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SYMMETRIC SHIFT REGISTERS, PART 2

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We study symmetric shift registers defined by

$$(x_1, \cdots, x_n) \longrightarrow (x_2, \cdots, x_n, x_{n+1})$$

where $x_{n+1} = x_1 + S(x_2, \cdots, x_n)$ **and** S **is a symmetric polynomial over the field** GF(2).

Introduction. In this paper we study symmetric shift registers over the field GF(2) = {0, 1}. In [2] we introduced the block structure of elements in $\{0, 1\}^n$ and developed a theory about this block structure. In this paper we will use the results in [2] about the block structure to determine the cycle structure of the symmetric shift registers.

The symmetric shift register θ_S corresponding to $S(x_2, \cdots, x_n)$ where S is a symmetric polynomial, is defined by

$$\theta_S(x_1, \cdots, x_n) = (x_2, \cdots, x_{n+1}) \quad \text{where} \quad x_{n+1} = x_1 + S(x_2, \cdots, x_n) .$$

q is the minimal period of $A \in \{0, 1\}^n$ with respect to θ_S if q is the least integer such that $\theta_S^q(A) = A$. Then $A \to \theta_S(A) \to \cdots \to \theta_S^q(A) = A$ is called the cycle corresponding to A. We will for all S solve the following three problems:

1. Determine the minimal period for each $A \in \{0, 1\}^n$.
2. Determine the possible minimal periods.
3. Determine the number of cycles corresponding to each minimal period.

Moreover, the problems will be solved in a constructive way, a way which will describe how the minimal periods and the number of cycles can be calculated. In [1] (see also [2]) we reduced all the problems to the case $S = E_k + \cdots + E_{k+p}$ where E_i is defined by

$$E_i(x_2, \cdots, x_n) = 1 \quad \text{if and only if} \quad \sum_{j=2}^{n} x_j = i .$$

In this paper we will only study $S = E_k + \cdots + E_{k+p}$.

I will now roughly describe the structure of the proof. First we need a definition. Suppose $\mathscr{M} \subset \{0, 1\}^n$ is a set such that for all $A \in \mathscr{M}$ there exists an $i > 0$ such that $\theta_S^i(A) \in \mathscr{M}$. Then we define Index: $\mathscr{M} \to \{1, 2, \cdots\}$ and $\psi: \mathscr{M} \to \mathscr{M}$ in the following way:

Let $i > 0$ be the least integer such that $\theta_S^i(A) \in \mathscr{M}$, then we define Index $(A) = i$ and $\psi(A) = \theta_S^i(A)$.

In the proof we need only consider certain subsets $\mathscr{M}$ which can be represented in a nice way. Each $A \in \mathscr{M}$ is uniquely determined by its block structure. In [2] we proved how we can determine the block structure of $\psi(A)$ by means of the block structure of A. We continue in this way and calculate the block structure of $\psi^2(A)$, $\psi^3(A)$, $\cdots$. Finally, we find a q such that A and $\psi^q(A)$ have the same block structure. Hence $A = \psi^q(A)$. Then

$$\text{Index}\,(A) + \text{Index}\,(\psi(A)) + \cdots + \text{Index}\,(\psi^{q-1}(A))$$
$$\text{is the minimal period of } A\ .$$

Next we give a short outline of the paper. Section 2 contains some definitions and notations. In § 3 we compute ψ for a certain subset $\mathscr{M}$ and describe the main ideas. In the §§ 4, 5 and 6 we solve the Problems 1, 2 and 3 respectively for the set $\mathscr{M}$. In § 7 we generalize the results to all $A \in \{0, 1\}^n$. This generalization will not be difficult.

2. Preliminaries. We must repeat some of the definitions from [2]. First we define the blocks of $A \in \{0, 1\}^n$ ([2], Def. 3.1). Intuitively an i-block is i consecutive 1's in A. 0_i denotes i consecutive 0's in A and 1_i denotes i consecutive 1's in A for $i \geqq 0$.

We need some notation. We write $a_1 \cdots a_n = (a_1, \cdots, a_n) \in \{0, 1\}^n$. If $A = a_1 \cdots a_n \in \{0, 1\}^n$, we define

$$f(a_i \cdots a_j) = (\text{the number of 1's in } a_i \cdots a_j)$$
$$- (\text{the number of 0's in } a_i \cdots a_j)\ .$$

If $r \leqq i \leqq j \leqq s$ and $(r \neq i$ or $j \neq s)$ we write $a_i \cdots a_j < a_r \cdots a_s$. Moreover, $a \wedge b$ denotes the minimum of a and b, and we define $w(\cdot)$ by $w(a_1 \cdots a_n) = \sum_{i=1}^n a_i$.

We divide the definition of blocks into two parts by first defining 1-structures and 0-structures of A. A 1-structure (0-structure) is a generalization of q consecutive 1's (respectively 0's) which is succeeded by q 0's (respectively 1's). We will say that a block B_i is on level i if it is contained in a chain $B_1 > B_2 > B_3 > \cdots > B_i$ of blocks.

DEFINITION 2.1, *Part* 1. Suppose $A = a_1 \cdots a_n \in \{0, 1\}^n$.

(a) Suppose $a_r = 1$. Let s be the maximal integer such that $D = a_r \cdots a_s$ satisfies

(1) $0 < f(a_r \cdots a_i) \leqq f(a_r \cdots a_s)$ for $i \in \{r, \cdots, s\}$

and

(2) If $r \leqq i \leqq j \leqq s$, then $f(a_i \cdots a_j) > -(p + 1)$.

By definition D is a 1-structure with respect to p.

(b) Suppose $a_r = 0$. Let s be the maximal integer such that $D = a_r \cdots a_s$ satisfies

$$0 > f(a_r \cdots a_i) \geqq f(a_r \cdots a_s) \quad \text{for} \quad i \in \{r, \cdots, s\} .$$

By definition D is a 0-structure.

DEFINITION 2.1, *Part* 2. (a) Suppose $A = a_1 \cdots a_n \in \{0, 1\}^n$. We define the blocks in A with respect to p by induction with respect to the level of the blocks in the following way: (The 1-structures are defined with respect to p.)

Level 1. We decompose A in the following way $A = 0_{i_1}B_1\, 0_{i_2}B_2 \cdots B_m 0_{i_{m+1}}$ where B_j is a 1-structure. By definition $B_1, \cdots, B_m$ are the blocks in A on level 1.

Level 2. Suppose B is a block on level 1. We decompose B in the following way

$$B = 1_{i_1}B_1\, 1_{i_2}B_2 \cdots B_m 1_{i_{m+1}} \quad \text{where} \quad B_j \text{ is a 0-structure} . \tag{2.1}$$

By definition $B_1, \cdots, B_m$ are the blocks in A on level 2 which are contained in B.

Level 3. Suppose B is a block on level 2. We decompose B in the following way

$$B = 0_{i_1}B_1\, 0_{i_2}B_2 \cdots B_m 0_{i_{m+1}} \quad \text{where} \quad B_j \text{ is a 1-structure} . \tag{2.2}$$

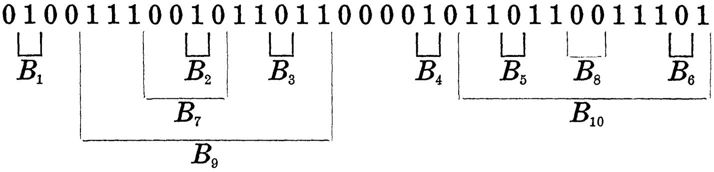
By definition $B_1, \cdots, B_m$ are the blocks in A on level 3 which are contained in B.

We continue in this way. If $i \in \{3, 5, 7, \cdots\}$ and B is a block on level i, we decompose B as in (2.1). If $i \in \{4, 6, 8, \cdots\}$ and B is a block on level i, we docompose B as in (2.2).

(b) Let B be a block in A on level i. Then we define level $(B) = i$, type $(B) = |f(B)| \wedge (p + 1)$ and $m(B) = |f(B)|$. Moreover, if type $(B) = q$ we say that B is a q-block or that B is a block of type q.

We illustrate Definition 2.1 by the example $p = 2$ and

$$A = 0\,1\,0\,0\,1\,1\,1\,0\,0\,1\,0\,1\,1\,0\,1\,1\,0\,0\,0\,0\,1\,0\,1\,1\,0\,1\,1\,0\,0\,1\,1\,1\,0\,1$$



where

$$B_1, B_2, B_3, B_4, B_5 \text{ and } B_6 \quad \text{are blocks of type 1}$$

B_7 and B_8	are blocks of type 2
B_9 and B_{10}	are blocks of type 3
B_1, B_9, B_4 and B_{10}	are blocks on level 1
B_7, B_3, B_5, B_8 and B_6	are blocks on level 2
B_2	is a block on level 3 .

We establish the convention that B always denotes a block. Moreover, we suppose k and p are fixed integers such that $0 \leqq k \leqq k + p \leqq n - 1$. The block structure is always determined with respect to p and we always work with $S = E_k + \cdots + E_{k+p}$. We write $\theta = \theta_S$. These conventions do not concern § 7.

If $A = a_1 \cdots a_n$, we write $l_A(a_i \cdots a_j) = i$ and $r_A(a_i \cdots a_j) = j$. Next we define $d(B)$ which measures how far the block B is to the left in A. Suppose $A = a_1 \cdots a_n$. We define

$$d_q(a_1 \cdots a_j) = j - \sum \{q \wedge \text{type}\,(B) \colon l_A(B) \leqq j\} \\ - \sum \{q \wedge \text{type}\,(B) \colon r_A(B) \leqq j\} \ .$$

If B is a block of A, then we define $d(B) = 0$ if $l_A(B) = 1$. Otherwise,

$$d(B) = d_q(a_1 \cdots a_j) \quad \text{where} \quad j = l_A(B) - 1 \text{ and } q = \text{type}\,(B) \ .$$

In our example in this section we get

$$(d(B_1), d(B_2), d(B_3), d(B_4), d(B_5), d(B_6)) = (1, 5, 6, 10, 11, 15) \\ (d(B_7), d(B_8)) = (3, 7) \\ (d(B_9), d(B_{10})) = (2, 4) \ .$$

3. Main ideas. In this section we let $\gamma_1, \cdots, \gamma_{p+1}$ be fix integers such that $\gamma_i \geqq 0$ for $i = 1, \cdots, p$ and $\gamma_{p+1} > 0$. Moreover, we will only work with $A \in \{0, 1\}^n$ which contains γ_i i-blocks for $i = 1, \cdots, p + 1$, and such that $w(A) = k + p + 1$. That is; A contains $(k + p + 1)$ 1's.

In [2] we described how the blocks move by applying the shift register. We will reformulate these results by introducing new notation. First we have to repeat a lot of the notation from [2]. Moreover, we will mention some of the problems we must solve and describe the main ideas on an example.

In [2] we defined $(i = 1, \cdots, p + 1)$

$$\text{(3.1)} \quad \begin{aligned} \alpha_i &= n + i - 2\gamma_1 - 4\gamma_2 - \cdots - 2i\gamma_i - 2i(\gamma_{i+1} + \cdots + \gamma_{p+1}) \ . \\ m &= k + p + 1 - \gamma_1 - 2\gamma_2 - 3\gamma_3 - \cdots - (p + 1)\gamma_{p+1} \ . \end{aligned}$$

Since α_i and m are very important constants, we will give an interpretation of them. To do this we define a subset $\mathscr{M} \subset \{0, 1\}^n$ in the following way

$$(3.2)\qquad A \in \mathscr{M} \Longleftrightarrow \begin{cases} w(A) = k + p + 1\ . \\ A \text{ starts with 0 or a } (p+1)\text{-block}\ . \\ A \text{ contains } \gamma_i\ i\text{-blocks for } i = 1, \cdots, p+1\ . \\ A \text{ ends with a } (p+1)\text{-block}\ . \end{cases}$$

In the §§ 3–6 we will study this subset, and in § 7 we reduce the general problem to $\mathscr{M}$. It can be proved that

$$(3.3)\qquad \alpha_i \geqq \max\{d(B) \colon B \text{ is an } i\text{-block in } A\}$$

for each $A \in \mathscr{M}$. For some $A \in \mathscr{M}$ we will have equality in (3.3). Next, we will give an interpretation of m. We use the function $f(\cdot)$ defined in § 2. From the definition of blocks we have $f(B) \geqq p + 1$ when type $(B) = p + 1$. We suppose $A \in \mathscr{M}$. Then it can be proved that

$$m = \sum \{f(B) - (p+1) \colon B \text{ is a } (p+1)\text{-block in } A\}\ .$$

m is in a way the sum of the superfluous 1's in the $(p + 1)$-blocks in A.

The subset $\mathscr{M}$ we defined in (3.2) is very important. We will now study the key map $\psi \colon \mathscr{M} \to \mathscr{M}$ defined by

$$(3.4)\qquad \begin{array}{l} \text{if } A \in \mathscr{M}, \text{ then } \psi(A) = \theta^i(A) \text{ where } i \text{ is the least integer} \\ \text{such that } \theta^i(A) \in \mathscr{M}. \text{ Moreover we define Index}\,(A) = i\ . \end{array}$$

In [2] we called this map $\varphi_{\min}$. Moreover, if $\gamma_{p+1} = 1$ then $\varphi = \varphi_{\min}$ in [2]. By Lemma 4.11 (the case $\gamma_{p+1} = 1$) and Lemma 4.13 in [2] there exists a bijective correspondence (which we also call ψ)

$$(3.5)\qquad \psi \colon \{\text{the blocks in } A\} \longrightarrow \{\text{the blocks in } \psi(A)\}$$

which satisfies Condition 4.9 in [2]. That implies that the map (3.5) have a lot of nice properties which we describe now. We have

$$\text{type}\,(B) = \text{type}\,(\psi(B)) \qquad \text{and} \qquad |f(B)| = |f(\psi(B))|$$

where f is as in § 2. In [2] we also write $m(B) = |f(B)|$. But the most important thing which Condition 4.9 in [2] gives us is the following: Let i be an integer such that $1 \leqq i \leqq p + 1$ and

$$B_1, \cdots, B_{\gamma_i}$$

are the i-blocks in A ordered from left to right. Then there exists an integer r (depending on i) such that

$$\psi(B_{r+1}), \psi(B_{r+2}), \cdots, \psi(B_{\gamma_i}), \psi(B_1), \cdots, \psi(B_r)$$

are the i-blocks in $\psi(A)$ ordered from left to right. Moreover, there exists an integer β (depending on i) such that

$$d(\psi(B_i)) = \begin{cases} d(B_i) - \beta & \text{when} \quad d(B_i) \leqq \beta \\ d(B_i) - \beta + \alpha_i & \text{otherwise} . \end{cases}$$

We calculated these integers r and β in [2]. Unfortunately, these calculations are very complicated. We will return to these calculations in Lemmas 3.3 and 3.4. Moreover, we proved in [2] (Lemma 4.1(b) in [2]) the following fundamental result:

(3.6) If $A, A' \in \mathscr{M}$ and there is a correspondence $B \longrightarrow B'$ between the blocks of respectively A and A' such that

$$d(B) = d(B') \quad \text{for each block } B$$

and

$$f(B) = f(B') \quad \text{for each } (p+1)\text{-block } B ,$$

then $A = A'$.

Now we need a simple way to describe the block structure. To each $A \in \mathscr{M}$ we define $(p + 1)$ vectors which contains all information about the block structure of A.

DEFINITION 3.1. Let $A \in \mathscr{M}$. Suppose $1 \leqq i \leqq p + 1$ and

$$B_1, \cdots, B_{\gamma_i}$$

are the i-blocks in A ordered from left to right. If $1 \leqq i \leqq p$, we define

$$D_i(A) = (d(B_1), \cdots, d(B_{\gamma_i})) .$$

If $i = p + 1$, then we define

$$D_{p+1}(A) = (d(B_1), \cdots, d(B_{\gamma_{p+1}})) \times (f(B_1) - (p + 1), \cdots, f(B_{\gamma_{p+1}}) - (p + 1))$$

where f is as in § 2. As a convention we let $D_i(A)$ be the empty vector if $\gamma_i = 0$.

The last part of $D_{p+1}(A)$, namely $(f(B_1) - (p + 1), \cdots, f(B_{\gamma_{p+1}}) - (p + 1))$ tells us how large each $(p + 1)$-block in A is. Let A be as in our example in § 2. Then $n = 34$ and by putting $p = 2$ and $k = 15$ we get $A \in \mathscr{M}$. Moreover, we get

(3.7)

$$\gamma_1 = 6 , \quad \gamma_2 = 2 , \quad \gamma_3 = 2 , \quad \alpha_1 = 15 , \quad \alpha_2 = 8 ,$$

$$\alpha_3 = 5 \quad \text{and} \quad m = 2 .$$

$$D_1(A) = (1, 5, 6, 10, 11, 15) , \quad D_2(A) = (3, 7) \quad \text{and}$$

$$D_3(A) = (2, 4) \times (1, 1) .$$

These results from [2] indicate that we must solve the following 3 problems: Let $A \in \mathscr{M}$.

1. Let i be an integer such that $1 \leqq i \leqq p+1$. How can we obtain $D_i(\psi^t(A)) = D_i(A)$?

2. How can we determine an integer t such that $D_i(\psi^t(A)) = D_i(A)$ for all $i \in \{1, \cdots, p+1\}$.

3. Suppose we have solved Problem 2. By (3.6) we have $\psi^t(A) = A$. How can we determine an integer "per" such that $\psi^t(A) = \theta^{\text{per}}(A)$?

By using Definition 3.1 we can define a map

$$g = D_1 \times D_2 \times \cdots \times D_{p+1} .$$

By (3.6) g is a bijective correspondence

$$g \colon \mathscr{M} \longrightarrow g(\mathscr{M}) .$$

One of the main ideas in this paper is that we work with $g(\mathscr{M})$ instead on $\mathscr{M}$. For example, later we will count some subsets of $\mathscr{M}$. Then we instead count the corresponding subset of $g(\mathscr{M})$. In [2] we described $g(\mathscr{M})$ in a nice way as in the following lemma.

LEMMA 3.2. (a) *If* $1 \leqq i \leqq p$, *then*

$$D_i(\mathscr{M}) = \{(t_1, \cdots, t_{\gamma_i}) \colon 1 \leqq t_1 \leqq t_2 \leqq \cdots \leqq t_{\gamma_i} \leqq \alpha_i\} .$$

We use the convention that $D_i(\mathscr{M}) = \{(\varnothing)\}$ *where* $(\varnothing)$ *is the empty vector, when* $\gamma_i = 0$.

(b)

$$\begin{aligned} D_{p+1}(\mathscr{M}) = \{&(t_1, \cdots, t_{\gamma_{p+1}}) \times (s_1, \cdots, s_{\gamma_{p+1}}) \colon t_i \geqq 0,\ s_i \geqq 0, \\ &s_1 + \cdots + s_{\gamma_{p+1}} = m,\ t_i + s_i \leqq t_{i+1}\ (i = 1, \cdots, \gamma_{p+1} - 1) \\ &\textit{and}\ t_{\gamma_{p+1}} + s_{\gamma_{p+1}} = \alpha_{p+1}\} . \end{aligned}$$

(c)

$$g(\mathscr{M}) = \mathop{\times}_{i=1}^{p+1} D_i(\mathscr{M}) .$$

PROOF. The lemma is a reformulation of Lemma 4.1(c).

Instead of $\psi \colon \mathscr{M} \to \mathscr{M}$ we will later use the corresponding map on $g(\mathscr{M})$. That is; we will find a map $\hat{\psi}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{M} & \xrightarrow{\ g\ } & g(\mathscr{M}) \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\hat{\psi}} \\ \mathscr{M} & \xrightarrow[\ g\ ]{} & g(\mathscr{M}) . \end{array}$$

$\hat{\psi}$ will be defined implicitly in Lemmas 3.3 and 3.4. We do not need an explicit definition of $\hat{\psi}$.

The next two lemmas describe how we calculate $D_i(\psi(A))$ from $D_i(A)$.

LEMMA 3.3. (a) *Suppose* $A \in \mathscr{M}$ *and* $\gamma_{p+1} = 1$. *We define* $r_p, \cdots, r_1$ *and* $\beta_p, \cdots, \beta_1$ *inductively in the following way*:

$$\begin{aligned} \beta_p &= 1 \\ r_p &= \text{the number of } p\text{-blocks } B \text{ in } A \text{ such that } d(B) \leqq \beta_p . \\ &\vdots \\ \beta_i &= (p + 1 - i) + 2r_{i+1} + 4r_{i+2} + 6r_{i+3} + \cdots + 2(p - i)r_p \\ r_i &= \text{the number of } i\text{-blocks } B \text{ in } A \text{ such that } d(B) \leqq \beta_i . \\ &\vdots \end{aligned}$$

Suppose $1 \leqq i \leqq p$ *and* $D_i(A) = (t_1, \cdots, t_{\gamma_i})$. *Then we have*

$$D_i(\psi(A)) = (t'_{r_i+1}, \cdots, t'_{\gamma_i}, t'_1, \cdots, t'_{r_i})$$

where

$$t'_j = \begin{cases} t_j + \alpha_i - \beta_i & \text{if } \ j \leqq r_i \\ t_j - \beta_i & \text{otherwise} . \end{cases}$$

Moreover, $D_{p+1}(\psi(A)) = D_{p+1}(A)$ *and* $0 \leqq \beta_i \leqq \alpha_i$ *for* $1 \leqq i \leqq p$ *and*

$$\text{Index}\,(A) = (n + p + 1) + 2r_1 + 4r_2 + \cdots + 2\cdot p\cdot r_p .$$

We also write $r_i(A) = r_i$ *and* $\beta_i(A) = \beta_i$.

PROOF. (a) $\varphi(A)$ in Lemma 4.11 in [2] is equal to $\psi(A)$. By Lemma 4.11(b) and (d) in [2] $\beta_i = x_i(A)$ and $r_i = r_i$ where $x_i(A)$ and r_i are used in Lemma 4.11. Then it is not difficult to see that this lemma is a reformulation of Lemma 4.11 in [2]. □

LEMMA 3.4. (a) *Suppose* $A \in \mathscr{M}$ *and* $\gamma_{p+1} > 1$. *We define* $r_{p+1}, \cdots, r_1$ *and* $\beta_{p+1}, \cdots, \beta_1$ *inductively in the following way*:

$$\begin{aligned} \beta_{p+1} &= d(B) + f(B) - (p + 1) \text{ where } B \text{ is the first } (p + 1)\text{-block in } A . \\ r_{p+1} &= 1 \\ \beta_p &= \beta_{p+1} + 2r_{p+1} \\ r_p &= \text{the number of } p\text{-blocks } B \text{ in } A \text{ such that } d(B) \leqq \beta_p . \\ &\vdots \\ \beta_i &= \beta_{p+1} + 2r_{i+1} + 4r_{i+2} + \cdots + 2(p + 1 - i)r_{p+1} \\ r_i &= \text{the number of } i\text{-blocks in } A \text{ such that } d(B) \leqq \beta_i . \\ &\vdots \end{aligned}$$

Suppose $1 \leqq i \leqq p$ *and* $D_i(A) = (t_1, \cdots, t_{r_i})$. *Then we have*

$$D_i(\psi(A)) = (t'_{r_i+1}, \cdots, t'_{r_i}, t'_1, \cdots, t'_{r_i})$$

where

$$t'_j = \begin{cases} t_j + \alpha_i - \beta_i & \text{if } j \leqq r_i \\ t_j - \beta_i & \text{otherwise} . \end{cases}$$

Suppose $D_{p+1}(A) = (t_1, \cdots, t_{r_{p+1}}) \times (s_1, \cdots, s_{r_{p+1}})$. *Then we have*

$$D_{p+1}(\psi(A)) = (t'_2, t'_3, \cdots, t'_{r_{p+1}}, t'_1) \times (s_2, \cdots, s_{r_{p+1}}, s_1)$$

where

$$t'_j = \begin{cases} t_j - \beta_{p+1} & \text{if } j \geqq 2 \\ t_1 + \alpha_{p+1} - \beta_{p+1} = \alpha_{p+1} - s_1 & \text{if } j = 1 . \end{cases}$$

Moreover, we have $0 < \beta_i < \alpha_i$ *for* $1 \leqq i \leqq p$ *and*

$$\text{Index}\,(A) = \beta_{p+1} + 2r_1 + 4r_2 + \cdots + 2(p+1)r_{p+1} .$$

We also write $r_i(A) = r_i$ *and* $\beta_i(A) = \beta_i$.

PROOF. Since ψ is equal to $\varphi_{\min}$ in [2] this is a reformulation of Lemma 4.13 in [2].

We will illustrate this lemma by our example in § 2. We get

$$\begin{array}{lll} \beta_3 = 2 + 1 = 3 & \beta_2 = 3 + 2\cdot 1 = 5 & \beta_1 = 3 + 2\cdot 1 + 4\cdot 1 = 9 \\ r_3 = 1 & r_2 = 1 & r_1 = 3 . \end{array}$$

Since $D_1(A) = (1, 5, 6, 10, 11, 15)$ and $\alpha_1 = 15$ we get

$$\begin{aligned} D_1(\psi(A)) &= (10 - \beta_1, 11 - \beta_1, 15 - \beta_1, 1 + \alpha_1 - \beta_1, 5 + \alpha_1 - \beta_1, 6 + \alpha_1 - \beta_1) \\ &= (1, 2, 6, 7, 11, 12) . \end{aligned}$$

Since $D_2(A) = (3, 7)$ and $\alpha_2 = 8$ we get

$$D_2(\psi(A)) = (7 - \beta_2, 3 + \alpha_2 - \beta_2) = (2, 6) .$$

Since $D_3(A) = (2, 4) \times (1, 1)$ and $\alpha_3 = 5$ we get

$$D_3(\psi(A)) = (4 - \beta_3, 2 + \alpha_3 - \beta_3) \times (1, 1) = (1, 4) \times (1, 1) .$$

In our forthcoming proofs we need not know what $\psi(A)$ looks like. But, if we want, we can successively construct

$$K_3 = K_3(\psi(A)) \longrightarrow K_2 = K_2(\psi(A)) \longrightarrow K_1(\psi(A)) = \psi(A)$$

as in the proof of Lemma 4.1 in [2]. We will only sketch this method:

$$K_3 = 01111000001111$$

since K_3 is the unique vector satisfying: K_3 contains only 3-blocks, $D_3(K_3) = D_3(A)$ and the length of $K_3 = n - 2\gamma_1 - 4\gamma_2 = 14$.

By putting in 1100 or 0011 between certain positions in K_3 we get a vector K_2 which only contains 2- and 3-blocks and satisfies: $D_i(K_2) = D_i(A)$ for $i = 2, 3$ and the length of $K_2 = n - 2\gamma_1 = 22$. we get

$$K_2 = 0111001110000011001111 .$$

By putting in 10 or 01 between certain positions in K_2 we finally get:

$$\psi(A) = K_3 = 01011011001110100100001101001011111 .$$

Next we will determine q such that $D_j(\psi^q(A)) = D_j(A)$. To do this we must be able to determine $D_j(\psi^q(A))$ directly from $D_j(A)$. We will develop a method in Lemma 3.6. First we need more notation.

DEFINITION 3.5. When it is clear which $A \in \{0, 1\}^n$ we are working with, we define $(s = 0, 1, 2, \cdots)$

$$\beta_j(s) = \beta_j(\psi^s(A)) \qquad \text{and} \qquad r_j(s) = r_j(\psi^s(A))$$
$$\mathscr{B}_j(s) = \beta_j(0) + \cdots + \beta_j(s-1) \qquad \text{and} \qquad \mathscr{R}_j(s) = r_j(0) + \cdots + r_j(s-1) .$$

LEMMA 3.6. *Suppose* $A \in \mathscr{M}$, $1 \leqq j \leqq p$ *and* $D_j(A) = (t_1, \cdots, t_{\gamma_j})$. *Then we determine* $D_j(\psi^s(A))$ *in the following way:*

We determine integers f *and* β^* *such that*

$$\mathscr{B}_j(s) = f \cdot \alpha_j + \beta^* \qquad \textit{and} \qquad 0 \leqq \beta^* < \alpha_j .$$

We let r^* = *the number of coordinates* t_i *in* $D_j(A)$ *such that* $t_i \leqq \beta^*$.

Then we have

$$D_j(\psi^s(A)) = (t'_{r^*+1}, \cdots, t'_{\gamma_j}, t'_1, \cdots, t'_{r^*}) \qquad \textit{where}$$

$$t'_i = \begin{cases} t_i + \alpha_j - \beta^* & \textit{when} \quad 1 \leqq i \leqq r^* \\ t_i - \beta^* & \textit{when} \quad i > r^* . \end{cases}$$

(*If* $r^* = \gamma_j$, *then* $D_j(\psi^s(A)) = (t'_1, \cdots, t'_{\gamma_j})$.) *Moreover,* $\mathscr{R}_j(s) = f \cdot \gamma_j + r^*$.

PROOF. We suppose the lemma is true for s, and we will prove that it is true for $(s + 1)$. We write

$$D_j(\psi^s(A)) = (u_1, \cdots, u_{\gamma_j}) .$$

By Lemma 3.3 or Lemma 3.4 we have $(\beta^{**} = \beta_j(s)$ and $r^{**} = r_j(s))$

$$D_j(\psi^{s+1}(A)) = (u'_{r^{**}+1}, \cdots, u'_{\gamma_j}, u'_1, \cdots, u'_{r^{**}}) \qquad \text{where}$$
$$u'_i = \begin{cases} u_i + \alpha_j - \beta^{**} & \text{for} \quad 1 \leqq i \leqq r^{**} \\ u_i - \beta^{**} & \text{for} \quad i > r^{**} . \end{cases}$$

We suppose $\beta^* + \beta^{**} \geqq \alpha_j$ (the case $\beta^* + \beta^{**} < \alpha_j$ is treated analogously). We observe

$$t'_{\gamma_j} = t_{\gamma_j} - \beta^* \leqq \alpha_j - \beta^* \leqq \beta^{**} .$$

Hence we get

$$D_j(\psi^s(A)) = (\underbrace{t'_{r^*+1}, \cdots, t'_{\gamma_j}, t'_1, \cdots, t'_v}, t'_{v+1}, \cdots, t'_{r^*})$$
$$= (\underbrace{u_1, \quad \cdots \qquad \cdots \quad u_{r^{**}}}, u_{r^{**}+1}, \cdots)$$

and

$$D_j(\psi^{s+1}(A)) = (t''_{v+1}, \cdots, t''_{\gamma j}, t''_1, \cdots, t''_v) \qquad \text{where}$$
$$t''_i = \begin{cases} t_i + \alpha_j - (\beta^* + \beta^{**} - \alpha_j) & \text{if} \quad 1 \leqq i \leqq v \\ t_i - (\beta^* + \beta^{**} - \alpha_j) & \text{if} \quad i > v . \end{cases}$$

(For example, if $1 \leqq i \leqq v$ we get: $t''_i = t'_i + \alpha_j - \beta^{**} = (t_i + \alpha_j - \beta^*) + \alpha_j - \beta^{**} = t_i + \alpha_j - (\beta^* + \beta^{**} - \alpha_j)$).

Now we will prove that this is in accordance with the lemma:

$$\mathscr{B}_j(s + 1) = f\alpha_j + \beta^* + \beta^{**} = (f + 1)\alpha_j + (\beta^* + \beta^{**} - \alpha_j) .$$

If $1 \leqq i \leqq v$, then we have

$$t_i = (t_i + \alpha_j - \beta^*) + \beta^* - \alpha_j = t'_i + \beta^* - \alpha_j \leqq \beta^{**} + \beta^* - \alpha_j .$$

If $v < i \leqq r^*$, then we have

$$t_i = (t_i + \alpha_j - \beta^*) + \beta^* - \alpha_j = t'_i + \beta^* - \alpha_j > \beta^{**} + \beta^* - \alpha_j .$$

If $v > r^*$, then we have

$$t_i > \beta^* \geqq \beta^* + \beta^{**} - \alpha_j .$$

Hence, $v =$ the number of coordinates t_i in $D_j(A)$ such that $t_i \leqq \beta^* + \beta^{**} - \alpha_j$.

We observe $v = r^* + r^{**} - \gamma_j$. Hence,

$$\mathscr{R}_j(s + 1) = \mathscr{R}_j(s) + r^{**} = f \cdot \gamma_j + r^* + r^{**} = (f + 1) \cdot \gamma_j + v$$

and the proof is complete.

Now we return to our example. We divide the treatment into 5 steps:

Step 1. We have $D_2(A) = (3, 7)$ and $\alpha_2 = 8$. If $\beta^* = 0, 1, 2, \cdots, 7$

respectively in Lemma 3.6 we get that $D_2(\psi^s(A))$ is equal to (3, 7), (2, 6), (1, 5), (4, 8), (3, 7), (2, 6), (1, 5), (4, 8) respectively. Hence, $\beta^* = 0$ or 4 gives $D_2(\psi^s(A)) = (3.7)$ and therefore

$$D_2(\psi^s(A)) = D_2(A) \Longleftrightarrow \mathscr{B}_2(s) \text{ is a multiple of } 4 . \tag{3.8}$$

Step 2. In the same way as in Step 1 we get

$$D_1(\psi^s(A)) = D_1(A) \Longleftrightarrow \mathscr{B}_1(s) \text{ is a multiple of } 5 . \tag{3.9}$$

Step 3. By using Lemma 3.4 we get

$$\begin{aligned} D_3(A) &= (2, 4) \times (1, 1) & \beta_3(A) &= 3 & r_3(A) &= 1 \\ D_3(\psi(A)) &= (1, 4) \times (1, 1) & \beta_3(\psi(A)) &= 2 & r_3(\psi(A)) &= 1 \\ D_3(\psi^2(A)) &= (2, 4) \times (1, 1) . \end{aligned}$$

Hence, we get $D_3(A) = D_3(\psi^2(A)) = D_3(\psi^4(A)) = \cdots$ and

$$\begin{aligned} &\mathscr{B}_3(2) = 5 , \quad \mathscr{B}_3(4) = 10, \cdots , \quad \mathscr{B}_3(2 \cdot X_3) = 5 \cdot X_3, \cdots \\ &\mathscr{R}_3(2) = 2 , \quad \mathscr{R}_3(4) = 4, \cdots , \quad \mathscr{R}_3(2 \cdot X_3) = 2 \cdot X_3, \cdots \end{aligned}$$

where X_3 is an integer.

Step 4. We will determine Y such that $D_i(\psi^Y(A)) = D_i(A)$ for $i = 2, 3$. By Step 3

$$Y = 2 \cdot X_3 \quad \text{for an integer} \quad X_3 .$$

By Lemma 3.4 and Step 3

$$\begin{aligned} \mathscr{B}_2(Y) &= \sum_{s=0}^{Y-1} \beta_3(s) + 2r_3(s) = \mathscr{B}_3(Y) + 2 \mathscr{R}_3(Y) \\ &= \mathscr{B}_3(2X_3) + 2 \mathscr{R}_3(2X_3) = 5X_3 + 4X_3 = 9X_3 . \end{aligned}$$

By (3.8) $\mathscr{B}_2(Y)$ must be a multiple of 4. Hence, the possible values of X_3 and $Y = 2 \cdot X_3$ are

$$X_3 = 4, 8, 12, \cdots \quad \text{and} \quad Y = 8, 16, 24, \cdots .$$

Direct calculation gives us

$$\mathscr{R}_2(8) = 9 , \quad \mathscr{R}_2(16) = 18 , \quad \mathscr{R}_2(24) = 27 , \quad \text{etc.}$$

Later, of course, we must do this in a more sofisticated way. But at the present stage, this will obscure the ideas.

Step 5. We will determine Y such that $D_i(\psi^Y(A)) = D_i(A)$ for $i = 1, 2, 3$. The possible values of Y are $Y = 8, 16, 24, \cdots$. By Lemma 3.4 we have

$$\mathscr{B}_1(Y) = \sum_{s=0}^{Y-1} \beta_3(s) + 2r_2(s) + 4r_3(s) = \mathscr{B}_3(Y) + 2\mathscr{R}_2(Y) + 4\mathscr{R}_3(Y) .$$

Hence, by Step 3 and Step 4 we get

$$\mathscr{B}_1(8) = \mathscr{B}_3(8) + 2\mathscr{R}_2(8) + 4\mathscr{R}_3(8) = 20 + 18 + 32 = 70$$

which is a multiple of 5. Hence $Y = 8$ is the least Y such that $\psi^Y(A) = A$.

Now I will try to sketch thoroughly the ideas on the case $S = E_k + E_{k+1} + E_{k+2}$. Instead I will delete the general proof of how the minimal periods are determined. We suppose $A \in \mathscr{M}$, $\gamma_{p+1} > 1$ and again we divide the treatment of A into 5 steps.

Step 1. Suppose $D_2(A) = (t_1, \cdots, t_{\gamma_2})$. We will find a formula similar to (3.8). To do this we define Λ_2 in the following way:

If $t_1 = \cdots = t_r = 1$ and $t_{r+1} > 1$ we define $\Lambda_2(t_1, \cdots, t_r, \cdots, t_{\gamma_2}) = (t_{r+1} - 1, \cdots, t_{\gamma_2} - 1, t_1', \cdots, t_r')$ where $t_1' = \cdots = t_r' = \alpha_2$.

By Lemma 3.4 we get

$$\begin{aligned} D_2(\psi(A)) &= \Lambda_2^{\beta_2(A)}(D_2(A)) \\ D_2(\psi^2(A)) &= \Lambda_2^{\beta_2(A)+\beta_2(\psi(A))}(D_2(A)) = \Lambda_2^{\mathscr{B}_2(2)}(D_2(A)) \\ &\ \ \vdots \\ D_2(\psi^s(A)) &= \quad \cdots \quad = \Lambda_2^{\mathscr{B}_2(s)}(D_2(A)) . \end{aligned}$$

The next problem is to determine when $\Lambda_2^\alpha(D_2(A)) = D_2(A)$. First we observe that this is true for $\alpha = \alpha_2$. Next we let α be the least α such that $\Lambda_2^\alpha(D_2(A)) = D_2(A)$. We will now describe how $D_2(A)$ looks in this case. We must have $\alpha_2 = r\alpha$ for an integer r. We let γ be the maximum integer such that $t_\gamma \leqq \alpha$. By definition of Λ_2^α we get

$$\begin{aligned} \Lambda_2^\alpha(D_2(A)) &= (t_{\gamma+1} - \alpha, \cdots, t_{\gamma_2} - \alpha, t_1 + \alpha_2 - \alpha, \cdots, t_\gamma + \alpha_2 - \alpha) \\ &= D_2(A) . \end{aligned}$$

Now we get obviously that $D_2(A)$ must have the form

$$D_2(A) = (\underbrace{t_1, \cdots, t_\gamma}_{\text{Part 1}}, \underbrace{t_1 + \alpha, \cdots, t_\gamma + \alpha}_{\text{Part 2}}, \cdots, \tag{3.10}$$
$$\underbrace{t_1 + (r-1)\alpha, \cdots, t_\gamma + (r-1)\alpha}_{\text{Part } r})$$

where $\alpha_2 = r\alpha$.

Now we will prove that (3.10) is a sufficient condition. Therefore we suppose (3.10) is true. Then we get by Lemma 3.2 that

$$t_{\gamma_2} = t_\gamma + (r-1)\alpha \leqq \alpha_2 \quad \text{and} \quad t_1 > 0 .$$

Hence

$$t_\gamma \leqq \alpha \quad \text{and} \quad t_{\gamma+1} > \alpha .$$

Hence, $\Lambda^\alpha(D_2(A)) = D_2(A)$.

We let α_2^* be the least α such that $\Lambda^\alpha(D_2(A)) = D_2(A)$. We get

$$D_2(\psi^s(A)) = D_2(A) \Longleftrightarrow \mathscr{B}_2(s) = X_2\alpha_2^* \quad \text{for an integer} \quad X_2 .$$

Moreover, if $\mathscr{B}_2(s) = X_2\alpha_2^*$, then

$$\mathscr{R}_2(s) = X_2\gamma_2^* \quad \text{where} \quad \gamma_2^* = \frac{\alpha_2^*}{\alpha_2}\gamma_2 . \tag{3.11}$$

We prove (3.11) as follows: If $0 \leqq z < r$, then by (3.10) the number of coordinates less than or equal to $z\cdot\alpha_2^*$ is $z\cdot\gamma_2^*$. We suppose $\mathscr{B}_2(s) = (wr + z)\alpha_2^* = w\alpha_2 + z\cdot\alpha_2^*$ where $0 \leqq z < r$. By Lemma 3.6 we get

$$\mathscr{R}_2(s) = w\gamma_2 + z\gamma_2^* = (wr + z)\gamma_2^*$$

and the proof of (3.11) is complete.

Step 2. Suppose $D_1(A) = (t_1, \cdots, t_{\gamma_1})$. Analoguosly with Step 1 we define Λ_1 in the following way:

If $t_1 = \cdots = t_r = 1$ and $t_{r+1} > 1$ we define $\Lambda_1(t_1, \cdots, t_{\gamma_1}) = (t_{r+1} - 1, t_{r+2} - 1, \cdots, t_{\gamma_1} - 1, t_1', \cdots, t_r')$ where $t_1' = \cdots = t_r' = \alpha_1$.

We let α_1^* be the least integer such that $\Lambda_1^{\alpha_1^*}(D_1(A)) = D_1(A)$. Analogously with Step 1 we get

$$D_1(\psi^s(A)) = D_1(A) \Longleftrightarrow \mathscr{B}_1(s) = X_1\alpha_1^* \quad \text{for an integer} \quad X_1$$

and

$$\text{If } \mathscr{B}_1(s) = X_1\alpha_1^*, \text{ then } \mathscr{R}_1(s) = X_1\gamma_1^* \text{ where } \gamma_1^* = \frac{\alpha_1^*}{\alpha_1}\gamma_1 .$$

Step 3. Suppose $D_3(A) = (t_1, \cdots, t_{\gamma_3}) \times (s_1, \cdots, s_{\gamma_3})$. Now we will determine when $D_3(\psi^q(A)) = D_3(A)$. Again we define a function Λ_3 in the following way:

$$\Lambda_3(t_1, \cdots, t_{\gamma_3}) \times (s_1, s_2, \cdots, s_{\gamma_3}) = (t_2', \cdots, t_{\gamma_3}', t_1') \times (s_2, \cdots, s_{\gamma_3}, s_1)$$

where

$$t_i' = \begin{cases} t_1 + \alpha_3 - (s_1 + t_1) = \alpha_3 - s_1 & \text{for} \quad i = 1 \\ t_i - (s_1 + t_1) & \text{for} \quad i = 2, 3, \cdots, \gamma_3 . \end{cases}$$

We observe by Lemma 3.4 that

$$D_3(\psi(A)) = \Lambda_3(D_3(A)), \cdots, D_3(\psi^q(A)) = \Lambda_3^q(D_3(A)), \cdots .$$

By definition of Λ_3 we have for $1 \leqq q \leqq \gamma_3$ that

$$(3.12)\quad \begin{cases} \Lambda_3^q(t_1, \cdots, t_{\gamma_3}) \times (s_1, \cdots, s_{\gamma_3}) \\ \qquad = (t''_{q+1}, \cdots, t''_{\gamma_3}, t''_1, \cdots, t''_q) \times (s_{q+1}, \cdots, s_{\gamma_3}, s_1, \cdots, s_q) \\ \text{where} \\ \qquad t''_i = \begin{cases} t_i + \alpha_3 - (s_q + t_q) & \text{for} \quad i = 1, \cdots, q \\ t_i - (s_q + t_q) & \text{for} \quad i = q + 1, \cdots . \end{cases} \end{cases}$$

For example if $q = 2$ and $i > 2$ we get

$$\begin{aligned} t''_i &= t'_i - (s_2 + t'_2) = t_i - (s_1 + t_1) - s_2 - (t_2 - (s_1 + t_1)) \\ &= t_i - (s_2 + t_2) . \end{aligned}$$

Specially, if $q = \gamma_3$ we get ($s_{\gamma_3} + t_{\gamma_3} = \alpha_3$ by Lemma 3.2)

$$t''_i = t_i + \alpha_3 - (s_{\gamma_3} + t_{\gamma_3}) = t_i \quad \text{for} \quad i = 1, \cdots, \gamma_3 .$$

Hence, $\Lambda^{\gamma_3}(D_3(A)) = D_3(A)$.
If $D_3(A) = (t_1, \cdots, t_{\gamma_3}) \times (s_1, \cdots, s_{\gamma_3})$ and $1 \leqq q \leqq \gamma_3$, we have by Lemma 3.4 that

$$D_3(\psi^q(A)) = (t''_{q+1}, \cdots, t''_{\gamma_3}, t''_1, \cdots) \times (s_{q+1}, \cdots, s_{\gamma_3}, s_1, \cdots, s_q)$$

where

$$t''_i = \begin{cases} t_i + \alpha_3 - (\beta_3(0) + \cdots + \beta_3(q-1)) \\ \qquad = t_i + \alpha_3 - \mathscr{B}_3(q) \quad \text{for} \quad 1 \leqq i \leqq q \\ t_i - (\beta_3(0) + \cdots + \beta_3(q-1)) \\ \qquad = t_i - \mathscr{B}_3(q) \qquad \text{for} \quad i > q . \end{cases}$$

Hence,

$$(3.13)\qquad \mathscr{B}_3(q) = s_q + t_q \quad \text{for} \quad 1 \leqq q \leqq \gamma_3 .$$

The next problem is to determine when $\Lambda^\gamma(D_3(A)) = D_3(A)$. Next we suppose γ is the least integer such that $\Lambda^\gamma(D_3(A)) = D_3(A)$. Then we have $\gamma_3 = r\gamma$ for an integer r, and by (3.12) we get that $D_3(A)$ has the form

$$(3.14)\quad \begin{aligned} D_3(A) = (&\underbrace{t_1, \cdots, t_\gamma}_{\text{Part 1}}, \underbrace{t_1 + \alpha, \cdots, t_\gamma + \alpha}_{\text{Part 2}}, \cdots, \\ &\underbrace{t_1 + (r-1)\alpha, \cdots, t_\gamma + (r-1)\alpha}_{\text{Part } r}) \\ &\times (\underbrace{s_1, \cdots, s_\gamma}_{\text{Part 1}}, \underbrace{s_1, \cdots, s_\gamma}_{\text{Part 2}}, \cdots, \underbrace{s_1, \cdots, s_\gamma}_{\text{Part } r}) \end{aligned}$$

where $\alpha r = \alpha_3$ (which is equivalent to $\alpha = s_\gamma + t_\gamma$). (We get directly from (3.12) that (3.14) is true with $\alpha = s_\gamma + t_\gamma$. But this is equivalent to $\alpha r = \alpha_3$ because $s_{\gamma_3} + t_{\gamma_3} = (s_\gamma + t_\gamma) + (r-1)\alpha = \alpha_3$ by Lemma 3.2.)

We let γ_3^* be the least integer γ such that $\Lambda_3^\gamma(D_3(A)) = D_3(A)$. Then we have

$$D_3(\psi^Y(A)) = D_3(A) \Longleftrightarrow Y = X_3\gamma_3^* \quad \text{for an integer} \quad X_3 \,.$$

Moreover, if $Y = X_3\gamma_3^*$, then

$$\mathscr{B}_3(Y) = X_3\alpha_3^* \quad \text{where} \quad \alpha_3^* = \frac{\gamma_3^*}{\gamma_3}\alpha_3 \,. \tag{3.15}$$

We prove (3.15) as follows: By (3.13) and (3.14) we have

$$\mathscr{B}_3(q\cdot\gamma_3^*) = t_{q\cdot\gamma_3^*} + s_{q\cdot\gamma_3^*} = q\alpha_3^* \quad \text{for} \quad 0 \leqq q < r \,,$$

where $r = \gamma_3/\gamma_3^*$, and

$$\mathscr{B}_3(r\gamma_3^*) = \mathscr{B}_3(\gamma_3) = s_{\gamma_3} + t_{\gamma_3} = \alpha_3 = r\alpha_3^* \,,$$

and (3.15) follows.

Step 4. Next, we will determine Y such that $D_i(\psi^Y(A)) = D_i(A)$ for $i = 2, 3$. By Step 3 we must have $Y = X_3\cdot\gamma_3^*$. Moreover in this case

$$\mathscr{B}_2(Y) = \mathscr{B}_3(Y) + 2\mathscr{R}_3(Y) = X_3\alpha_3^* + 2X_3\gamma_3^* \,.$$

Moreover, by Step 1, we must have

$$\mathscr{B}_2(Y) = X_2\alpha_2^* \quad \text{for an integer} \quad X_2 \,.$$

Hence, we get the equation $X_2\alpha_2^* = X_3\alpha_3^* + 2X_3\gamma_3^*$.

Step 5. Next, we will determine Y such that $D_i(\psi^Y(A)) = D_i(A)$ for $i = 1, 2, 3$. By Step 2 this is true for $i = 2, 3$ if and only if there exist integers X_2 and X_3 such that $X_2\alpha_2^* = X_3\alpha_3^* + 2X_3\gamma_3^*$ and $Y = X_3\gamma_3^*$. Moreover by the previous steps we have

$$\mathscr{B}_3(Y) = X_3\alpha_3^* \,, \qquad \mathscr{R}_3(Y) = X_3\gamma_3^* \,, \qquad \mathscr{B}_2(Y) = X_2\alpha_2^* \qquad \text{and}$$
$$\mathscr{R}_2(Y) = X_2\gamma_2^* \,.$$

Hence,

$$\mathscr{B}_1(Y) = \mathscr{B}_3(Y) + 2\mathscr{R}_2(Y) + 4\mathscr{R}_3(Y) = X_3\alpha_3^* + 2X_2\gamma_2^* + 4X_3\gamma_3^* \,.$$

Moreover, by Step 2 we must have

$$\mathscr{B}_1(Y) = X_1\alpha_1^* \quad \text{for an integer} \quad X_1 \,.$$

Hence, we get the equation

$$X_1\alpha_1^* = X_3\alpha_3^* + 2X_2\gamma_2^* + 4X_3\gamma_3^* .$$

Conclusion. $\psi^Y(A) = A \Leftrightarrow D_i(\psi^Y(A)) = D_i(A) \quad i = 1, 2, 3 \Leftrightarrow$ There exists integers X_1, X_2 and X_3 such that

$$\begin{aligned} X_2\alpha_2^* &= X_3\alpha_3^* + 2X_3\gamma_3^* \\ X_1\alpha_1^* &= X_3\alpha_3^* + 2X_2\gamma_2^* + 4X_3\gamma_3^* \\ Y &= X_3\gamma_3^* . \end{aligned}$$

Let X_1, X_2, X_3 be the least integral solution. Then ($\mathscr{R}_1(Y) = X_1\gamma_1^*$ follows from Step 2)

$$\begin{aligned} \sum_{s=0}^{Y-1} \text{Index}\,(\psi^s(A)) &= \sum_{s=0}^{Y-1} \beta_3(s) + 2r_1(s) + 4r_2(s) + 6r_3(s) \\ &= \mathscr{B}_3(Y) + 2\mathscr{R}_1(Y) + 4\mathscr{R}_2(Y) + 6\mathscr{R}_3(Y) \\ &= X_3\alpha_3^* + 2X_1\gamma_1^* + 4X_2\gamma_2^* + 6X_3\gamma_3^* \end{aligned}$$

which is the minimal period of A.

If $A \in \mathscr{M}$ and $\gamma_{p+1} = 1$ we must use Lemma 3.3 instead of Lemma 3.4. Then we have always $D_3(\psi(A)) = D_3(A)$. Hence, we need only to modify Steps 4 and 5 as follows.

Step 4. By Lemma 3.3 we get $\mathscr{B}_2(Y) = Y$. We must have $\mathscr{B}_2(Y) = Y = X_2\alpha_2^*$ for an integer X_2. In this case $\mathscr{R}_2(Y) = X_2\gamma_2^*$.

Step 5. By Lemma 3.3 we get

$$\mathscr{B}_1(Y) = \sum_{s=0}^{Y-1} (2 + 2r_2(s)) = 2Y + 2\mathscr{R}_2(Y) = 2Y + 2X_2\gamma_2^* .$$

We must have $\mathscr{B}_1(Y) = 2Y + 2X_2\gamma_2^* = X_1\alpha_1^*$ for an integer X_1. In this case $\mathscr{R}_1(Y) = X_1\gamma_1^*$.

Conclusion. $A = \psi^Y(A) \Leftrightarrow$ There exist integers X_1 and X_2 such that $X_2\alpha_2^* = Y$ and $X_1\alpha_1^* = 2Y + 2X_2\gamma_2^*$. Suppose X_1, X_2 is the least solution. Then we get

$$\begin{aligned} \sum_{s=0}^{Y-1} \text{Index}\,(\psi^s(A)) &= \sum_{s=0}^{Y-1} [(n+3) + 2r_1(s) + 4r_2(s)] \\ &= Y(n+3) + 2\mathscr{R}_1(Y) + 4\mathscr{R}_2(Y) \\ &= Y(n+3) + 2X_1\gamma_1^* + 4X_2\gamma_2^* \end{aligned}$$

which is the minimal period.

4. The minimal periods. Now I will formulate the results from § 3 for a general p and very roughly sketch the proof. As before

$$A \in \mathscr{M} \Longleftrightarrow \begin{cases} w(A) = k + p + 1 \\ A \text{ starts with 0 or a } (p+1)\text{-block} \\ A \text{ contains } \gamma_i \text{ } i\text{-blocks for } i = 1, \cdots, p+1 \\ A \text{ ends with a } (p+1)\text{-block} . \end{cases}$$

The blocks in A are determined with respect to p. $D_i(A)$ $(i = 1, \cdots, p + 1)$ is defined in Definition 3.1.

DEFINITION 4.1. Let $A \in \mathscr{M}$ be given.

(a) Suppose $1 \leqq j \leqq p$ and $D_j(A) = (t_1, \cdots, t_{\gamma_j})$. We define Λ_j in the following way:

If $t_1 = \cdots = t_r = 1$ and $t_{r+1} > 1$ we define
$\Lambda_j(t_1, \cdots, t_{\gamma_j}) = (t_{r+1} - 1, \cdots, t_{\gamma_j} - 1, t'_1, \cdots, t'_r)$
where $t'_1 = \cdots = t'_r = \alpha_j$.

Let α_j^* be the least integer such that

$$\Lambda_j^{\alpha_j^*}(D_j(A)) = D_j(A) .$$

(b) Suppose $D_{p+1}(A) = (t_1, \cdots, t_{\gamma_{p+1}}) \times (s_1, \cdots, s_{\gamma_{p+1}})$. We define Λ_{p+1} in the following way:

$$\Lambda_{p+1}(t_1, \cdots, t_{\gamma_{p+1}}) \times (s_1, \cdots, s_{\gamma_{p+1}}) = (t'_2, \cdots, t'_{\gamma_{p+1}}, t'_1) \times (s_2, \cdots, s_{\gamma_{p+1}}, s_1)$$

where

$$t'_i = \begin{cases} \alpha_{p+1} - s_1 & \text{for } i = 1 \\ t_i - (s_1 + t_1) & \text{for } i > 1 . \end{cases}$$

Let γ_{p+1}^* be the least integer such that

$$\Lambda_{p+1}^{\gamma_{p+1}^*}(D_{p+1}(A)) = D_{p+1}(A) .$$

(c) If $1 \leqq i \leqq p$, we define $\gamma_i^* = \gamma_i \cdot \alpha_i^* / \alpha_i$. Moreover, we define $\alpha_{p+1}^* = \alpha_{p+1} \cdot \gamma_{p+1}^* / \gamma_{p+1}$.

As in the previous section we can prove that γ_i^* $(1 \leqq i \leqq p)$ and α_{p+1}^* are integers.

THEOREM 4.2. *Suppose $A \in \mathscr{M}$. We associate p equations to A in the following way:*

$$\begin{aligned} (p) \qquad & \alpha_p^* \cdot X_p = \alpha_{p+1}^* X_{p+1} + 2\gamma_{p+1}^* X_{p+1} \\ (p-1) \qquad & \alpha_{p-1}^* X_{p-1} = \alpha_{p+1}^* X_{p+1} + 2\gamma_p^* X_p + 4\gamma_{p+1}^* X_{p+1} \\ & \vdots \\ (1) \qquad & \alpha_1^* X_1 = \alpha_{p+1}^* X_{p+1} + 2\gamma_2^* X_2 + 4\gamma_3^* X_3 + \cdots + 2p\gamma_{p+1}^* X_{p+1} . \end{aligned}$$

If $\gamma_i = 0$, *we replace equation* (i) *by* $X_i = 0$. *We let* $X_1, \cdots, X_{p+1}$ *be the least integral solution of the equations.*

Then $X_{p+1}\alpha^*_{p+1} + \sum_{i=1}^{p+1} 2i \cdot \gamma^*_i \cdot X_i$ *is the minimal period of* A *with respect to the shift register* $(x_1, \cdots, x_n) \to (x_2, \cdots, x_{n+1})$ *where*

$$x_{n+1} = x_1 + (E_k + \cdots + E_{k+p})(x_2, \cdots, x_n) .$$

If $\gamma_i = 0$ for $i = 1, \cdots, p$, we observe that the minimal period $= X_{p+1}\alpha^*_{p+1} + 2(p+1)\gamma^*_{p+1}X_{p+1} = \alpha^*_{p+1} + 2(p+1)\gamma^*_{p+1} = (\gamma^*_{p+1}/\gamma_{p+1})(\alpha_{p+1} - 2(p+1)\gamma_{p+1}) = (\gamma^*_{p+1}/\gamma_{p+1})(n + p + 1)$.

The existence of the minimal solution $X_1, \cdots, X_{p+1}$ is proved as indicated in § 3 in [2].

Proof. We only sketch the proof since it is only a generalization of the case $p = 2$ which we treated in § 3.

First we suppose $\gamma_{p+1} > 1$.

We get

$$D_{p+1}(\psi^Y(A)) = D_{p+1}(A) \Longleftrightarrow Y = X_{p+1}\gamma^*_{p+1} \quad \text{for an integer} \quad X_{p+1} .$$

In this case $\mathscr{B}_{p+1}(Y) = X_{p+1}\alpha^*_{p+1}$ and $\mathscr{R}_{p+1}(Y) = X_{p+1}\gamma^*_{p+1}$. If $1 \leqq j \leqq p$ we get (if $\gamma_j \neq 0$)

$$D_j(\psi^Y(A)) = D_j(A) \Longleftrightarrow \mathscr{B}_j(Y) = X_j\alpha^*_j \quad \text{for an integer} \quad X_j .$$

In this case we have $\mathscr{R}_j(Y) = X_j\gamma^*_j$.

Suppose $X_1, \cdots, X_{p+1}$ satisfy the equations. Put $Y = X_{p+1}\gamma^*_{p+1}$. We prove by induction that

$$\mathscr{B}_i(Y) = X_i\alpha^*_i \quad \text{when} \quad \gamma_i \neq 0 \quad \text{and} \quad 1 \leqq i \leqq p . \tag{4.1}$$

Suppose (4.1) is true for $i = p, p-1, \cdots, j+1$. Then we have

$$\begin{aligned}\mathscr{B}_j(Y) &= \mathscr{B}_{p+1}(Y) + 2\mathscr{R}_{j+1}(Y) + \cdots + 2(p+1-j)\mathscr{R}_{p+1}(Y)\\ &= X_{p+1}\alpha^*_{p+1} + 2\gamma^*_{j+1}X_{j+1} + \cdots + 2(p+1-j)\gamma^*_{p+1}X_{p+1} = \alpha^*_j X_j .\end{aligned}$$

Hence (4.1) is true for $j = 1, \cdots, p$. Then we get $\psi^Y(A) = A$ and $\psi^Y(A) = \theta^t(A)$ where

$$\begin{aligned} t &= \mathscr{B}_{p+1}(Y) + 2\mathscr{R}_1(Y) + \cdots + 2(p+1)\mathscr{R}_{p+1}(Y)\\ &= X_{p+1}\alpha^*_{p+1} + \sum_{i=1}^{p+1} 2i \cdot \gamma^*_i \cdot X_i .\end{aligned}$$

Moreover, it is easily seen that all Y such that $\psi^Y(A) = A$ is obtained in this way.

Finally, we suppose $\gamma_{p+1} = 1$ and $\gamma_i \neq 0$ for at least one $i < p+1$. We only sketch the proof since the proof is analogous with the case

$\gamma_{p+1} > 1$. We get

$$\psi^Y(A) = A \Longleftrightarrow \mathscr{B}_i(Y) = X_i \cdot \alpha_i^* \quad \text{when} \quad \gamma_i \neq 0 \quad \text{and} \quad 1 \leqq i \leqq p\ .$$

In the same way as in § 3 (the case $\gamma_{p+1} = 1$) this is equivalent to: $X_1, \cdots, X_p, Y$ satisfy the equations $(1)', \cdots, (p)'$ given by

$$(q)'\begin{cases} X_q \cdot \alpha_q^* = Y(p+1-q) + \sum_{t=q+1}^{p} 2(t-q)X_t\gamma_t^* & \text{if} \quad \gamma_q \neq 0 \\ X_q = 0 & \text{if} \quad \gamma_q = 0\ . \end{cases}$$

Let $X_1, \cdots, X_p, Y$ be the least solution of the equations $(1)', \cdots, (p)'$. Then Y is the least Y such that $\psi^Y(A) = A$. We calculate the minimal period of A in the following way

$$\begin{aligned} \sum_{s=0}^{Y-1}\left[(n+p+1) + 2\sum_{i=1}^{p} i \cdot r_i(s)\right] &= Y(n+p+1) + 2\sum_{i=1}^{p} i \cdot \mathscr{B}_i(Y) \\ &= Y(n+p+1) + 2\sum_{i=1}^{p} i \cdot \gamma_i^* \cdot X_i\ . \end{aligned}$$

The proof will be complete if we can prove the following claim: Suppose $X_1, \cdots, X_{p+1}$ is the least solutions $(1), \cdots, (p)$. Let

$$Y = X_{p+1} \qquad \text{and} \qquad \hat{X}_t = \begin{cases} 0 & \text{if} \quad \gamma_t = 0 \\ X_t - Y \cdot \dfrac{\gamma_t}{\gamma_t^*} & \text{if} \quad \gamma_t \neq 0\ . \end{cases}$$

Then $\hat{X}_1, \cdots, \hat{X}_p, Y$ is the least solution of the equations $(1)', \cdots, (p)'$, and

$$Y(n+p+1) + \sum_{i=1}^{p} 2i \cdot \hat{X}_i \cdot \gamma_i^* = X_{p+1}\alpha_{p+1}^* + \sum_{i=1}^{p+1} 2i \cdot X_i \cdot \gamma_i^*\ .$$

Now we will prove this claim. Since $\gamma_{p+1} = \gamma_{p+1}^* = 1$, then $\alpha_{p+1} = \alpha_{p+1}^*$. We use the definition of α_{p+1} and get

$$\begin{aligned} & X_{p+1}\alpha_{p+1}^* + \sum_{i=1}^{p+1} 2i \cdot X_i \cdot \gamma_i^* \\ &= Y\left(n+p+1-\sum_{i=1}^{p+1} 2i\gamma_i\right) + \sum_{i=1}^{p} 2i\gamma_i^*\left(\hat{X}_i + Y\frac{\gamma_i}{\gamma_i^*}\right) + 2(p+1)\gamma_{p+1}Y \\ &= Y(n+p+1) + \sum_{i=1}^{p} 2i \cdot \gamma_i^* \cdot \hat{X}_i\ . \end{aligned}$$

Next we prove that the following 3 equations are equivalent (we use $\alpha_i^* \cdot \gamma_i/\gamma_i^* = \alpha_i$):

$$\alpha_i^* X_i = X_{p+1}\alpha_{p+1}^* + \sum_{t=i+1}^{p+1} 2(t-i)\gamma_i^* X_i$$

$$\alpha_i^* \hat{X}_i + \alpha_i Y = Y\alpha_{p+1} + \sum_{t=i+1}^{p} 2(t-i)\gamma_i^* \hat{X}_i + Y\sum_{t=i+1}^{p+1} 2(t-i)\gamma_i$$

$$\hat{X}_i\alpha_i^* = Y(p+1-i) + \sum_{t=i+1}^{p} 2(t-i)\gamma_i^*\hat{X}_i + Z$$

where

$$Z = Y\Big(-\alpha_i + \alpha_{p+1} + \sum_{t=i+1}^{p+1} 2(t-i)\gamma_i + i - (p+1)\Big).$$

$Z = 0$ follows from the definition of α_{p+1} and α_i. Hence, the proof of the claim is complete.

Finally we will include an alternative way to determine α_i^* and γ_i^*:

PROPOSITION 4.3. *Let* $A \in \mathscr{M}$.

(a) *Suppose* $1 \leqq j \leqq p$. *We define the map* ρ_j *in the following way: If* $D_j(A) = (t_1, \cdots, t_{\gamma_j})$, *then*

$$\rho_j(D_j(A)) = (d_1, \cdots, d_{\gamma_j})$$

where

$$d_i = \begin{cases} t_1 + \alpha_j - t_{\gamma_j} & \text{for} \quad i = 1 \\ t_{i+1} - t_i & \text{for} \quad i > 1 . \end{cases}$$

Then γ_j^* *is the cycle period of* $(d_1, \cdots, d_{\gamma_j})$, *that is;* γ_j^* *is the least integer such that*

$$(d_{\gamma_j^*+1}, \cdots, d_{\gamma_j}, d_1, \cdots, d_{\gamma_j^*}) = (d_1, \cdots, d_{\gamma_j}) .$$

(b) *Suppose* $D_{p+1}(A) = (t_1, \cdots, t_{\gamma_{p+1}}) \times (s_1, \cdots, s_{\gamma_{p+1}})$. *Then we define*

$$\eta_{p+1}(D_{p+1}(A)) = (d_1, \cdots, d_{\gamma_{p+1}}) \times (s_1, \cdots, s_{\gamma_p+1})$$

where

$$d_i = \begin{cases} t_1 + \alpha_{p+1} - (t_{\gamma_{p+1}} + s_{\gamma_{p+1}}) = t_1 & \text{for} \quad i = 1 \\ t_{i+1} - (t_i + s_i) & \text{for} \quad i > 1 . \end{cases}$$

Then γ_{p+1}^* *is the least cycle period of* $(d_1, \cdots, d_{\gamma_{p+1}}) \times (s_1, \cdots, s_{\gamma_{p+1}})$. *That is;* γ_{p+1}^* *is the least integer such that*

$$\begin{aligned}(d_{\gamma_{p+1}^*+1}, \cdots, d_{\gamma_{p+1}}, d_1, \cdots, d_{\gamma_{p+1}^*}) \times (s_{\gamma_{p+1}^*+1}, \cdots, s_{\gamma_{p+1}}, s_1, \cdots, s_{\gamma_{p+1}^*}) \\ = (d_1, \cdots, d_{\gamma_{p+1}}) \times (s_1, \cdots, s_{\gamma_{p+1}}) .\end{aligned}$$

Proof. (a) By (3.10) we have that γ_j^* is the least integer such that $D_j(A)$ has the form

$$D_j(A) = (\underbrace{t_1, \cdots, t_{\gamma_j^*}}_{\text{Part 1}}, \underbrace{t_1 + \alpha_j^*, \cdots, t_{\gamma_j^*} + \alpha_j^*}_{\text{Part 2}}, \cdots, \tag{4.2}$$
$$\underbrace{t_1 + (r-1)\alpha_j^*, \cdots, t_{\gamma_j^*} + (r-1)\alpha_j^*}_{\text{Part } r}) \quad \text{and}$$

$$\alpha_j = r\alpha_j^* .$$

Moreover, this is equivalent to that $\rho_j(D_j(A))$ has the form

$$(4.3)\quad \begin{aligned}&\rho_j(D_j(A)) = (\underbrace{d_1, \cdots, d_{\gamma_j^*}}_{\text{Part 1}}, \underbrace{d_1, \cdots, d_{\gamma_j^*}}_{\text{Part 2}}, \cdots, \underbrace{d_1, \cdots, d_{\gamma_j^*}}_{\text{Part } r}) \quad \text{and}\\ &d_1 + \cdots + d_{\gamma_j^*} = \alpha_j^* .\end{aligned}$$

We indicate how this is proved: Suppose (4.2) is satisfied, then

$$\begin{aligned} d_1 &= t_1 + \alpha_j - t_{\gamma_j} = t_1 + \alpha_j - (t_{\gamma_j^*} + (r-1)\alpha_j^*) \\ &= t_1 + \alpha_j^* - t_{\gamma_j^*} = t_{\gamma_j^*+1} - t_{\gamma_j^*} = d_{\gamma_j^*+1} , \quad \text{etc.} \end{aligned}$$

Suppose (4.3) is satisfied, then

$$t_{\gamma_j^*+1} = \sum_{i=2}^{\gamma_j^*+1} (t_i - t_{i-1}) + t_1 = \sum_{i=2}^{\gamma_j^*+1} d_i + t_1 = \alpha_j^* + t_1 , \quad \text{etc.}$$

Since (4.2) is equivalent to (4.3), (a) follows easily.

(b) We define ρ_j for $j = p + 1$ as in (a). Since (3.14) is analogous with (3.10) we get as in (a) that γ_{p+1}^* is the least common cycle period for $\rho_{p+1}(D_{p+1}(A))$ and $(s_1, \cdots, s_{\gamma_{p+1}})$. This is equivalent with that γ_{p+1}^* is the least cycle period of $\eta_{p+1}(D_{p+1}(A))$.

5. The possible periods. By Theorem 4.2 the minimal periods of $A \in \mathscr{M}$ are completely determined by $(\gamma_1^*, \cdots, \gamma_{p+1}^*)$ since $\alpha_i^* = (\gamma_i^*/\gamma_i)\alpha_i$. We define

$$\begin{aligned}&\mathrm{PER}\,(\gamma_1^*, \cdots, \gamma_{p+1}^*) \\ &\qquad = X_{p+1}\alpha_{p+1}^* + 2X_1\gamma_1^* + 4X_2\gamma_2^* + \cdots + 2(p+1)\gamma_{p+1}^* X_{p+1}\end{aligned}$$

where $X_1, \cdots, X_{p+1}$ is the least solution of the equations corresponding to $(\gamma_1^*, \cdots, \gamma_{p+1}^*)$ in Theorem 4.2. Moreover, we let

$$m = k + p + 1 - \gamma_1 - 2\gamma_2 - \cdots - (p+1)\gamma_{p+1} .$$

THEOREM 5.1. (a) *The possible periods of the elements in* $\mathscr{M}$ *are*:

$$\{\mathrm{PER}\,(\gamma_1^*, \cdots, \gamma_{p+1}^*) : (\gamma_1^*, \cdots, \gamma_{p+1}^*) \text{ corresponds to an } A \in \mathscr{M}\} .$$

(b) *There exists* $A \in \mathscr{M}$ *corresponding to* $(\gamma_1^*, \cdots, \gamma_{p+1}^*)$ *if and only if*

$$\frac{\gamma_i}{\gamma_i^*} \quad (i = 1, \cdots, p+1) , \qquad \alpha_i \cdot \frac{\gamma_i^*}{\gamma_i} \quad (i = 1, \cdots, p+1) \qquad \text{and}$$

$$m \cdot \frac{\gamma_{p+1}^*}{\gamma_{p+1}} \quad \text{are integers.}$$

Proof. (a) is obvious. We let $\rho_1, \cdots, \rho_p, \eta_{p+1}$ be as in Proposition 4.3. By Lemma 3.2 we get easily

$$\rho_1 \times \rho_2 \times \cdots \times \rho_p \times \eta_{p+1}\left\{\bigtimes_{i=1}^{p+1} D_i(A)\colon A \in \mathscr{M}\right\} = \bigtimes_{i=1}^{p+1} \mathscr{N}_i$$

where

$$\begin{aligned}\mathscr{N}_i = \{&(d_1, \cdots, d_{\gamma_i})\colon d_1 > 0,\ d_j \geqq 0\ (j = 2, \cdots, \gamma_i) \text{ and} \\ &d_1 + \cdots + d_{\gamma_i} = \alpha_i\} \quad \text{for} \quad 1 \leqq i \leqq p \qquad \text{and} \\ \mathscr{N}_{p+1} = \{&(d_1, \cdots, d_{\gamma_{p+1}}) \times (s_1, \cdots, s_{\gamma_{p+1}})\colon d_i \geqq 0,\ s_i \geqq 0, \\ &d_1 + \cdots + d_{\gamma_{p+1}} = \alpha_{p+1} - m \text{ and } s_1 + \cdots + s_{\gamma_{p+1}} = m\}\end{aligned}$$

where $m = k + p + 1 - \gamma_1 - 2\gamma_2 - \cdots - (p + 1)\gamma_{p+1}$.

By Proposition 4.3 we get {the possible $(\gamma_1^*, \cdots, \gamma_{p+1}^*)$} is equal to the set

$$\bigtimes_{i=1}^{p+1} \{\text{the cycle periods of elements in } \mathscr{N}_i\} .$$

Finally, we get easily that {the possible cycle periods of elements in $\mathscr{N}_i$} is equal to the set

$$\left\{\gamma_i^*\colon \frac{\gamma_i}{\gamma_i^*} \text{ and } \alpha_i \cdot \frac{\gamma_i^*}{\gamma_i} \text{ are integers}\right\}$$

for $1 \leqq i \leqq p$. Moreover, we get

$$\{\text{the possible cycle periods of elements in } \mathscr{N}_{p+1}\}$$

is equal to the set

$$\left\{\gamma_{p+1}^*\colon \frac{\gamma_{p+1}}{\gamma_{p+1}^*},\ \alpha_{p+1} \cdot \frac{\gamma_{p+1}^*}{\gamma_{p+1}} \text{ and } m \cdot \frac{\gamma_{p+1}^*}{\gamma_{p+1}} \text{ are integers}\right\}$$

and the proof is complete.

6. The number of cycles. In this section we will count the number of cycles $\mathscr{C}$ in

$$\bar{\mathscr{M}} = \{A \in \{0, 1\}^n\colon \exists\, i \text{ such that } \theta^i(A) \in \mathscr{M}\}$$

corresponding to a given $(\gamma_1^*, \cdots, \gamma_{p+1}^*)$. That means: If $A \in \mathscr{C} \cap \mathscr{M}$, then $(\gamma_1^*, \cdots, \gamma_{p+1}^*)$ corresponds to A. We let $\sharp$ denote "the number of elements in". Moreover, we let $\mathscr{N}_i$ $(i = 1, \cdots, p + 1)$ be as in § 5. That is;

$$\begin{aligned}\mathscr{N}_i = \{&(d_1, \cdots, d_{\gamma_i})\colon d_1 > 0,\ d_j \geqq 0\ (j = 2, \cdots, \gamma_i) \text{ and} \\ &d_1 + \cdots + d_{\gamma_i} = \alpha_i\} \quad \text{for} \quad 1 \leqq i \leqq p \qquad \text{and} \\ \mathscr{N}_{p+1} = \{&(d_1, \cdots, d_{\gamma_{p+1}}) \times (s_1, \cdots, s_{\gamma_{p+1}})\colon d_i \geqq 0,\ s_i \geqq 0, \\ &d_1 + \cdots + d_{\gamma_{p+1}} = \alpha_{p+1} - m \text{ and } s_1 + \cdots + s_{\gamma_{p+1}} = m\} .\end{aligned}$$

THEOREM 6.1. *Suppose $X_1, \cdots, X_{p+1}$ is the least solution of the equations corresponding to $(\gamma_1^*, \cdots, \gamma_{p+1}^*)$ in Theorem 4.2. Then the number of cycles in $\bar{\mathscr{M}}$ corresponding to $(\gamma_1^*, \cdots, \gamma_{p+1}^*)$ is*

$$\prod_{i=1}^{p+1} w_i / X_{p+1}\gamma_{p+1}^*$$

where

$$w_{p+1} = \#\{\textit{the elements in } \mathscr{N}_{p+1} \textit{ with cycle period } \gamma_{p+1}^*\}$$

and for $1 \leqq j \leqq p$

$$w_j = \sum_{t=1}^{\alpha_j^*} t \cdot w_{j,t}$$

where

$$w_{j,t} = \#\{(d_1, \cdots, d_{r_j}) \in \mathscr{N}_j \textit{ with cycle period } \gamma_j^* \textit{ and } d_1 = t\} .$$

Proof. Suppose $A \in \mathscr{M}$ corresponds to $(\gamma_1^*, \cdots, \gamma_{p+1}^*)$. In the proof of Theorem 4.2 we prove that $Y = X_{p+1}\gamma_{p+1}^*$ is the least integer such that $\psi^Y(A) = A$. Hence, there are $X_{p+1}\gamma_{p+1}^*$ elements in $\mathscr{M}$ on the same cycle as A. Hence, the proof will be complete if we can prove

$$\#\{A \in \mathscr{M} : A \text{ corresponds to } (\gamma_1^*, \cdots, \gamma_{p+1}^*)\} = \prod_{i=1}^{p+1} w_i .$$

We get by Lemma 3.2 that

$$\begin{aligned}&\#\{A \in \mathscr{M} : A \text{ corresponds to } (\gamma_1^*, \cdots, \gamma_{p+1}^*)\} \\ &\quad = \prod_{i=1}^{p+1} \#\{D_i(A) : D_i(A) \text{ corresponds to } \gamma_i^* \text{ and } A \in \mathscr{M}\} .\end{aligned}$$

Hence, the proof will be complete if we can prove $(1 \leqq i \leqq p + 1)$

$$(6.1) \qquad \#\{D_i(A) : D_i(A) \text{ corresponds to } \gamma_i^* \text{ and } A \in \mathscr{M}\} = w_i .$$

First we will prove that (6.1) is true for $i = p + 1$. It is sufficient to prove that the map

$$\eta_{p+1} : \{D_{p+1}(A) : A \in \mathscr{M}\} \longrightarrow \mathscr{N}_{p+1}$$

defined in Proposition 4.3 is bijective: Let $(d_1, \cdots, d_{r_{p+1}}) \times (s_1, \cdots, s_{r_{p+1}}) \in \mathscr{N}_{p+1}$. Then there exists one and only one $D_{p+1}(A)$ such that

$$\eta_{p+1}(D_{p+1}(A)) = (d_1, \cdots, d_{r_{p+1}}) \times (s_1, \cdots, s_{r_{p+1}}) .$$

This $D_{p+1}(A) = (t_1, \cdots, t_{r_{p+1}}) \times (s_1, \cdots, s_{r_{p+1}})$ is given by $t_1 = d_1$, $t_2 = d_2 + t_1 + s_1$, $t_3 = d_3 + t_2 + s_2$, etc.

Next we will prove (6.1) in the case $i < p + 1$, and we do the following observation ($i = 1, \cdots, p$):

> To each $(d_1, \cdots, d_{\gamma_i}) \in \mathscr{N}_i$ there exists exactly d_1 elements $D = D_i(A)$ such that $\rho_i(D) = (d_1, \cdots, d_{\gamma_i})$ where ρ_i is as in Proposition 4.3.

These elements are

$$\left(s, s + d_2, s + d_2 + d_3, \cdots, s + \sum_{j=2}^{\gamma_i} d_j\right) \text{ where } s = 1, \cdots, d_1 .$$

(6.1) follows from this observation in the case $i < p + 1$.

The next theorem gives us a way of calculating w_{p+1} and $w_{j,t}$.

THEOREM 6.2. (a) *We let $\sigma(r, s, t)$ = the number of elements in*

$$\mathscr{C}(r, s, t) = \{(d_1, \cdots, d_s) : d_i \geqq 0,\ d_1 = r,\ d_1 + \cdots + d_s = t \text{ and } (d_1, \cdots, d_s) \text{ has trivial period } s\} .$$

Then $\sigma(r, s, t)$ can be calculated inductively by the following formula:

$$\sigma(r, s, t) = \binom{t + s - r - 2}{s - 2} - \sum \left\{\sigma\left(r, \frac{s}{s'}, \frac{t}{s'}\right) : \frac{s}{s'} \text{ and } \frac{t}{s'} \text{ are integers}\right\} .$$

( ) is the binomial coefficient.

(b) *We let $\sigma(s, t)$ = the number of elements in*

$$\mathscr{C}(s, t) = \{(d_1, \cdots, d_s) : d_i \geqq 0,\ d_1 + \cdots + d_s = t \text{ and } (d_1, \cdots, d_s) \text{ has trivial period } s\} .$$

Then $\sigma(s, t)$ can be calculated inductively by the following formula:

$$\sigma(s, t) = \binom{t + s - 1}{s - 1} - \sum \left\{\sigma\left(\frac{s}{s'}, \frac{t}{s'}\right) : \frac{s}{s'} \text{ and } \frac{t}{s'} \text{ are integers}\right\} .$$

(c) *The number of elements in*

$$\mathscr{Q}(s, t) = \{(d_1, \cdots, d_s) : d_i \geqq 0 \text{ and } d_1 + \cdots + d_s = t\}$$

is $$\binom{s + t - 1}{s - 1} .$$

(d) $w_{i,t} = \sigma(t, \gamma_i^*, \alpha_i^*)$ *for* $1 \leqq i \leqq p$ *and* $1 \leqq t \leqq \alpha_i^*$.

(e) *Let* $m^* = m \cdot \gamma_{p+1}^* / \gamma_{p+1}$. *Then we have*

$$w_{p+1} = r_1 \cdot q_1 + r_2 \cdot q_2 - r_1 \cdot r_2$$

where

$$r_1 = \sigma(\gamma^*_{p+1}, \alpha^*_{p+1} - m^*) \quad \textit{and} \quad q_1 = \binom{m^* + \gamma^*_{p+1} - 1}{\gamma^*_{p+1} - 1}$$

$$r_2 = \sigma(\gamma^*_{p+1}, m^*) \quad \textit{and} \quad q_2 = \binom{\alpha^*_{p+1} - m^* + \gamma^*_{p+1} - 1}{\gamma^*_{p+1} - 1} .$$

Proof. (a)

$$\begin{aligned}
\{(d_1, \cdots, d_s)&: d_i \geqq 0,\ d_1 = r \text{ and } d_1 + \cdots + d_s = t\}^\# \\
&= \{(d_2, \cdots, d_s): d_i \geqq 0 \text{ and } d_2 + \cdots + d_s = t - r\}^\# \\
&= \text{the number of ways to divide } (t - r) \text{ 1's into } (s - 1) \text{ groups} \\
&= \text{the number of ways to put } s - 2 \text{ 0's into } (t + s - r - 2) \text{ positions} \\
&= \binom{t + s - r - 2}{s - 2} .
\end{aligned}$$

We subtract those $(d_1, \cdots, d_s)$ with trivial period less than s. For each s' such that s/s' and t/s' are integers, $(d_1, \cdots, d_s) \to (d_1, \cdots, d_{s/s'})$ is a bijective correspondence between

$$\{(d_1, \cdots, d_s): 0 \leqq d_i,\ d_1 = r,\ d_1 + \cdots + d_s = t \text{ and } (d_1, \cdots, d_s) \text{ has trivial period } s/s'\}$$

and

$$\mathscr{C}(r, s/s', t/s') .$$

By using these correspondences (a) follows.

(b) and (c) are proved in the same way.

(d) By definition $w_{i,t}$ is the number of elements in the set

$$\mathscr{A}_1 = \{(d_1, \cdots, d_{\gamma_i}) \in \mathscr{N}_i;\ d_1 = t \text{ and } (d_1, \cdots, d_{\gamma_i}) \text{ has cycle period } \gamma_i^*\} .$$

The map from $\mathscr{A}_1$ into $\mathscr{C}(t, \gamma_i^*, \alpha_i^*)$ given by

$$(d_1, \cdots, d_{\gamma_i}) \longrightarrow (d_1, \cdots, d_{\gamma_i^*})$$

is bijective, and (d) follows.

(e) By definition w_{p+1} is the number of elements in the set

$$\mathscr{A}_2 = \{(d_1, \cdots, d_{\gamma_{p+1}}) \times (s_1, \cdots, s_{\gamma_{p+1}}) \in \mathscr{N}_{p+1} \text{ which has cycle period } \gamma^*_{p+1}\} .$$

We define

$$\begin{aligned}
\mathscr{A}_3 = \{(d_1, \cdots, d_{\gamma^*_{p+1}}) \times (s_1, \cdots, s_{\gamma^*_{p+1}})&: d_i \geqq 0,\ s_i \geqq 0, \\
d_1 + \cdots + d_{\gamma^*_{p+1}} = \alpha^*_{p+1} - m^*,\ & s_1 + \cdots + s_{\gamma^*_{p+1}} = m^* \text{ and} \\
(d_1, \cdots, d_{\gamma^*_{p+1}}) \text{ or } (s_1, \cdots, s_{\gamma^*_{p+1}}) & \text{ has cycle period } \gamma^*_{p+1}\} .
\end{aligned}$$

The map from $\mathscr{A}_2$ into $\mathscr{A}_3$ given by

$$(d_1, \cdots, d_{r_{p+1}}) \times (s_1, \cdots, s_{r_{p+1}}) \longrightarrow (d_1, \cdots, d_{r^*_{p+1}}) \times (s_1, \cdots, s_{r^*_{p+1}})$$

is bijective. We observe that

$$\#\mathscr{A}_3 = r_1 \cdot q_1 + r_2 \cdot q_2 - r_1 \cdot r_2$$

where

$$r_1 = \#\mathscr{C}(\gamma^*_{p+1}, \alpha^*_{p+1} - m^*) \quad \text{and} \quad q_1 = \mathscr{Q}(\gamma^*_{p+1}, m^*)$$
$$r_2 = \#\mathscr{C}(\gamma^*_{p+1}, m^*) \quad \text{and} \quad q_2 = \mathscr{Q}(\gamma^*_{p+1}, \alpha^*_{p+1} - m^*)$$

and (e) follows.

7. The reduction. We will reduce the cycle structure problem to the set studied in the §§ 3–6. First we need two lemmas. $C < D$ means C contained in D and $C \neq D$. If $D = a_r \cdots a_s$, we define $(t \in D \Leftrightarrow r \leqq t \leqq s)$ and $f_D(t) = f(a_r \cdots a_t)$.

We need more precise notation. If we are working with A we write

$$\alpha_i(A),\ \gamma_i(A) \text{ and } m_A \text{ instead of } \alpha_i,\ \gamma_i \text{ and } m .$$

LEMMA 7.1. *Suppose* $A = 0_{i_1}B_1C_10_{i_2}B_2C_2 \cdots 0_{i_f}\ B_f$ *where* B_i *is a block on level* 1. *Moreover, we suppose* $f(C_i) = -type\,(B_i)$ *and* $0 > f_{C_i}(t) \geqq -type\,(B_i)$ *for* $t \in C_i$.

Then we have

$$n + type\,(B_f) = \left(\sum_{i=1}^{p+1} 2i\gamma_i\right) + m_A + (i_1 + \cdots + i_f) ,$$

and if $type\,(B_f) \geqq type\,(B_i)$ *for* $i = 1, \cdots, f$ *then*

$$\alpha_{type\,(B_f)}(A) = m_A \Longleftrightarrow i_1 + \cdots + i_f = 0 .$$

Proof. We let $C_f = 0_{\text{type}\,(B_f)}$ and consider $A^* = AC_f = 0_{i_1}B_1C_1 \cdots 0_{i_f}B_fC_f$.

As in the proof of Lemma 4.13 in [2] we get

$$\text{the length of } \ B_i = f(B_i) + \sum \{2 \cdot \text{type}\,(B^*) \colon B^* < B_i\} ,$$
$$\text{the length of } \ C_i = \text{type}\,(B_i) + \sum \{2 \cdot \text{type}\,(B^*) \colon B^* < C_i\} .$$

If $\text{type}\,(B_i) = p + 1$, we therefore have

$$\text{the length of } \ B_iC_i = [f(B_i) - (p + 1)] + \sum \{2 \cdot \text{type}\,(B^*) \colon B^* < B_iC_i\} .$$

Otherwise,

$$\text{the length of } \ B_iC_i = \sum \{2 \cdot \text{type}\,(B^*) \colon B^* < B_iC_i\} .$$

Hence,

$$\text{the length of}\quad A^* = \sum \{f(B_i) - (p+1)\colon \text{type}\,(B_i) = p+1\} + \sum \{2\cdot\text{type}\,(B^*)\colon B^* \text{ a block}\} + (i_1 + \cdots + i_f) = m_A + \left(\sum_{i=1}^{p+1} 2i\gamma_i\right) + (i_1 + \cdots + i_f)\ .$$

The equivalence follows by the definition of $\alpha_{\text{type}\,(B_f)}(A)$. □

We write

$$\theta_{k,p} = \theta_{E_k + \cdots + E_{k+p}}\ . \tag{7.1}$$

LEMMA 7.2. *We suppose the block structure of* $A \in \{0, 1\}^n$ *is determined with respect to* p. *Moreover, we suppose* $w(A) = k + p + 1$. *Then we have*

$$([\gamma_{p+1}(A) \neq 0 \textit{ and } \alpha_{p+1}(A) = m_A] \textit{ or } [z = \sup_i \{i\colon \gamma_i(A) \neq 0\} < p + 1 \textit{ and } \alpha_z(A) = 0]) \Longleftrightarrow \theta^j_{k,p}(A) = \theta^j_{k,p'}(A) \quad \textit{for} \quad p' > p \quad \textit{and every} \quad j\ .$$

Proof. We suppose first $\gamma_{p+1}(A) \neq 0$. By Lemma 4.4 in [2] there exists q such that $\bar{A} = \theta^q_{k,p}(A)$ satisfies $\gamma_i(A) = \gamma_i(\bar{A})$, $\alpha_i(A) = \alpha_i(\bar{A})$, $m_A = m_{\bar{A}}$, $\bar{A}$ ends with a $(p+1)$-block, $\bar{A}$ starts with 0 or a $(p+1)$-block and $w(\bar{A}) = k + p + 1$.

Moreover, $\bar{A}$ has the form

$$\bar{A} = 0_{i_1} B_1 C_1 0_{i_2} B_2 C_2 \cdots 0_{i_f} B_f \quad \text{as in Lemma 7.1}\ .$$

(If $f = 1$, then $\bar{A} = 0_{i_1} B_1$.)

We suppose $\theta^j_{k,p}(A) = \theta^j_{k,p'}(A)$ for $p' > p$. If $i_1 \neq 0$, then $w(\theta_{k,p+1}(A)) = k + p + 2 \neq w(\theta_{k,p}(A))$. Hence, $i_1 = 0$. By Lemma 5.7 in [2] we have

$$w(\theta^s_{k,p}(\bar{A})) = k + p + 1 \quad \text{where} \quad s = \text{length of } B_1 C_1\ .$$

In the same way we prove $i_1 = \cdots = i_f = 0$. By Lemma 7.1 $\alpha_{p+1}(\bar{A}) = m_{\bar{A}}$. Hence, $\alpha_{p+1}(A) = m_A$.

Next we suppose $\alpha_{p+1}(A) = m_A$. Hence, $\alpha_{p+1}(\bar{A}) = m_{\bar{A}}$. By Lemma 7.1 we have $i_1 + \cdots + i_f = 0$. Hence, $\text{type}\,(B_1) = p + 1$. Moreover. let $j = \inf\{i > 1\colon \text{type}\,(B_i) = p + 1\}$. Put $C_1{''}{=}{''}C_1 B_2 C_2 \cdots B_{j-1} C_{j-}$ and $B_2{''}{=}{''}B_j$. By continuing in this way we can suppose $\text{type}\,(B_1) = \cdots = \text{type}\,(B_f) = p + 1$. Hence, by Lemma 5.6(c) in [2] we get $\theta^j_{k,p}(\bar{A}) = \theta^j_{k,p'}(\bar{A})$ for $p' > p$.

Finally we treat the case $z = \sup_i \gamma_i(A) < p + 1$. By Lemma 5.6 (a) in [2] we have $\theta^j_{k,p}(A) = \theta^j_{k_1,p_1}(A)$ where $k_1 = p + 1 - z$ and $p_1 = z - 1$. By Lemma 4.4 in [2] there exists q such that $\bar{A} = \theta^q_{k,p}(A)$ satisfies:

$\gamma_i(A) = \gamma_i(\bar{A})$, $\alpha_i(A) = \alpha_i(\bar{A})$, $m_A = m_{\bar{A}} = 0$, $\bar{A}$ ends with a z-block, $\bar{A}$ starts with 0 or a z-block and $w(\bar{A}) = k + p + 1$. Moreover, $\bar{A}$ has the form

$$\bar{A} = 0_{i_1}B_1C_10_{i_2}B_2C_2 \cdots 0_{i_f}B_f \quad \text{as in Lemma 7.1.}$$

We suppose $\theta^j_{k,p}(A) = \theta^j_{k,p'}(A)$ for $p' > p$. As in the case $\gamma_{p+1}(A) \neq 0$ we prove $i_1 = \cdots = i_f = 0$. By Lemma 7.1 $\alpha_z(A) = m_A = 0$.

Next we suppose $\alpha_z(A) = 0$. Hence, $\alpha_z(\bar{A}) = m_{\bar{A}} = 0$. By Lemma 7.1 we have $i_1 + \cdots + i_f = 0$. As before we can suppose type $(B_1) = \cdots =$ type $(B_f) = z$. Hence, by Lemma 5.6 (c) we get $\theta^j_{k,p}(\bar{A}) = \theta^j_{k,p'}(\bar{A})$ for $p' > p$. □

Previously in this paper we have not mentioned the possible values of $(\gamma_1, \cdots, \gamma_{p+1})$. However, by Lemma 4.1 in [2] we have the following result (k, p and n are given)

$(\gamma_1, \cdots, \gamma_{p+1})$ is a possible vector if and only if

$$\exists m \geqq 0 \text{ such that } m + \sum_{i=1}^{p+1} i \cdot \gamma_i = k + p + 1$$

and

$$m + 2 \cdot \sum_{i=1}^{p+1} i \cdot \gamma_i \leqq n + p + 1$$

(m corresponds to m defined previously).

The results obtained in this paper give a complete description of the cycle structure of $\mathscr{M}$ where

(7.2) $\mathscr{M} =$ the union of all $\mathscr{M}$ defined in (3.2) corresponding to the possible vectors $(\gamma_1, \cdots, \gamma_{p+1})$ satisfying $\gamma_{p+1} \neq 0$.

Now we start the reduction process. For $\mathscr{A} \subset \{0, 1\}^n$, we define the closure of $\mathscr{A}$ with respect to θ by

$$\bar{\mathscr{A}} = \{\theta^i(A) \colon A \in \mathscr{A}\}.$$

We let $\theta = \theta_{k,p}$ and we define

$$\mathscr{F} = \{A \colon k \leqq w(\theta^i(A)) \leqq w(A) \leqq k + p + 1 \ \forall\, i\}.$$

If $A \notin \bar{\mathscr{F}}$, then $\theta^i(A) = C^i(A)$ $\forall\, i$, where $C(a_1, \cdots, a_n) = a_2 \cdots a_n a_1$ is the pure cycling register. Hence, it is enough to study $\bar{\mathscr{F}}$. We define

$$\mathscr{D}(i, j) = \{A \in \mathscr{F} \colon k + i = \inf_s w(\theta^s(A)) \leqq w(A) = k + j\}.$$

Then we have obviously that

$$\bar{\mathscr{F}} = \bigcup_{i \leq j} \overline{\mathscr{D}(i, j)}$$

is a disjoint union. Hence, it is sufficient to determine the cycle structure of the sets $\overline{\mathscr{D}(i, j)}$. First we need an observation:

Observation 7.3. Suppose $\theta = \theta_{k,p}$, $w(A) = k + p + 1$ and $0 \leqq p' < p$. Then we have

$$\begin{aligned} &\gamma_{p'+1} \neq 0 \quad \text{ and} \\ &\gamma_{p'+2} = \cdots = \gamma_{p+1} = 0 \Longleftrightarrow \inf_s w(\theta^s(A)) = k + p - p' . \end{aligned}$$

Proof. This follows directly from the definition of the blocks, or for example from Lemma 5.1 in [2].

We also need very precise notation. If we are working with p we write α_i^p, γ_i^p and m^p instead of α_i, γ_i and m.

Case 1. $\overline{\mathscr{D}(0, p + 1)} = \bar{\mathscr{M}}$ where $\mathscr{M}$ is as in (7.2).

Proof. Let $A \in \mathscr{D}(0, p+1)$. By Observation 7.3 we have $\gamma_{p+1} \neq 0$. By Lemma 4.4 in [2] there exists s such that $\theta^s(A) \in \mathscr{M}$ and the claim follows.

Case 2. If $0 \leqq i < j < p + 1$, we can determine $\overline{\mathscr{D}(i, j)}$ in the following way: Let $k' = k + i$, $p' = j - i - 1$ and let $\mathscr{M}$ be as in (7.2) with respect to k' and p'. Then

$$\begin{aligned} \overline{\mathscr{D}(i, j)} &= \overline{\{A \in \mathscr{M} : \alpha_{p'+1} = 0\}} \quad \text{if} \quad i > 0 \\ \overline{\mathscr{D}(i, j)} &= \overline{\{A \in \mathscr{M} : \alpha_{p'+1} = m\}} \quad \text{if} \quad i = 0 \end{aligned}$$

where $\alpha_{p'+1}$ and m are determined with respect to p'. Moreover, the closure of $\mathscr{D}(i, j)$ with respect to $\theta_{k,p}$ and $\theta_{k',p'}$ respectively are equal.

Proof. Let $p'' = j - 1$ and $A \in \mathscr{D}(i, j)$. By Lemma 7.2 there are two possibilities:

(1) If $\gamma_{p''+1}^{p''} \neq 0$, then $\alpha_{p''+1}^{p''} = m^{p''}$.

(2) If $\gamma_z^{p''} \neq 0$ and $\gamma_{z+1}^{p''} = \cdots = \gamma_{p''+1}^{p''} = 0$, then $\alpha_z^{p''} = 0$.

We suppose first that $i > 0$. By Observation 7.3 we are in Case 2 with $z = j - i$ since

$$k + p'' + 1 - (j - i) = k + i \leqq w(\theta^s(A)) \leqq k + p'' + 1 .$$

Hence, we have $\alpha_z^{p''} = \alpha_{p'+1}^{p''} = 0$ and $\gamma_z^{p''} = \gamma_{p'+1}^{p''} \neq 0$. Since, $\gamma_{z+1}^{p''} = \cdots = \gamma_{p''+1}^{p''} = 0$ we have

$$\alpha^{p'}_{p'+1} = \alpha^{p''}_{p'+1} = 0 \quad \text{and} \quad \gamma^{p'}_{p'+1} = \gamma^{p''}_{p'+1} \neq 0 \ .$$

By Lemma 4.4 in [2] there exists s such that $\theta^s_{k',p'}(A) \in \mathscr{M}$ where $\mathscr{M}$ is defined as in (7.2) with respect to k' and p'.

Next we suppose $i = 0$. Then we are in Case 1 and $p'' = p'$. Hence, we have $\alpha^{p'}_{p'+1} = m^{p'}$ and $\gamma^{p'}_{p'+1} \neq 0$. By Lemma 4.4 in [2] there exists s such that $\theta^s_{k',p'}(A) \in \mathscr{M}$ where $\mathscr{M}$ is defined as in (7.2) with respect to k' and p'.

Case 3. If $0 < i < j = p + 1$, then

$$\overline{\mathscr{D}(i, j)} = \overline{\{A \in \mathscr{M} : m = 0\}}$$

where $\mathscr{M}$ and m is defined with respect to $k' = k + i$ and $p' = p - i$. Moreover, the closure of $\mathscr{D}(i, j)$ with respect to $\theta_{k,p}$ and $\theta_{k',p'}$ respectively are equal.

Proof. Let $A \in \mathscr{D}(i, j)$. By Observation 7.3 we have

$$(*) \qquad \gamma^{p'}_{p'+2} = \cdots = \gamma^{p'}_{p+1} = 0 \ .$$

Hence, $m^{p'} = 0$. Namely, if $m^{p'} \neq 0$, then (*) would not be true.

Moreover, by Lemma 5.6 in [2] we have

$$\theta^s_{k,p'}(A) = \theta^s_{k,p}(A) \quad \forall\, s$$

and there exists s such that $\theta^s_{k',p'}(A) \in \mathscr{M}$ where $\mathscr{M}$ is defined with respect to k' and p'. Hence the proof of Case 3 is complete.

Case 4. If $i = j$, then $\mathscr{D}(i, i) = \varnothing$ except in the following case: If $k + p + 1 = n$, then $\overline{\mathscr{D}(p + 1, p + 1)} = \{A = 1_n\}$.

The proof of Case 4 is obvious.

Finally we will mention how to determine the minimal period for $A \in \{0, 1\}^n$ with respect to $\theta_{k,p}$ in the following 4 steps:

1. If $w(A) \notin \{k, \cdots, k + p + 1\}$, then $\theta_{k,p}(A) = \xi(A)$ where $\xi(a_1 \cdots a_n) = (a_2 \cdots a_n a_1)$ and the problem is trivial. We therefore suppose $w(A) \in \{k, \cdots, k + p + 1\}$.

2. We calculate $w(A), w(\theta_{k,p}(A)), \cdots, w(\theta^{2n}_{k,p}(A))$ and choose j such that $A^* = \theta^j_{k,p}(A)$ satisfies

$$w(A^*) = \sup_{1 \leq i \leq 2n} w(\theta^i_{k,p}(A)) = \sup_i w(\theta^i_{k,p}(A)) \ .$$

3. Put $p' = w(A^*) - k - 1$. Then we can use $\theta_{k,p'}$ instead of $\theta_{k,p}$ (Lemma 5.6 (b) in [2]). We have $w(A^*) = k + p' + 1$.

4. Next we determine the block structure of A^* with respect to p'. We put $j = \sup\{i : \gamma^{p'}_i(A) \neq 0\}$, and $k'' = p' - j$ and $p'' = j - 1$. Then we can use $\theta_{k'',p''}$ instead of $\theta_{k,p}$ (Lemma 5.6 (a) in [2]). More-

over, we have $w(A^*) = k'' + p'' + 1$ and $\gamma^{p''}_{p''+1}(A^*) \neq 0$. Hence, we can use Theorem 4.2.

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