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DUALITY CONDITION AND PROPERTY (S)

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We consider some geometric aspects of Borel's density theorem and property (S) of Selberg for simply connected complete Riemannian manifolds of nonpositive curvature. We also have some results on simply connected complete Kähler manifolds of nonpositive curvature.

A subgroup Γ of a topological group G is said to have property (S) in G if for each neighborhood U of the identity e of G and each element g in G there exists and integer $n > 0$ such that $g^n \in U \cdot \Gamma \cdot U$. In [3], Borel has proved the density theorem for subgroup Γ of property (S) in a connected semi-simple Lie group G without compact factors. Intuitively, it means that Γ is the product of some simple factors $\{G_i\}$ of $G = \prod_{i=1}^{k} G_i$ by a discrete group in the product of other simple $\{G_j\}$ of G (see p. 179 of [3]).

In [5], the duality condition for a group Γ of isometries of a simply connected complete Riemannian manifold M of nonpositive curvature was introduced. Γ satisfies the duality condition if for each infinite geodesic σ of M there is a sequence $\{\gamma_n\} \subset \Gamma$ such that $\gamma_n(p) \to \sigma(\infty)$ and $\gamma_n^{-1}(p) \to \sigma(-\infty)$ for each p in M. If the quotient space M/Γ is compact or has finite volume, then Γ satisfies the duality condition [6], [7].

In this paper, we shall prove that if Γ is any subgroup of the isometry group $I(M)$ satisfying the duality condition and if M is a simply connected complete visibility manifold (see [6]) then either Γ is discrete or M is a rank one symmetric space of noncompact type and $(\bar{\Gamma})_0 = I_0(M)$ or $\bar{\Gamma}$ is of finite index (less than $[I(M), I_0(M)]$ in $I(M)$. This is an analogue of Borel's density theorem. In fact, the theorem is true if M satisfies a weaker condition of [1] and [8], that is, some geodesic σ of M does not bound an imbedded flat totally geodesic half plane. We shall compare the duality condition with the property (S). The duality condition is apparently weaker than property (S) for noncompact symmetric spaces.

In [12], Heintze has proved that a subgroup Γ of property (S) of the noncompact semi-simple Lie group G satisfies the duality condition. We shall prove that the duality condition is equivalent to a condition on the set of axial transformations (or transvections [18]) in G similar to the property (S). The last part of this paper is concerned with the complex version of several main theorems in [5]. These results seem to be interesting in the area of simply connected complete Kähler manifolds of nonpositive curvature investigated by

Greene and Wu [10], [11].

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1. Preliminaries. Let M be a simply connected complete Riemannian manifold of nonpositive curvature. Any noncompact symmetric spaces has nonpositive curvature [13]. Any two distinct points of M can be joined by a geodesic. Two geodesics σ_1 and σ_2 (with the same speed) are asymptotic in M if $d(\sigma_1(t), \sigma_2(t)) \leqq c$ for some constant $c > 0$ and all $t \geqq 0$. An equivalence class of asymptotes is a point at infinity for M and $M(\infty)$ denotes the set of points at infinity. The space $\bar{M} = M \cup M(\infty)$ with the cone topology of Eberlein and O'Neill [6] is homeomorphic to the closed ball.

Any geodisc σ can be extended to the boundary $M(\infty)$. The asymptote classes of σ and its reverse are denoted by $\sigma(\infty)$ and $\sigma(-\infty)$. If any two points $x \neq y$ in $M(\infty)$ can be joined by a geodesic then M is said to be a visibility manifold [6], [7].

Let $I(M)$ and $I_0(M)$ denote the isometry group of M and its identity component. These isometry groups are Lie groups. For each element ϕ of $I(M)$, we have a displacement function $g_\phi: p \to d(p, \phi p)$. An isometry ϕ is called elliptic, hyperbolic or parabolic if g_ϕ has zero, positive or no minimum respectively. A subgroup Γ of $I(M)$ determines a limit set $L(\Gamma) \subseteqq M(\infty)$ which is closed in $M(\infty)$ and is invariant under Γ. $L(\Gamma)$ is the set of points in $M(\infty)$ that are accumulation points of an orbit Γp of some point p in M.

Let $M = M_1 \times \cdots \times M_k$ be the de Rham decomposition of M into irreducible factors. Let Γ be a subgroup of $I(M)$ satisfying the duality condition and preserving the factors. Let Γ_i be the projection of Γ into $I(M_i)$, $1 \leqq i \leqq k$. Then each Γ_i also satisfies the duality condition.

2. Property (S). The property (S) and the duality condition are satisfied by all discrete subgroups Γ of a connected semi-simple Lie group G of noncompact type such that Γ/G is compact or has finite invariant measure. In [12], Heintze has proved that if Γ satisfies property (S), then Γ satisfies the duality condition. The question is whether they are equivalent or one is stronger than the other. In this section, we shall show that property (S) is apparently stronger than the duality condition by finding an equivalent condition which is weaker than property (S).

Let us recall some basic facts about geodesic flows in noncompact symmetric spaces [16]. This is needed, because the duality condition can be stated in terms of geodesic flows [1], [6], [7]. Let $M = G/K$ be a noncompact symmetric space where G is a connected semi-simple Lie group with compact factors and with finite center and K is a maximal compact subgroup of G. The Lie algebra $\mathfrak{G}$ of G has the Cartan decomposition $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P}$, where $\mathfrak{P}$ is the orthogonal subspace to the subalgebra $\mathfrak{K}$ for K with respect to the Killing form. Thus $\mathfrak{P}$ can be identified with the tangent space to G/K at the point K. For each $X \in \mathfrak{P}$, $\exp(tX)$ is a 1-parameter subgroup of G and $(\exp tX)K$ if a geodesic through K in G/K. Conversely, every geodesic is of this form. Let F_t denote the geodesic flow in the unit tangent bundle $T_1(M) = T_1(G/K)$ of G/K. There is a natural action of G on $T_1(G/K)$. For each $g \in G$, the geodesic through the point gK of G/K in the direction of the unit tangent vector gX is $g[\exp tX]K$. The geodesic flow F_t associated to G/K is given by $F_t(gX) = g[\exp tX]X$, where $X \in T_1(G/K)$, $g \in G$ and $t \in R$.

The following theorem gives equivalent conditions to the duality condition [5]. The first two equivalences are valid in any complete simply connected Riemannian manifold of nonpositive curvature. The condition (2) is used in Ballmann's work [1].

THEOREM 1. *Let $M = G/K$ be a noncompact symmetric space where G is a connected semisimple Lie group without compact factors and with finite center. Let Γ be a subgroup of G. Then the following are equivalent.*

(1) *Γ satisfies the duality condition.*

(2) *For each unit tangent vector v in $T_1(G/K)$ there exist sequences*

$$\{\gamma_n\} \subseteqq \Gamma\,, \qquad \{t_n\} \subseteqq R \quad \text{and} \quad \{v_n\} \subseteqq T_1(G/K)$$

such that $t_n \to +\infty$, $v_n \to v$ and $\gamma_n F_{t_n} v \to v$ as $n \to \infty$.

(3) *There exists a dense subset Z of $G \times \mathfrak{P}$ (with the product topology) such that for every $(g, X) \in Z$ the transvection $\phi = g(\exp X)g^{-1}$ has the following property: given a neighborhood U of e in G there exists an integer $n \geqq 1$ and elements ω, γ and k of G such that*

$$\phi^n = \gamma\omega k$$

where $\omega \in U$, $\gamma \in \Gamma$ and k fixes the tangent vector $gX \in T(G/K)$.

REMARK 1. In condition (3) it would be nicer to say that there exists a dense subset T^* of the set T of all transvections such that ϕ satisfies the properties of (3) for every $\phi \in T^*$. Clearly the existence of a dense subset $Z \subseteqq G \times \mathfrak{P}$ with the properties of (3) implies the existence of a dense subset $T^* \subseteqq T$ with the same properties.

However, if T^* is given as a dense subset of T it is not clear how one then obtains a dense subset $Z \subseteqq G \times \mathfrak{P}$. In particular, if $\{\phi_n\}$ is a sequence of transvections converging to a transvection $\phi = g(\exp X)g^{-1}$ for some $g \in G$, $X \in \mathfrak{P}$ can one find sequences $\{g_n\} \subseteqq G$ and $\{X_n\} \subseteqq \mathfrak{P}$ such that $g_n \to g$, $X_n \to X$ and $\phi_n = g_n(\exp X_n)g_n^{-1}$? Part of the difficulty is that the representation of a transvection as $g(\exp X)g^{-1}$ is not unique.

Actually, one would need only a weaker result to use T^* in the statement of (3) instead of Z. It would suffice to know that if $\{\phi_n\}$ is a sequence of transvections converging to a transvection ϕ, and if σ_n, σ are geodesics translated by ϕ_n, ϕ then $\sigma_n(\infty) \to \sigma(\infty)$ and $\sigma_n(-\infty) \to \sigma(-\infty)$ as $n \to \infty$. This would follow immediately if one could show that $\sigma_n'(0) \to \sigma'(0)$ as $n \to \infty$ for a suitable choice of σ_n and σ.

Proof of Theorem 1. The equivalence of (1) and (2) is proved by Eberlein in Proposition 3.7 of [7]. We prove (3) $\to$ (1) and (2) $\to$ (3).

(3) $\to$ (1). Let σ be an arbitrary geodesic of M and let p denote $\sigma(0) = gK$ for some $g \in G$. Then

$$\sigma(t) = g(\exp tX)K = g(\exp tX)g^{-1}(p)$$

for a suitable $X \in \mathfrak{P}$. By hypothesis there exists a sequence $(g_n, X_n) \subseteqq Z$ that converges to (g, X). Let σ_n be the geodesic of $M = G/K$ given by

$$\sigma_n(t) = g_n(\exp tX_n)K = g_n(\exp tX_n)g_n^{-1}(p_n) ,$$

where $p_n = g_nK \to p$ as $n \to \infty$. Clearly $\sigma_n'(0) \to \sigma'(0)$ and it follows that $\sigma_n(\infty) \to \sigma(\infty)$ and $\sigma_n(-\infty) \to \sigma(-\infty)$ as $n \to +\infty$.

It suffices to prove that $\sigma_n(\infty)$ is dual to $\sigma_n(-\infty)$ relative to Γ for every n. Assuming this to be proved let $\{U_k\}$, $\{V_k\}$ be neighborhood bases in $\bar{M} = M \cup M(\infty)$ for $\sigma(\infty)$, $\sigma(-\infty)$. For each integer $k \geqq 1$ the points $\sigma_n(\infty)$, $\sigma_n(-\infty)$ lie in U_k, V_k for sufficiently large n. Since $\sigma_n(\infty)$ is dual to $\sigma_n(-\infty)$, there exists $\phi_k \in \Gamma$ so that $\phi_k(p) \in U_k$ and $\phi_k^{-}(p) \in V_k$, and this proves that $\sigma(\infty)$ is dual to $\sigma(-\infty)$.

It remains to show that if $\sigma(t) = g(\exp tX)K$, where $(g, X) \in Z$, then $\sigma(\infty)$ is dual to $\sigma(-\infty)$. If $\phi = g(\exp X)g^{-1}$ then by the hypothesis of (3) there exist sequences $\{n_i\} \subseteqq \boldsymbol{Z}$, $\{\omega_i\} \subseteqq G$, $\{\gamma_i\} \subseteqq \Gamma$ and $\{k_i\} \subseteqq G$ such that

$$\phi^{n_i} = \gamma_i \omega_i k_i$$

for every i, where $\omega_i \to 1$ and $k_i(gX) = gX \in T(G/K)$ for every i.

We consider first the case that $\{n_i\}$ has a subsequence converging to some integer m. Since k_i fixes $gK = p \in M$ for every i we may choose a subsequence of $\{k_i\}$ that converges to an element k that fixes gX. By passing to an appropriate subsequence we find that $\phi^m = \gamma k$ for some $\gamma \in \Gamma$. It follows that γ translates the geodesic σ and hence $\sigma(\infty) = \lim_{n\to\infty} \gamma^n(p)$ is dual to $\sigma(-\infty) = \lim_{n\to\infty} \gamma^{-n}(p)$.

Suppose next that $n_i \to +\infty$ as $i \to +\infty$. Then $\{\gamma_i\}$ can have no convergent subsequence and in particular $d(\gamma_i p, p) \to \infty$ as $i \to +\infty$. We show that $\gamma_i(p) \to \sigma(\infty)$ as $i \to +\infty$. Since $\phi^{n_i} = \gamma_i \omega_i k_i$ for each i, where $\omega_i \to 1$ and $k_i(gX) = gX \in T(G/K)$ for each i, we have

$$\phi^{n_i}(p) = \phi^{n_i}(gK) = \gamma_i \omega_i k_i(gK) = \gamma_i \omega_i gK = \gamma_i \omega_i(p)$$

and

$$g(\exp n_i X)K = g(\exp n_i X)g^{-1}(gK) = \phi^{n_i}(p) = \gamma_i \omega_i(p) \ .$$

It follows that $\sigma(n_i) = \gamma_i \omega_i p$. From $d(\gamma_i(p), p) \to \infty$ as $i \to +\infty$, we have

$$\begin{aligned}\sphericalangle_p(\sigma(\infty), \gamma_i p) &\leqq \sphericalangle_p(\sigma(\infty), \gamma_i\omega_i(p)) + \sphericalangle_p(\gamma_i\omega_i(p), \gamma_i(p)) \\ &= \sphericalangle_p(\sigma(\infty), \sigma(n_i)) + \sphericalangle_p(\gamma_i\omega_i(p), \gamma_i(p)) \to 0 \ .\end{aligned}$$

Thus $\gamma_i(p) \to \sigma(\infty)$ as $i \to +\infty$.

To show that $\gamma_i^{-1}(p) \to \sigma(-\infty)$ as $i \to \infty$ one first needs to observe that ϕ commutes with k_i for each i; otherwise k_i^{-1} is in the wrong position in the formula for ϕ^{-n_i}. Note that the 1-parameter group of transvections $(k_i g)(\exp tX)(k_i g)^{-1}$ translates the geodesic with initial velocity $k_i gX = gX$ and hence must equal the 1-parameter group $g(\exp tX)g^{-1}$. One now sees that $\phi^{n_i}k_i^{-1} = k_i^{-1}\phi^{n_i} = \gamma_i\omega_i$ and hence $\phi^{-n_i}k_i = \omega_i^{-1}\gamma_i^{-1}$ and $\phi^{-n_i} = \omega_i^{-1}\gamma_i^{-1}k_i^{-1}$. One now applies the same argument as above to conclude that $\gamma_i^{-1}(p) \to \sigma(-\infty)$ as $i \to \infty$. This shows that (3) $\to$ (1).

(2) $\to$ (3). We need the following fact which may be found, for example, on p. 464 of P. Eberlein, "Lattices in Spaces of Nonpositive Curvature," Annals of Math., **111** (1980), 435–476. Let $A \subseteq T_1M$, where $M = G/K$, be the set of vectors v such that there exists a sequence $\{t_n\} \to +\infty$ and a sequence $\{\gamma_n\} \subseteq \Gamma$ such that $(\gamma_n)F_{t_n}v \to v$ as $n \to \infty$. Then A is dense in T_1M if Γ satisfies the duality condition.

Next we need a result whose proof is postponed temporarily.

LEMMA. *Let $I_0 = [a, b]$ be any interval with $a > 0$ and let $\{t_m\} \to +\infty$ be any divergent sequence. Then there exists a point $x \in I_0$ and divergent sequences $\{m_k\}$, $\{r_k\}$ of positive integers such that*

$$|t_{m_k} - r_k x| \longrightarrow 0 \quad as \quad k \longrightarrow +\infty \ .$$

Assuming this lemma, we continue the proof of (2) $\to$ (3). *Let* $Z = \{(g, X) \in G \times \mathfrak{P}: \phi = g(\exp X)g^{-1}$ *satisfies the conditions of* (3)$\}$. *We show that* Z *is dense in* $G \times \mathfrak{P}$. *Let* (g, X) *be an arbitrary element of* $G \times \mathfrak{P}$ *and let* $v = gX \in TM$. *Let* $A^* = \{ta: a \in A, t \geqq 0\}$. *Then* A^* *is dense in* TM *and every nonzero element of* A^* *satisfies the same* Γ*-recurrence property as expressed in the definition of* A. *By the density of* A^* *we can choose a sequence* (g_n, X_n) *converging to* (g, X) *such that* $v_n = g_n X_n \in A^*$ *for every* n.

Now let V, W be arbitrary neighborhoods in G, of g, X respectively. Then $g_n \in V$ and $X_n \in W$ for n sufficiently large. Fix such an integer n and let $g^* = g_n$, $X^* = X_n$ and $v^* = v_n = g^* X^*$. By the definition of A and A^* we can find a sequence $\{t_m\} \to +\infty$ and a sequence $\{\gamma_m\} \subseteqq \Gamma$ such that $(\gamma_m)F_{t_m}v^* \to v^*$ as $m \to +\infty$. Choose an integer $M > 0$ so large that $\xi X^* \in W$ for every $\xi \in I_0 = [1 - (1/M), 1 + (1/M)]$. Applying the lemma above to I_0 we obtain divergent integer sequences $\{m_k\} \subseteqq \boldsymbol{Z}^+$ and $\{r_k\} \subseteqq \boldsymbol{Z}^+$ and a point $\xi \in I_0$ such that $|t_{m_k} - r_k \xi| \to 0$ as $k \to +\infty$. Clearly $(g^*, \xi X^*) \in V \times W$ and we assert that $(g^*, \xi X^*) \in Z$, which will prove that Z is dense in $G \times \mathfrak{P}$ and complete the proof of (2) $\to$ (3).

Let $\phi = g^*(\exp \xi X^*)g^{*-1}$. We show by passing to a subsequence if necessary that $\phi^{r_k} = \gamma_{m_k}^{-1}\omega_k \xi_0$, where ξ_0 fixes the vector $g^*(\xi X^*) \in TM$, $\omega_k \to 1$ and $\{\gamma_k\} \subseteqq \Gamma$, $\{m_k\} \subseteqq \boldsymbol{Z}^+$ and $\{r_k\} \subseteqq \boldsymbol{Z}^+$ are the sequences constructed above. By hypothesis $(\gamma_m)F_{t_m}v^* = \gamma_m g^*(\exp t_m X^*)g^{*-1} \times (g^* X^*) \to g^* X^* = v^*$ as $m \to +\infty$. If $\psi_k = \gamma_{m_k} g^*(\exp t_{m_k} X^*)g^{*-1}$, then $\psi_k(g^* X^*) \to g^* X^*$ as $k \to \infty$ and hence ψ_k converges to ξ_0 fixing $g^* X^*$ (and $g^*(\xi X^*)$) by passing to a subsequence. Moreover $(\gamma_{m_k}\phi^{r_k})^{-1}\psi_k = g^*(\exp[t_{m_k} - r_k \xi]X^*)g^{*-1} \to 1$ as $k \to \infty$. Hence $\gamma_{m_k}\phi^{r_k} \to \xi_0$ by passing to an appropriate subsequence. Therefore, we may write $\gamma_{m_k}\phi^{r_k} = \omega_k \xi_0$, where $\omega_k \to 1$ as $k \to \infty$ and it follows that $\phi^{r_k} = \gamma_{m_k}^{-1}\omega_k \xi_0$. Hence $(g^*, \xi X^*) \in Z$.

We complete the proof of (2) $\to$ (3) by proving the lemma stated earlier. Let $I_0 = [a, b]$ and $\{t_m\} \to +\infty$ be given where $a > 0$. We proceed by induction. Let $\xi_0 \in (a, b)$ be an arbitrary point of (a, b). Choose δ_1 so that $0 < \delta_1 < 1$ and $I_1 = (\xi_0 - \delta_1, \xi_0 + \delta_1) \subseteqq I_0 = [a, b]$. We let nI_1 denote $\{n\xi: \xi \in I_1\} = (n\xi_0 - n\delta_1, n\xi_0 + n\delta_1)$ for any positive integer n. Choose n so large that $n\delta_1 > b$. The integer multiples of ξ_0 partition R into intervals of length $\xi_0 < b$. Hence, we may choose an integer $m_1 > 1$ and an integer $r_1 > n$ so that $|t_{m_1} - r_1 \xi_0| < b$. It follows that $t_{m_1} \in r_1 I_1$ or equivalently that $t_{m_1} = r_1 \xi_1$ for some $\xi_1 \in I_1$.

Choose δ_2 so that $0 < \delta_2 < 1/2$, $I_2 = (\xi_1 - \delta_2, \xi_1 + \delta_2) \subseteqq I_1$ and $|t_{m_1} - r_1 \xi| < 1/2$ for all $\xi \in I_2$. Choose an integer n so that $n\delta_2 > b$. Choose integers $m_2 > 2$ and $r_2 > n$ so that $|t_{m_2} - r_2 \xi_1| < b$. Then

$t_{m_2} \in r_2 I_2$ or $t_{m_2} = r_2 \xi_2$ for some $\xi_2 \in I_2$.

Proceeding in this fashion one may construct sequences $\{\delta_k\} \subseteqq R$, $\{\xi_k\} \subseteqq I_0$, $\{r_k\} \subseteqq Z^+$ and $\{m_k\} \subseteqq Z^+$ so that for every k, $0 < \delta_k < 1/k$, $\xi_k \in I_k = (\xi_{k-1} - \delta_k, \xi_{k-1} + \delta_k) \subseteqq I_{k-1}$, $t_{m_k} = r_k \xi_k$ and $|t_{m_k} - r_k \xi| < 1/k + 1$ for all $\xi \in I_{k+1}$. The intervals $\{I_k\}$ are nested and have lengths $2\delta_k$ that decrease to zero. Therefore $\bigcap_{k=1}^{\infty} I_k$ is a single point $\xi \in I_0$. Finally, $|t_{m_k} - r_k \xi| < 1/k + 1$ for every k since $\xi \in I_{k+1}$. This completes the proof of the lemma.

3. A density theorem. In this section, we shall prove the following result which is an analogue of Borel's density theorem for simply connected complete visibility manifolds.

THEOREM 2. *Let M be a simply connected complete visibility manifold and let $\Gamma \subseteqq I(M)$ be any subgroup that satisfies the duality condition. Then either Γ is discrete or M is a rank one symmetric space of noncompact type, $(\bar{\Gamma})_0 = I_0(M)$ and $\bar{\Gamma}$ is of finite index in $I(M)$.*

Proof. $I(M)$ satisfies the duality condition because Γ does. Either $I(M)$ is discrete or $I_0(M)$ is not trivial. In the first case, Γ is obviously discrete. If $I_0(M)$ is not trivial, then it satisfies the duality condition. This follows from the argument in the proof of Theorem 5.8 of [5]. Then Corollary 4.14 of the same paper implies that M is a rank one symmetric space of noncompact type. The subgroup Γ is not necessarily a Lie subgroup of $I(M)$. But its closure $\bar{\Gamma}$ is a Lie subgroup. Either $\bar{\Gamma}$ is discrete (hence Γ is discrete) or $(\bar{\Gamma})_0$ is not trivial. By the argument in [4], we can prove that $(\bar{\Gamma})_0 = I_0(M)$. The main idea comes from a decomposition theorem of Mostow [17]. Since $I_0(M) = (\bar{\Gamma})_0 \subseteqq \bar{\Gamma}$ and the index $[I(M), I_0(M)]$ is finite, $\bar{\Gamma}$ is of finite index in $I(M)$ and $\bar{\Gamma} \supseteqq I_0(M)$.

4. Bergman-Kobayashi metric and hyperbolic manifold. In this section, we recall some basic facts about two classes of complex manifolds which will be studied in the next section. We intend to obtain some results for these two classes of manifolds with the additional condition of nonpositive curvature.

Let M be a complex manifold and $H(M)$ be the group of holomorphic automorphisms of M. In general, $H(M)$ may be infinite dimensional. If M is a bounded domain D in C^n, then $H(D)$ is a subgroup of the isometry group $I(D)$ with respect to the Bergman metric which is Kähler on D. S. Kobayashi [15] has generalized the Bergman metric for D to a complex manifold M satisfying the very ampleness assumption on the complex Hilbert space of holomorphic n-forms which are square integrable. In the sequel, such a manifold will be called a Kobayashi manifold. This Bergman-Kobayashi metric is Kähler and the group $H(M)$ of holomorphic automorphisms is a closed subgroup of $I(M)$ of isometries of M.

Another class of complex manifolds called hyperbolic manifolds has also been introduced by S. Kobayashi [14]. By considering holomorphic mapping of the open unit disc into a complex manifold M, one denies a pseudo-distance d_M on M. If d_M is a distance, then M is called a hyperbolic manifold. Hermitian manifolds of holomorphic sectional curvature bounded above by a negative constant are hyperbolic. For hyperbolic manifolds, the group $H(M)$ of holomorphic automorphisms is a closed subgroup of the Lie group $I(M)$ of isometries of M with respect to d_M.

5. Complete Kähler manifolds of nonpositive curvature. Let M be a simply connected complete Kähler manifold of nonpositive curvature. In particular, M is a Riemannian manifold and hence theorems in [5] are applicable to M. We shall refer to the corresponding results in [5].

THEOREM 3. *Let M be a simply connected complete Kähler manifold of sectional curvature bounded above by a negative constant. Then*:

(1) *If $H(M)$ acts transitively on M and satisfies the duality condition, then M is the unit ball in C^n (or the hermitian symmetric space of rank one).*

(2) *If M/Γ is a complete Kähler manifold and the covering group Γ satisfies the duality condition, then either $H(M)$ is discrete or M is the unit ball in C^n (or the hermitian symmetric space of rank one). Γ admits no solvable subgroup of finite index.*

Proof. Since M has sectional curvature bounded above by a negative constant, its holomorphic sectional curvature is bounded above by the same negative constant and M is a hyperbolic manifold. Thus group $H(M)$ of holomorphic automorphisms of M is a closed subgroup of the isometry group $I(M)$. To prove (1), we note that $H(M)$ (hence $I(M)$) acts transitively on M and satisfies the duality condition. By Theorem 5.4 of [5], M is the product of a complex Euclidean space and a hermitian symmetric space. This is due to the fact that M has complex structure. But the Euclidean factor has to be trivial, because there is no complex line in a hyperbolic manifold ([14], p. 49, [15], p. 81). The proof of (1) is finished. The proof of (2) goes as follows. The group $\Gamma \subset H(M) \subset I(M)$ satisfies the duality condition. By Corollary 5.9 of [5], either $I(M)$ is discrete or M is a rank one symmetric space. The first case implies that $H(M)$ is discrete; while the second case gives that M is the unit ball in C^n. That Γ admits no solvable subgroup of finite index follows from Theorem 5.1 of [5].

REMARK 2. This theorem is related to [10] and [11].

REMARK 3. A Kähler manifold M is holomorphically negatively pinched if its holomorphic sectional curvature is negatively pinched. According to Berger [2], its Riemannian sectional curvature is also negatively pinched. Consequently, we may change the assumption in Theorem 3 to Kähler manifolds which are holomorphically negatively pinched. The conclusions remain to be true.

The following result concerns with Kobayashi manifolds of nonpositive curvature, that is, the Bergman-Kobayashi metric has always nonpositive sectional curvature.

THEOREM 4. *Let M be a simply connected complete Kobayashi manifold of nonpositive curvature.*

(i) *If M admits a group Γ of holomorphic automorphisms with the duality condition, than either $H(M)$ is discrete or the identity component $H_0(M)$ in a noncompact semi-simple Lie group. Moreover Γ admits no solvable subgroup of finite index. If in addition M is homogeneous, then M is a hermitian symmetric space.*

(ii) *If D is a bounded homogeneous domain in C^n of nonpositive curvature and D admits a group Γ of holomorphic automorphisms with the duality condition, then D is a bounded symmetric domain in C^n.*

Proof. The proof is similar to that of Theorem 3. We note that Kobayashi manifolds do not admit parallel vector fields and do not contain complex lines [15]. Thus, Kobayashi manifolds can not be flat. $H(M)$ is a closed subgroup of $I(M)$. By Proposition 2.5 of [5], $H_0(M)$ is either $\{1\}$ or a noncompact semi-simple Lie group.

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