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ON g-METRIZABILITY

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We show that a regular topological space is g-metrizable if and only if it is weakly first countable and admits a σ-locally finite k-network and that a g-metrizable space need not be g-developable.

0. **Introduction.** G-metrizable spaces were defined in [8], where it was also shown that a space admits a countable weak base if and only if it is weakly first countable and has a countable k-network. In this paper we provide the corresponding result for g-metrizable spaces and give an example of a g-metrizable space which is not g-developable. The former result is in response to a question in [8], the latter answers a question in [6]. All spaces are at least regular.

1. **Definition.**

1.1. Let X be a space. If Γ is a family of subsets of X and $\zeta: \Gamma \to \mathscr{P}(X)$ is a function, then the pair $\langle \Gamma, \zeta \rangle$ is a weak base for X if, in addition, the following hold:

(a) For every member G of Γ, $\zeta(G)$ is a subset of G.

(b) If G_1 and G_2 are members of Γ and x is an element of $\zeta(G_1) \cap \zeta(G_2)$, then there is a member G_3 of Γ so that x is in $\zeta(G_3)$ and G_3 is a subset of $G_1 \cap G_2$.

(c) A subset U of X is open if and only if for every element x of U there is a member G of Γ so that x is in $\zeta(G)$ and U contains. G.

This definition of weak base differs from that of [1], namely, a collection $\mathscr{B} = \cup\{T_x: x \in X\}$ is a weak base for X if a set U is open in X precisely when for each point $x \in U$ there exists $B \in T_x$ such that $B \subset U$. It is easy to see that our definition is equivalent to this, for if B is as above, we let $\Gamma = \mathscr{B}$ and for $G \in \Gamma$, let $\delta(G) = \{x: G \in T_x\}$ and if $\langle \Gamma, \delta \rangle$ is a weak base by 1.1, then we let $T_x = \{G: x \in \delta(G)\}$ and $\mathscr{B} = \cup\{T_x: x \in X\}$.

1.2. A space X is g-metrizable if it has a weak base $\langle \Gamma, \zeta \rangle$ where Γ is a σ-locally finite family. X is weakly first countable if X has a weak base $\langle \Gamma, \zeta \rangle$ so that the family $\{\zeta(G): G \in \Gamma\}$ is point countable or, equivalently, there is a function $B: \omega \times X \to \mathscr{P}(X)$ (called a wfc system for X) so that

(a) for all $n < \omega$ and $x \in X$, $B(n+1, x) \subset B(n, x)$;

(b) for all x in X, $x \in \cap\{B(n, x): n < \omega\}$

(c) a subset U of X is open if and only if for every x in U there is an $n < \omega$ so that U contains $B(n, x)$.

If x is an element of a space X, then a subset S of X is said to be weak neighborhood of x if every sequence converging to x is eventually in S. One may show that if X is weakly first countable with weak base $\langle \Gamma, \zeta \rangle$ so that $\{\zeta(G): G \in \Gamma\}$ is point countable, then S is a weak neighborhood of x if and only if S contains a member G of Γ so that $x \in \zeta(G)$. Thus weakly first countable spaces are sequential [4].

1.3. If X is a space, a collection Γ of subsets of X is said to be a k-network [7] for X if for any compact subset K of X and any neighborhood U of K, there is a finite subcollection Γ' of Γ so that $K \subset \cup \Gamma' \subset U$.

2. g-metrizability and k-networks.

LEMMA 2.1. *If X is a space in which points are G_δ and if $\langle \Gamma, \zeta \rangle$ is a weak base for X, then Γ is a k-network for X.*

Proof. Let K be a compact subset of X and U an open neighborhood of K. As K is closed, $\langle \Gamma', \zeta' \rangle$ given by $\Gamma' = \{G \cap K: G \in \Gamma\}$ and $\zeta'(G \cap K) = \zeta(G) \cap K$ for all G in Γ, is a weak base for K. Thus since K is Fréchet, for every G in Γ $\zeta'(G \cap K) \subset \mathrm{int}_K (G \cap K)$. Consequently if Γ^* is a subcollection of Γ so that $K \subset \cup \{\zeta(G): G \in \Gamma^*\}$ and $\cup \Gamma^* \subset U$, then a finite subfamily of Γ^* convers K.

THEOREM 2.2 [3]. *A regular space with a σ-locally finite k-network has a σ-dicrete k-network.*

LEMMA 2.3. *Suppose X hase $\langle \Gamma, \zeta \rangle$ so that $\Gamma = \cup \{\Gamma_n: n < \omega\}$ where every Γ_n ts a closure-preserving family of closed sets. If $\{F_\alpha: \alpha \in I\}$ is a discrete collection of subsets of X, then there is a pairwise disjoint collection $\{N_\alpha: \alpha \in I\}$ so that for every $\alpha \in I$ and $x \in F_\alpha$, there is a G in Γ so that $x \in \zeta(G)$ and $G \subset N_\alpha$.*

Proof. For each $n < \omega$ and each $\alpha \in I$, let

$$G(n, \alpha) = \cup \{G \in \Gamma_n: G \cap (\cup \{F_\beta: \beta \neq \alpha\}) = \varnothing\}$$

For each $\alpha \in I$, let

$$N_\alpha = \bigcup_{n<\omega} [G(n, \alpha) \setminus \cup \{G(m, \beta): m \leqq n, \beta \neq \alpha\}] .$$

Of course $\{N_\alpha: \alpha \in I\}$ is pairwise disjoint; we now verify that $\{N_\alpha: \alpha \in I\}$ is the desired collection. Let $\alpha \in I$ and let $x \in F_\alpha$. Find an

$n<\omega$ and a G_1 in Γ_n so that $x \in \zeta(G_1)$ and so that G_1 misses the closed set $\cup\{F_\beta: \beta \neq \alpha\}$. Pick $G_2 \in \Gamma$ so that $x \in \zeta(G_2)$ and so that G_2 misses the closed set $\cup\{G(m, \beta): m \leqq n, \beta \neq \alpha\}$. Now there is a $G_3 \in \Gamma$ with $x \in \zeta(G_3)$ so that G_3 is a subset of $G_1 \cap G_2$, hence $G_3 \subset N_\alpha$, as desired.

We are now in a position to prove the main result of this section.

THEOREM 2.4. *A regular space is g-metrizable if and only if it is weakly first countable and admits a σ-locally finite k-network.*

Proof. The necessity follows from Lemma 2.1. For the sufficiency: by Theorem 2.2, for each $n < \omega$ let Λ_n be a discrete collection of closed subsets of X so that $\Lambda = \cup\{\Lambda_n: n < \omega\}$ is closed under finite intersections and is a k-network for X. Let

$$\Gamma = \{\cup \Lambda^*: \Lambda^* \text{ is a finite subset of } \Lambda \text{ so that } \cap \Lambda^* \neq \varnothing\} .$$

For Λ^* a finite subset of Λ with $\cap \Lambda^* \neq \varnothing$, let

$$\zeta(\cup \Lambda^*) \simeq \{x \in \cap \Lambda^*: \cup \Lambda^* \text{ is a weak neighborhood of } x\} .$$

Note that $\{(G): G \in \Gamma\}$ is point-countable. We now show that $\langle \Gamma, \zeta\rangle$ is a weak base for X. One easily verifies that (a) and (b) of 1.1 are satisfied. For (c), observe that if U is a subset of X so that for every $x \in U$ there is a $G \in \Gamma$ so that $x \in \zeta(G)$ and U contains G, then U is sequentially open, hence open. Conversely, suppose U is open and there is an element x of U so that U contains no member G of Γ such that $x \in \zeta(G)$, i.e. the union of no finite subset of $\{L_j: j < \omega\} = \{L \in \Lambda: x \in L, L \subset U\}$ is a weak neighborhood of x. Let B a wfc system for X so that $B(1, x) \subset U$. Inductively pick a sequence $\{x_n: n < \omega\}$ so that $x_n \in B(n, x) \setminus \cup\{L_j: j \leqq n\}$. The sequence $\{x_n: n < \omega\}$ converges to x, hence $\{x\} \cup \{x_n: n < \omega\}$ is compact. Let Λ' be a finite subset of Λ so that $\{x\} \cup \{x_n: n < \omega\} \subset \cup \Lambda' \subset U$ and let $\Lambda^* = \{L \in \Lambda': x \in L\}$. The closed set $\cup(\Lambda' \setminus \Lambda^*)$ omits x, so there is an $m < \omega$ so that $\{x\} \cup \{x_n: n \geqq m\} \subset \cup \Lambda^*$. Also $\Lambda^* \subset \{L \subset \Lambda: x \in L, L \subset U\}$, so there is an $r \geqq m$ so that $\Lambda^* \subset \{L_j: j \leqq r\}$, which implies that $x_r \in \cup \Lambda^* \subset \cup\{L_j: j \leqq r\}$. This contradicts the fact that x_r was picked in the complement of $\cup\{L_j: j \leqq r\}$. Thus if U is open, then for all $x \in U$, U contains a $G \in \Gamma$ so that $x \in \zeta(G)$; so $\langle \Gamma, \zeta\rangle$ is a weak base for X.

Note that if $n < \omega$,

$$\Gamma_n = \{\cup \Lambda^*: \Lambda^* \text{ is a finite subset of } \cup\{\Lambda_j: j \leqq n\} \text{ so that } \cap \Lambda^* \neq \varnothing\}$$

is a closure-preserving collection, hence $\Gamma = \cup\{\Gamma_n: n < \omega\}$ is σ-conservative.

For every finite subset S of ω, let

$$\Lambda_S = \{\Lambda^*: \text{ for } n < \omega \ \Lambda^* \cap \Lambda_n \neq \emptyset \text{ iff } n \in S; \ \cap \Lambda^* \neq \emptyset\}$$

and write $\Lambda_S = \{\Lambda^*_\alpha: \alpha \in I(S)\}$. Further, as $\{\cap \Lambda^*_\alpha: \alpha \in I(S)\}$ is a discrete collection, use Lemma 2.3 to find a pairwise disjoint collection $\{N_\alpha: \alpha \in I(S)\}$ so that for every α in $I(S)$ N_α is a weak neighborhood of $\cap \Lambda^*_\alpha$.

Now if $n < \omega$, S is a finite subset of ω, and if $\alpha \in I(S)$, let

$$G(n, \alpha) = \cup \{G \in \Gamma_n: G \subset (\cup \Lambda^*_\alpha) \cap N_\alpha\} .$$

and let

$$\zeta'(G(n, \alpha)) = \cup \{\zeta(G) \cap \zeta(\cup \Lambda^*_\alpha): G \in \Gamma_n, G \subset (\cup \Lambda^*_\alpha) \cap N_\alpha\} .$$

If $n < \omega$ and if S is a finite subset of ω, let

$$\Gamma(n, S) = \{G(n, \alpha): \alpha \in I(S)\} .$$

The collections $\Gamma(n, S)$ are conservative and, since $G(n, \alpha) \subset N_\alpha$ for every $\alpha \in I(S)$, pairwise disjoint, hence discrete. Let Γ' be the family of all intersections of finite subcollections of $\cup \{\Gamma(n, S): n < \omega$, S a finite subset of $\omega\}$ and extend ζ' to Γ' by $\zeta'(\bigcap_{i=1}^k G(n_i, \alpha_i)) = \bigcap_{i=1}^k \zeta'(G(n_i, \alpha_i))$. Observe that Γ' is σ-discrete; we will show that $\langle \Gamma', \zeta' \rangle$ is a weak base for X, completing the proof.

Conditions (a) and (b) of 1.1 are easily verified. Recalling that $\{\zeta(G): G \in \Gamma\}$ is point countable, the remarks in 1.2 give that for all $G \in \Gamma$ G is a weak neighborhood of $\zeta(G)$ so that if $n < \omega$, S is a finite subset of ω and if $\alpha \in I(S)$, then $G(n, \alpha)$ is a weak neighborhood of $\zeta'(G(n, \alpha))$. Consequently if $G' \in \Gamma'$, then G' is a weak neighborhood of $\zeta(G')$. Hence if U is a subset of X such that for every member x of U there is a member G' of Γ' with $x \in \zeta(G')$ and $G' \subset U$, then U is a weak neighborhood of each of its elements, thus sequentially open, and so U is open. To complete the proof of (c), let U be an open subset of X, and let $x \in U$. Since $\langle \Gamma, \zeta \rangle$ is a weak base for X, there is a finite subset Λ^* of Λ so that $x \in \zeta(\cup \Lambda^*) \subset \cap \Lambda^* \subset \cup \Lambda^* \subset U$. Find a finite subset S of ω and an $\alpha \in I(S)$ so that $\Lambda^* = \Lambda^*_\alpha$. Since $\cup \Lambda^*_\alpha$ is a member of Γ, $\cup \Lambda^*_\alpha$ is a weak neighborhood of $\zeta(\cup \Lambda^*_\alpha)$, hence of x; N_α is a weak neighborhood of $\cap \Lambda^*_\alpha$, hence of x; thus $(\cup \Lambda^*_\alpha) \cap N_\alpha$ is a weak neighborhood of x. Again since $\{\zeta(G): G \in \Gamma\}$ is point-countable, we have that there is an $n < \omega$ and a $G \in \Gamma_n$ so that $x \in \zeta(G)$ and $G \subset (\cup \Lambda^*_\alpha) \cap N_\alpha$. Thus $x \in \zeta'(G(n, \alpha))$ and $G(n, \alpha) \subset \cup \Lambda^*_\alpha \subset U$. Thus (c) is established.

3. g-developable spaces. Generalizing a characterization of developability given in [5], Lee [6] defined g-developable spaces to be those weakly first countable spaces X which have a wfc system satisfying the following: if $x \in X$ and if $\{x_n: n < \omega\}$ and $\{y_n: n < \omega\}$ are sequences in X so that for every $n < \omega$ x and x_n are elements of $B(n, y_n)$, then the sequence $\{x_n: n < \omega\}$ converges to x.

PROPOSITION 3.1. *A σ-discrete weakly first countable space X is g-developable.*

Proof. Write $X = \dot{\cup}\{D_n: n < \omega\}$, where D_n is a closed discrete set for every $n < \omega$. X is symmetrizable [1], so let d be a compatible symmetric function. We define $B: w \times X \to \mathscr{P}(X)$ as follows: if m and n are finite ordinals and if $x \in D_m$, let

$$B(n, x) = \{y \in X: d(x, y) < 1/n\} \backslash \cup \{D_k: k < m\} .$$

One easily checks that B is a wfc system for the topology of X. To see that B satisfies the defining condition for g-developability let $x \in X$ and let $\{x_n: n < \omega\}$ and $\{y_n: n < \omega\}$ be sequences in X so that for every $n < \omega$ x and x_n are in $B(n, y_n)$. If $m < \omega$ so that $x \in D_m$, then there is a $j < \omega$ so that $\{y \in X: d(x, y) < 1/j\} \cap (\cup \{D_k: k \leqq m\}) = \{x\}$. The fact that $x \notin \cup \{B(j, y): y \neq x\}$ implies that if $n \geqq j$, then $y_n = x$. Thus for all $n \geqq j$ we have $x_n \in B(n, x)$, hence $\{x_n: n < \omega\}$ converges to x, as desired.

The definition of g-developable inspires the question to which the following is a negative answer.

THEOREM 3.2. *There is a g-metrizable space which is not g-developable.*

Proof. Let $\boldsymbol{R}$ denote the set of real numbers $\boldsymbol{Q}$ the set of rationals. Choose a countable quasibase Λ for the Euclidean topology of $\boldsymbol{R}$ consisting of closed sets. Let $X = \{\langle x, y\rangle \in \boldsymbol{R}^2$: either $y = 0$, or $x \in \boldsymbol{Q}$ & $1/y \in \omega\}$, and view $\boldsymbol{R}$ as $\{\langle x, y\rangle \in X: y = 0\}$. For every $q \in \boldsymbol{Q}$ and $m < \omega$, define $A(m, q) = \{r \in \boldsymbol{R}: |r - q| \leqq 1/m\} \cup \{\langle q, 1/n\rangle: n > m\}$. Let

$$\Gamma = \{A(m, q): m < \omega, q \in \boldsymbol{Q}\} \cup \Lambda \cup \{\{\langle q, 1/n\rangle\}: q \in \boldsymbol{Q}, n < \omega\} ,$$

and define

$$\zeta(A(m, q)) = \{q\} , \quad \text{if } m < \omega \text{ and } q \in \boldsymbol{Q} ;$$
$$\zeta(L) = \{r \in \boldsymbol{R}\backslash\boldsymbol{Q}: r \text{ is in the Euclidean interior of } L\} , \quad \text{if } L \in \Lambda ;$$
$$\zeta(\{\langle q, 1/n\rangle\}) = \{\langle q, 1/n\rangle\} , \quad \text{if } q \in \boldsymbol{Q} \text{ and } n < \omega .$$

Give X the topology for which $\langle \Gamma, \zeta\rangle$ is a weak base. Certainly Γ is countable, so, as X is easily seen to be regular, X is g-metrizable. To show that X is not g-developable, assume that B is a wfc-system for X satisfying the defining condition for g-developability.

Define a function $\phi: \boldsymbol{R}\backslash\boldsymbol{Q} \rightarrow \omega$ so that if $r \in \boldsymbol{R}\backslash\boldsymbol{Q}$, then $r \notin \bigcup\{B(\phi(r), q): q \in \boldsymbol{Q}\}$. This is possible, for if there is an $r \in \boldsymbol{R}\backslash\boldsymbol{Q}$ so that for every $n < \omega$ there is a $q_n \in \boldsymbol{Q}$ so that $r \in B(n, q_n)$, then find, for each $n < \omega$, an $x_n \in X\backslash\boldsymbol{R} \cap B(n, q_n)$. This would imply that for every $n < \omega$, r and x_n are in $B(n, q_n)$, but $\{x_n: n < \omega\}$ does not converge to r, a contradiction.

Since $\boldsymbol{R}\backslash\boldsymbol{Q} = \bigcup\{\{r \in \boldsymbol{R}\backslash\boldsymbol{Q}: \phi(r) \leqq n\}: n < \omega\}$, there is an $m < \omega$ so that the Euclidean closure $\mathrm{cl}_R\,\{r \in \boldsymbol{R}\backslash\boldsymbol{Q}: \phi(r) \leqq m\}$ contains a Euclidean open set U. Choose a $p \in \boldsymbol{Q} \cap U$. As $B(m, p) \cap \boldsymbol{R}$ is a Euclidean neighborhood of p in R, there is an $r \in \boldsymbol{R}\backslash\boldsymbol{Q} \cap B(m, p)$ so that $\phi(r) \leqq m$, that is $r \notin \bigcup\{B(m, q): q \in \boldsymbol{Q}\}$; this contradiction completes the proof.

References

1. A. Arhangel'skii, *Mappings and spaces*, Russ. Math. Surveys, **21** (1966), no. 4, 115-162.
2. A. Arhangel'skii and S. Franklin, *Ordinal invariants for topological spaces*, Michigan Math. J., **15** (1968), 313-320.
3. L. Foged, *Separation in k-and-$\aleph$ spaces*, to appear.
4. S. Franklin, *Spaces which sequence suffice*, Fund. Math., **57** (1965), 107-115.
5. R. Heath, *Arcwise connectedness in semi-metric spaces*, Pacific J. Math., **12** (1962), 1301-1319.
6. K. Lee, *On certain g-first countable spaces*, Pacific J. Math., **65** (1976), 113-118.
7. E. Michael, *$\aleph_0$-space*, J. Math. Mech., **15** (1966), 983-1002.
8. F. Siwiec, *On defining a space by a weak base*, Pacific J. Math., **52** (1974), 233-245.

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