characters.

Recall that $\phi \in (\mathscr{P}_d)_\chi$ iff ϕ is in the kernel of the linear functional λ given by

$$\lambda: \sum a_{\pm i}\, \phi_{\pm i} \longmapsto \sum a_{\pm i} \ .$$

To prove the proposition, we must determine whether or not λ is continuous (i.e. whether $\ker \lambda$ has $\|\cdot\|^u$-closure of codimension 1 or 0).

LEMMA 3.5. (i) $(\mathscr{P}_d)_\chi$ *is* $\|\cdot\|^u$-*dense in* $\ker \lambda$;
(ii) λ *is* $\|\cdot\|^u$-*continuous* iff $s, s' \in (-1, 0)$ *and* $s + s' < -1$.

Proof. (i) is trivial. For (ii), suppose $s, s' \in (-1, 0)$. Consider the orthonormal basis formed by the elements $\phi^u_{\pm i} = (\|\phi_{\pm i}\|^u)^{-1}\phi_{\pm i}$; so if $\phi = \sum a_{\pm i}\phi_{\pm i}$, then $\phi = \sum (a_{\pm i}\|\phi_{\pm i}\|^u)\phi^u_{\pm i}$. Thus $\lambda\phi = \sum a_{\pm i} = \sum(a_{\pm i}\|\phi_{\pm i}\|^u)(\|\phi_{\pm i}\|^u)^{-1}$, so the continuity of λ is equivalent to the convergence of $\sum(\|\phi_{\pm i}\|^u)^{-2}$.

By (3.4) and (3.5), ignoring finitely many terms, $\sum(\|\phi_{\pm i}\|)^{-2} \sim \sum_{h=1}^{\infty} 2(1 - 1/q)q^h \cdot q^{hs} \cdot q^{hs'} = 2(1 - 1/q)\sum q^{h(1+s+s')}$. This converges iff $1 + s + s' < 0$; i.e. $s + s' < -1$. If either s or $s' \in i\boldsymbol{R}$, we replace it in the above norm calculations with 0 (its real part), and the sum obviously diverges. □

All that remains of the proposition is to discuss the action of G on $\mathscr{P}_d^\perp$ in the case when it non-trivial. Note that in this case $(\mathscr{P}_d^\perp)_\chi$, the projection of $\mathscr{P}_d^\perp$ on $L^2(C_\pi \times C_\pi)^u_\chi$, has dimension 1 for each χ. If we can show that $\mathscr{P}_d^\perp$ contains the complementary series representation $T_{s+s'+1}$, then we shall be done, since by considering the representation of C_π on $\mathscr{P}_d^\perp$ we see that $\mathscr{P}_d^\perp$ cannot contain more than this one representation.

We argue that if $s + s' < -1$, then $T_s \otimes T_{s'}$ must contain some complementary series representation. Indeed, if u and w are unit K-fixed vectors in T_s and $T_{s'}$ respectively, and $v = u \otimes w$, consider the coefficient function on G given by $\phi(g) = \langle T_s \otimes T_{s'}(g)v, v\rangle = \langle T_s(g)u, u\rangle \langle T_{s'}(g)w, w\rangle$.

The spherical functions $\langle T_s(g)u, u\rangle$ and $\langle T_{s'}(g)w, w\rangle$ can be calculated (cf. [1], pp. 174–176), and we note that T_s is "of class L^q" iff $q > 2/(1 + s)$ (i.e. the spherical function is in $L^q(G)$). We also see that if $|a| = q^n$, $n \geqq 0$, then $\phi\begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix} = \text{const.}\ q^{-n(2+s+s')}$, so $\phi \in L^q(G)$ iff $q > 2/(s + s' + 1)$. In particular, $T_s \otimes T_{s'}$ is not of class $L^{2+\varepsilon}$ for arbitrarily small $\varepsilon > 0$. But since the representation of G on the closure of $\mathscr{P}_d$ is isomorphic to $\mathrm{Ind}_A^G 1$, which *is* of class $L^{2+\varepsilon}$, we must have that the representation of G on $\mathscr{P}_d^\perp$ is not of class $L^{2+\varepsilon}$, and hence must contain complementary series representations, since all other irreducible unitary representations are of class $L^{2+\varepsilon}$. But since it can contain at most one such, and must be of class L^q iff $q > 2/(s+s'+1)$, this one must be $T_{s+s'+1}$, as claimed. $\square$

THEOREM 3.6. (i) *If* $s \in (-1, 0)$ *and* σ' *is a unitary character of* $\mathscr{K}^\times$, *then*

$$T_s \otimes T_{\sigma'} \approx T_0 \otimes T_{\sigma'} ;$$

(ii) *If* $s, s' \in (-1, 0)$ *and* $s + s' \geqq -1$, *then*

$$T_s \otimes T_{s'} \approx T_0 \otimes T_0 ;$$

(iii) *If* $s, s' \in (-1, 0)$ *and* $s + s' < -1$, *then*

$$T_s \otimes T_{s'} \approx T_0 \otimes T_0 \oplus T_{s+s'+1} .$$

Proof. We apply Schur's Lemma ([4], Lemma 3.1) to the map F. It tells us that the representation of G on the closure of domain(F) is isomorphic to $\mathrm{Ind}_A^G(\sigma^*(\sigma'^*)^{-1})$. Parts (i) and (ii) follow since domain(F) is dense in these cases. Part (iii) follows since G acts on $\mathscr{P}_d^\perp$ by $T_{s+s'+1}$ and on the closure of $\mathscr{P}_d$ by $\mathrm{Ind}_A^G 1$. $\square$

REMARK. The decompositions of $T_0 \otimes T_{\sigma'}$ and $T_0 \otimes T_0$ can be found in [3], Theorem 3.

4. The special representation.

To extend our results to include the special representation T_{-1}, we must study the above situation in the case when s or s' is allowed to equal -1, still subject to (3.3). In this case, G acts unitarily not on $L^2(C_\pi \times C_\pi)$ with a new norm, but on a quotient of it. We let $L^2(C_\pi \times C_\pi)^u$ be the Hilbert space which has an orthogonal basis $\psi_{\pm i} \otimes \psi_{\pm j}$, such that $\|\psi_{\pm i} \otimes \psi_{\pm j}\| = \|\psi_{\pm i}\|_s \|\psi_{\pm j}\|_{s'}$, where $\|\cdot\|_s$ is as before if $s \neq -1$, and, when $s = -1$, $\|\psi_{-0}\|_{-1} = (2/(1+1/q))^{1/2}$, $\|\psi_{\pm i}\|_{-1} = q^{h/2}$, $\|\psi_{+0}\|_{-1} = 1$; here h is as before, and we have (in effect) used [5], (3.15), and we have just defined $\|\psi_{+0}\|_{-1}$ arbitrarily. Note that G does not act unitarily on $L^2(C_\pi \times C_\pi)^u$, just on the quotient by $\{\psi_{+0}\} \otimes L^2(C_\pi)$ and/or $L^2(C_\pi) \otimes \{\psi_{+0}\}$ according as s and/or s' equals -1.

Then the identity map $J: L^2(C_\pi \times C_\pi)^u \to L^2(C_\pi \times C_\pi)$ is bounded and we may define the map F as before. Moreover, Lemma 3.1 and Propositions 3.2 and 3.3 still hold in this case. The analogue of Proposition 3.4 is this:

PROPOSITION 4.1. *Let at least one of* s, s' *equal* -1. *Then* F *is defined on the G-space generated by* $\mathscr{P}$. *For each* $\chi \in \hat{C}_\pi$, *the closure of* $(\mathscr{P})_\chi$ *has codimension one in* $L^2(C_\pi \times C_\pi)_\chi^u$ iff *neither* s *nor* $s' \in i\boldsymbol{R}$; *otherwise* $\mathscr{P}$ *is dense.*

Proof. In the notation of Lemma 3.5, λ is continuous exactly in this case (same argument). $\square$

Next we let $V \subset L^2(C_\pi \times C_\pi)^u$ be the G-invariant subspace spanned by $\{\psi_{+0} \otimes \psi_{\pm i}\}$ and/or $\{\psi_{\pm i} \otimes \psi_{+0}\}$, according as s and/or s' is -1. So G acts unitarily on $L^2(C_\pi \times C_\pi)^u/V$, as $T_{-1} \otimes T_{-1}$ or $T_\sigma \otimes T_{-1}$. Then F induces an injective map $F_1: L^2(C_\pi \times C_\pi)^u/V \to L^2(\mathscr{K} \times \mathscr{K})/V'$ where $V' = F(V \cap \text{domain } F)$.

PROPOSITION 4.2. (i) *F_1 is a closed map.*
(ii) *If only one of s, s' is -1, then $V' = \{0\}$.*

Proof. (i) Suppose F_1 is not closed, then for some χ the map which F_1 induces, $(F_1)_\chi: L^2(C_\pi \times C_\pi)^u_\chi/V_\chi \to L^2(\mathscr{K} \times \mathscr{K})_\chi/V'_\chi$ is not closed. So there exist $f_n \in \text{domain } F$ such that $Ff_n \to f$, $f \neq 0$ modulo V'_χ and $f_n \to 0$ modulo V_χ. By adding elements of V'_χ to f and of $V_\chi \cap \text{domain}(F)$ to f_n, we may assume $0 \neq f \in (V'_\chi)^\perp$ and $Ff_n \in (V'_\chi)^\perp$. So $Ff_n \to f$. Also, there exist $v_n \in V_\chi$ such that $f_n - v_n \to 0$.

As in Proposition 3.3, we see that any $\phi \in \mathscr{P}$ is in domain(F^{*-1}), so for any $\phi \in \mathscr{P}$, $\langle f_n, \phi\rangle = \langle Ff_n, F^{*-1}\phi\rangle$, which converges. Thus we see that $\langle v_n, \phi\rangle$ converges for any $\phi \in \mathscr{P}$. Since $v_n \in V_\chi$, which is a finite-dimensional space, and inner products with $\mathscr{P}$ give all of its dual, this implies that v_n converges, to v, say, and so $f_n \to v$.

But since F is closed and $Ff_n \to f$, we must have $v \in \text{domain } F$ and $f = Fv \in V'$, a contradiction, as desired.

(ii) Suppose $V \cap \text{domain } F \neq 0$; V is a G-invariant space, isomorphic to T_σ. If F were defined on V, there would be an embedding of T_σ into $\text{Ind}_A^G(\sigma^*(\sigma'^*)^{-1})$, which is not possible. $\square$

PROPOSITION 4.3. *If $s = s' = -1$, then V is the span of $\{\psi_{+0} \otimes \psi_{\pm i}, \psi_{\pm i} \otimes \psi_{+0}\}$. Also $\psi_{+0} \otimes \psi_{+0} \notin$ domain F, so F induces a map on $V/\langle \psi_{+0} \otimes \psi_{+0}\rangle$. This space is of course two copies of T_{-1}; exactly one of them is in domain(F).*

Proof. If $\psi_{+0} \otimes \psi_{+0} \in \text{domain } F$, then the trivial representation would be embedded in $\text{Ind}_A^G(\sigma^*(\sigma'^*)^{-1})$, which is impossible. F is defined on $\mathscr{P}_d$, which contains $\psi_{+0} \otimes \psi_{\pm i} - \psi_{\pm i} \otimes \psi_{+0}$, for each $\pm i$. These vectors span a G-invariant subspace of $V/\langle \psi_{+0} \otimes \psi_{+0}\rangle$ on which G acts as T_{-1}.

Now either domain F contains *all* of $V/\langle \psi_{+0} \otimes \psi_{+0}\rangle$ or only this one copy of T_{-1}. We claim that the latter case obtains. Indeed, let us show that $\psi_{+0} \otimes \psi_{+i}$ is *not* in domain F. Now $J(\psi_{+0} \otimes \psi_{+i}) = q^{-h/2}\psi_{+0} \otimes \psi_{+i}$, where h is as before; and $L \circ E \circ J(\psi_{+0} \otimes \psi_{+i})(x, y) = (1/2)q^{-h/2}\psi_{+i}((1 + \sqrt{\pi}\, y)/(1 - \sqrt{\pi}\, y))$, which is *not* in $L^2(\mathscr{K} \times \mathscr{K})$.

Since $\hat{}$ is unitary and S (in this case) is the identity transformation, $\psi_{+0} \otimes \psi_{+i}$ is not in domain F. $\square$

THEOREM 4.4. (i) *If* $s \neq -1$, *then*

$$T_\sigma \otimes T_{-1} \approx T_\sigma \otimes T_0 \approx T_\sigma \otimes T_{s'}\ , \quad \forall s' \in i\boldsymbol{R}\ ;$$

(ii)
$$\begin{aligned} T_{-1} \otimes T_{-1} &\approx T_0 \otimes T_0 \ominus T_{-1} \\ &= 3T_{-1} \oplus \cdots\ . \end{aligned}$$

Proof. We note that by Proposition 4.1, $\mathscr{P}/V \cap \mathscr{P}$ is dense in $L^2(C_\pi \times C_\pi)^u/V$. Then part (i) is immediate from Proposition 4.2.

For part (ii), we note that by Proposition 4.3, V' is one copy of T_{-1}, so $\mathrm{Ind}_A^G(\sigma^*(\sigma'^*)^{-1})/V'$ is $T_0 \otimes T_0 \ominus T_{-1}$, and we are done. $\square$

5. **Tensor products with supercuspidals.** As is remarked, for example, in [6], every supercuspidal representation T is induced from some maximal compact subgroup, $K = K_T$, say. Let T' be any unitary representation of G. A simple application of Mackey's Subgroup Theorem says that if T is induced from the representation η of K, then

$$T \otimes T' \approx \mathrm{Ind}_K^G(\eta \otimes (T'|_K))\ .$$

To study the tensor product of T with a complementary series representation or the special representation, we must therefore study the restrictions of these latter representations to K_T. We shall compare the K_T-decomposition of a complementary series or special representation with the K_T-decomposition of a class one principal series representation. Since the structure of the tensor product of T with a class one principal series representation is known from [3], we shall then be able to describe the tensor product of a supercuspidal representation with a complementary series or special representation.

For convenience, in this section we let $K = K_T$ be *either* maximal compact subgroup of G.

LEMMA 5.1. *Let* $s \in (-1, 0)$. *Then*

$$T_s|_K \approx T_0|_K \approx T_{s'}|_K\ , \quad \forall s' \in i\boldsymbol{R}\ ,$$

i.e. *these representations have the same K-types.*

Moreover, $T_{-1}|_K \oplus 1_K \approx T_0|_K$, i.e. *the K-type of the special representation plus the trivial representation is the same as the class one principal series.*

Proof. The class one principal series and the complementary series representations can all be realized on Hilbert spaces of functions on K which are invariant under left multiplication by $(AN)\cap K$; K acts by right translation. The subspace consisting of locally constant functions is dense in each of these Hilbert spaces, and is obviously K-invariant; the actions of K on these dense subspaces are isomorphic, so the whole spaces are K-isomorphic. Furthermore, the special representation is realized on a completion of the quotient by the space of constant functions. □

Combining this with Mackey's Subgroup Theorem, we find

THEOREM 5.2. *Let T be any supercuspidal representation of G. Then*:

$$T_s \otimes T \approx T_0 \otimes T \qquad \forall s \in (-1, 0)$$
$$T_{-1} \otimes T \approx (T_0 \otimes T) \ominus T ,$$

i.e. *to get the decomposition of $T_{-1} \otimes T$, remove one copy of T from the decomposition of $T_0 \otimes T$.*

Proof. Suppose $T = \mathrm{Ind}_K^G \eta$. Then by Mackey's Subgroup Theorem and Lemma 5.1,

$$T_s \otimes T \approx \mathrm{Ind}_K^G [\eta \otimes T_s|_K] = \mathrm{Ind}[\eta \otimes T_0|_K] \approx T_0 \otimes T ,$$

and

$$\begin{aligned} T_{-1} \otimes T &\approx \mathrm{Ind}\,[\eta \otimes T_{-1}|_K] \\ &\approx \mathrm{Ind}\,[\eta \otimes (T_0|_K \ominus 1_K)] \\ &= \mathrm{Ind}\,[\eta \otimes T_0|_K] \ominus \mathrm{Ind}(\eta \otimes 1_K) \\ &= T_0 \otimes T \ominus T , \quad \text{as claimed} . \end{aligned}$$ □

Since the tensor product of a supercuspidal representation with a class one principal series representation is completely described in [3], we can read off the results for the special representation and complementary series representations too.

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