PACIFIC JOURNAL OF MATHEMATICS
Vol. 99, No. 2, 1982

TRANSVERSALS TO LAMINATIONS

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The stable and unstable manifolds of an Anosov diffeomorphism are not leaves of C^1-foliation. Instead, their unions comprise two laminations; that is, two C^0-foliations which have C^1-smooth leaves and continuous nonsingular tangent plane fields. Recently C. Ennis has shown that laminations have transversals at every point. In this note, the existence of transversals is shown to require plane field continuity.

For these purposes, a C^0-foliation with C^1-smooth leaves will be called an *erratic lamination*. These may contain infinite sequences of points, $\{p_k\} \to p_0$ having tangent planes which do not limit on the tangent plane through p_0.

The example of Theorem 2 is of a 1-dimensional erratic lamination of $\boldsymbol{R}^2$ containing a leaf having no differentiable transversals. Though higher-dimensional, lower codimentional analogues most certainly do exist, the discussion and definitions to follow will be limitted to 1-dimensional foliations.

A C^0-imbedd $(n-1)$-disk D contained in an n-manifold is *topologically transverse* to the leaf of a C^0-foliation if at each point of their intersection, the leaf crosses the disk in a single point. The terms "strictly ingressing" or "strictly egressing" are used similarly in flow theory [3]. D is *topologically transverse* to a C^0-foliation if it is topologically transverse to every leaf. A C^1-imbedded disk is *differentiably transverse* to an erratic lamination if it is differentiably transverse to every leaf. Erratic laminations are the most general foliations for which differentiably transverse disks may exist. A good reference for further definitions and theorems is B. Lawson's survey article, [5].

The Existence of transversals.

The following two theorems distinguish laminations from erratic laminations by the behavior of their topological transversals.

THEOREM 1 (*Ennis* [2], 1979): *Any C^0-imbedded $(n-1)$-disk, D, topologically transverse to a* 1-*dimensional lamination, $\mathscr{L}$ of M^n, can be C^0-approximated by a C^1-imbedded, differentiably transverse disk.*

THEOREM 2. *There exists a* 1-*dimensional lamination $\mathscr{L}$ of $\boldsymbol{R}^2$*

containing a leaf, l_0*, with a point* $p_0 \in l_0$ *such that all* C^1*-imbedded disks, differentiably transverse to* $\mathscr{L}$ *are disjoint from* l_0*. Furthermore, all such disks topologically transverse to* $\mathscr{L}$ *are disjoint from* $l_0 \backslash p_0$.

In a sense, l_0 is a "barrier" to differentiably transverse disks, and p_0 is the only "leak point" of topologically transverse disks through l_0.

The proof of Theorem 1 requires the following theorem.

THEOREM 3 (*Wilson* [6], 1969). *Let* $f\colon M^n \to \boldsymbol{R}$ *be continuous and let* X *be a continuous nonsingular vector field on* M *with unique trajectories. Assume* Xf *(the derivative of* f *along trajectories of* X*) exists and is continuous. Then for all* $\varepsilon > 0$*, there exists a* C^∞*-function* $g\colon M \to \boldsymbol{R}$ *which* ε*-approximates* f *in its* X*-derivative; that is, for all* $p \in M$, $|f(p) - g(p)| < \varepsilon$ *and* $|Xf(p) - Xg(p)| < \varepsilon$.

Proof of Theorem 1. Denote by l_p, the leaf of $\mathscr{L}$ through p. Let X be a normalized, nonsingular tangent vector field to $\mathscr{L}$. Designate $\mathscr{L}(D) = \{l \in \mathscr{L} : l \cap D \neq \varnothing\}$. Let N be a small C^0-foliation chart-neighborhood about D and denote by $\tilde{N} = \bigcup_{l \in \mathscr{L}(D)} (l \cap N)$. For $p \in \tilde{N}$, define $f(p)$ to be the arc-length along l_p from D to p, taken positively in the X-direction and negatively, counter the X-direction. It is assumed that N is sufficiently "box-like" that these arc-segments, l_p, lie entirely within $\tilde{N}$. Thus $f^{-1}(0) = D$.

f is continuous along integral curves of X, the leaves of $\mathscr{L}$; however, there is some question as to the continuity of f as a whole. This is a consequence of the continuity of X: Let $p \in \tilde{N}$ with $f(p) > 0$ and let $\{p_k\} \to p$ be an infinite sequence in $\tilde{N}$. Denote by $\bar{l}_k$ (resp. $\bar{l}_p$) the leaf segment from D to p_k (resp. p) within $\tilde{N}$. $\bar{l}_p$ is contained within a thin C^0 chart-neighborhood, $N_p \subset \tilde{N}$. For p_k sufficiently near p, $\bar{l}_k \subset N_p$. Thus, as $k \to \infty$, $\bar{l}_k \to \bar{l}_p$ in the C^0-sense. For each $x \in \bar{l}_p$ and k large enough, there are well-defined $x_k \in \bar{l}_k$ such that $x_k \to x$ as $k \to \infty$. X continuous then implies $X(x_k) \to X(x)$. And since X is unit,

$$f(p_k) = \int_{\bar{l}_k} X ds \longrightarrow \int_{\bar{l}_p} X ds = f(p)$$

as desired.

Again X unit implies that $Xf(p) = 1$ for all $p \in \tilde{N}$. f may be extended to all of M maintaining that $Xf = 1$ in a neighborhood of $\tilde{N}$ and that $f^{-1}(0) \cap \tilde{N} \supset D$.

Now Theorem 3 may be applied to this extended f: There exists a C^0-close, C^∞-map g near f such that $Xg > 1/2$ on a neighborhood of D. Thus $g^{-1}(0) \cap \tilde{N}$ contains a C^∞-imbedded disk which C^0-approximates D. g being in fact Lipshitz assures that $D_X g = Xg > 0$. So the rank of Dg near D is 1 implying that 0 is a regular value of g. Thus $g^{-1}(0)$ is transverse to $\mathscr{L}$ as desired.

Construction of the erratic lamination. In the discussion to follow, all smooth unit-speed arcs differentiably (resp. topologically) transverse to the lamination or erratic lamination in question will be called *pathways* (resp. *topological pathways*).

Let $S = \{(x, y) \in \boldsymbol{R}^2 : 0 \leqq x \leqq 1\}$. A pathway $\gamma: [0, 1] \to S$ such that $x(\gamma(0)) = 0$ and $x(\gamma(1)) = 1$ is *reversing* if there exists $t_0 \in (0, 1)$ such that $\gamma'(t_0) = -|\gamma'(t_0)|(\partial/\partial x)$. Such points, $\gamma(t_0)$, are called

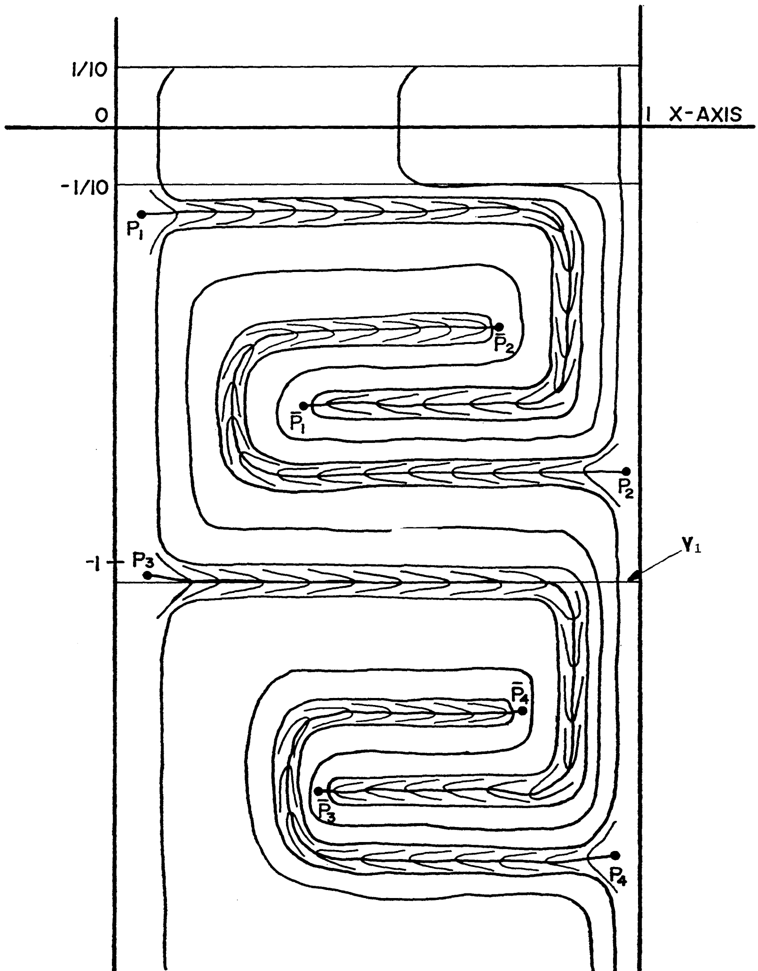

FIGURE 1

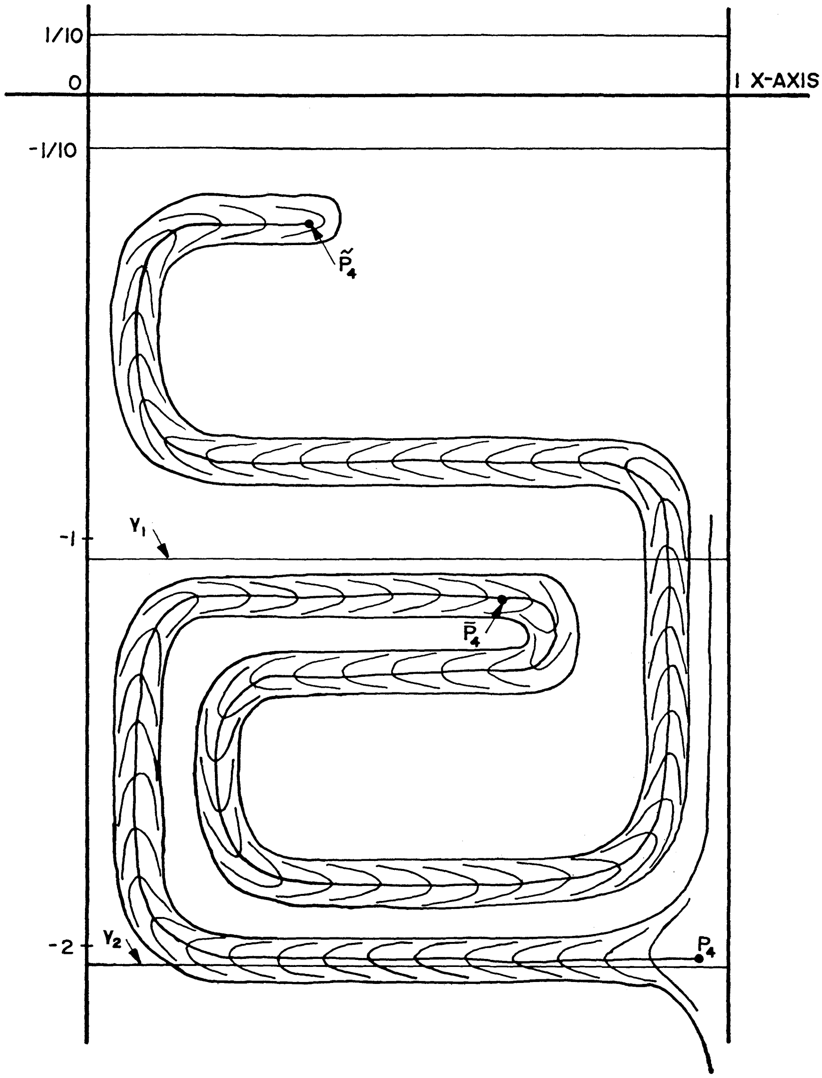

FIGURE 2

reversing points.

S is to be C^1-foliated by $\mathscr{G}$, an isotope of the product foliation, $\mathscr{G}_0 = \{x = \text{const.}\}$. Figures 1, 2, and 3 depict stages of this smooth isotopy, h_t which is symmetrical across the x-axis and has support off $\partial S \cup \{x\text{-axis}\}$. Pathways through $h_t\mathscr{G}_0$ are forced to bend. $\mathscr{G} = h_1\mathscr{G}_0$ is such that topological pathways are either reversing or meet the rectangle $B = [0, 1] \times [1/10, 1/10]$.

In Figure 1, points designated p_1, p_2, p_3, and p_4 have been pushed by h_t, $t \in [0, 1/3]$, along arcs to new locations; $\bar{p}_1$, $\bar{p}_2$, $\bar{p}_3$, and $\bar{p}_4$. In the process, $\mathscr{G}_0$ has been distorted to contain two pairs of hooking "tracer-protrusions"; if you will. These pairs are repeated down the

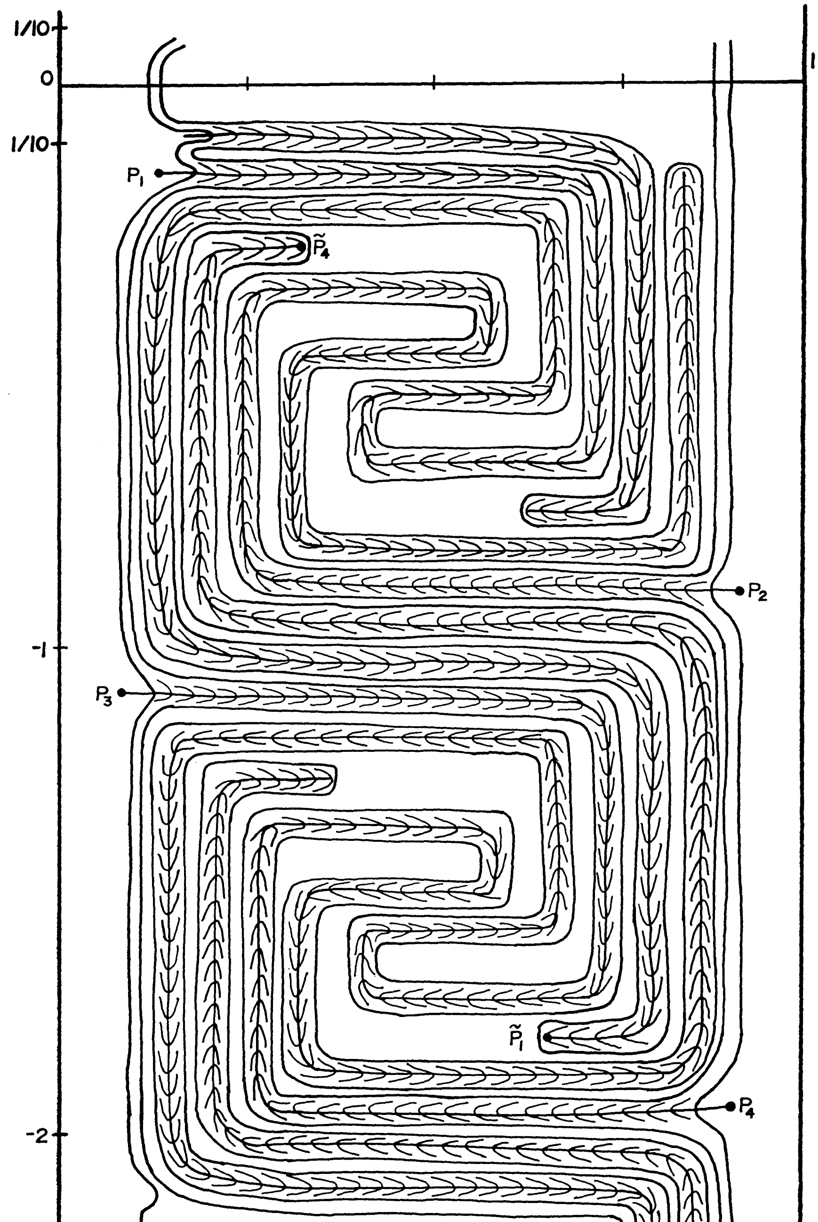

FIGURE 3

y-axis and are evenly-spaced.

The arc γ_1 of Figure 1 is a nonreversing topological pathway which is disjoint from B. In order to force γ_1 to bend, h_t, $t \in [1/2, 2/3]$, pushes $\bar{p}_4$ and its tracer-protrusion back on itself and through γ_1 (Figure 2). However, the topological pathway γ_2 of

Figure 2 is disjoint from B and nonreversing. To bend such pathways, all pairs of tracer-protrusions, up and down the y-axis, are folded back on themselves (Figure 3).

It is now claimed that $\mathscr{G} = h_1\mathscr{G}_0$ and $h = h_1^{-1}$ have the following properties:

LEMMA 1. (i) *The x-axis is topologically transverse to $\mathscr{G}$.*

(ii) *h carries $\mathscr{G}$ onto the product foliation, $\{x = \text{const.}\}$.*

(iii) *The C^0-size of h is bounded and so proportionate to the width of S.*

(iv) *Every nonreversing topological pathway through $\mathscr{G}$ intersects $B = [0, 1] \times [1/10, 1/10]$.*

In a sense, condition (iv) means that nonreversing pathways through $\mathscr{G}$ are "funneled" to within 1/10 of the x-axis (in y-coordinate). This distance will be referred to as the *funnel width* of S.

$\mathscr{L}$ of Theorem 2 is now constructed by mapping an infinite sequence of "replicas" of $\mathscr{G}$ into $\boldsymbol{R}^2$ in such a way that their boundary leaves converge C^1 onto the cubic leaf $l_0 = \{x, x^3\}$. These may be thought of as an infinite sequence of "gates" by which the desired behavior of topological pathways is forced. The point p_0 referred to in Theorem 2 is the origin.

For $c \geqq 0$, the leaf of $\mathscr{L}$ through $(c, 0)$ has the form $\{(x, (x - c)^3)\}$. A sequence of smooth arcs, $\{l_k : k \in \boldsymbol{Z}^+\}$ form the left-hand boundaries of the replicas of $\mathscr{G}$. The $\{l_k\}$ are presumed to have the following properties:

(1) $l_k \cap \{x - \text{axis}\} = -1/2^k$.

(2) Each l_k is differentiably transverse to the x-axis.

(3) $l_k \to l_0$ in the C^1-sense.

(4) $l_k = \{(g_k(0, y), y)\}$ where $g_k : \{y\text{-axis}\} \to \boldsymbol{R}$ is smooth.

(5) $\{l_k\}$ are pairwise disjoint. In fact, for all $k \in \boldsymbol{Z}^+$,

$$\inf_y \{g_{k+1}(0, y) - g_k(0, y)\} > 1/2^{k+1} .$$

$A_k : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ is given by $A_k(x, y) = (x/2^{k+2}, y)$. $B_k : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ is a diffeomorphism given by $B_k(x, y) = (x, b_k(y))$ where $b_k(y) = y$ when $|y| \geqq 1$ and $b_k(y) = (1/2^{k+2})^6 y$ when $|y| \leqq 1/2$. On $\{1/2 < |y| < 1\}$, b_k is the usual bump function. B_k will be applied to $\mathscr{G}$ on S to assure strong enough compression of the funnel widths as the replicas of $\mathscr{G}$ converge on l_0.

Let

$$S_k = \{(x, y) : g_k(0, y) \leqq x \leqq g_k(0, y) + 1/2^{k+2}\} .$$

Extend g_k onto $\{0 \leqq x \leqq 1/2^{k+2}\}$ by $\tilde{g}_k(x, y) = (g_k(0, y) + x, y)$. Now, $\mathscr{G}_k = \mathscr{L} \cap S_k = \tilde{g}_k A_k B_k \mathscr{G}$. Conditions (1) and (5) on $\{l_k\}$ assure that $\{S_k\}$ are pairwise disjoint; the bands in $\boldsymbol{R}^2$ between each S_k and S_{k+1} is smoothly foliated by graphs of the y-axis each differentiably transverse to the x-axis. The union of these leaves comprise $\mathscr{L}$.

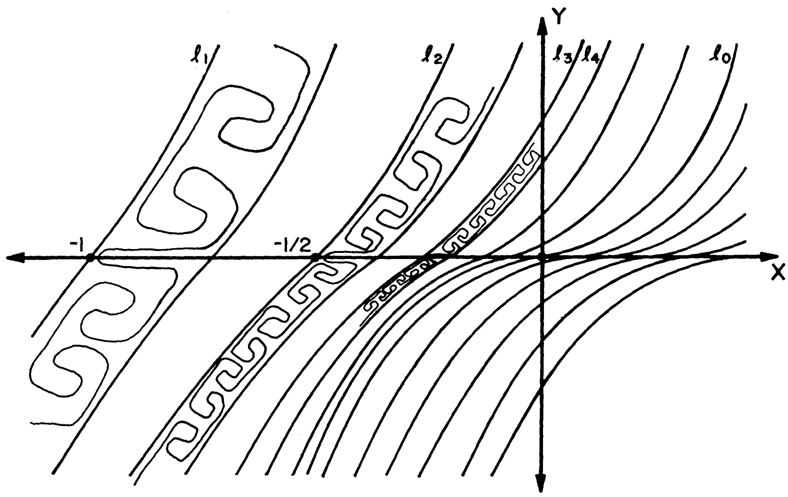

FIGURE 4

Proof of Theorem 2. $\mathscr{L}$ is the union of (C^1) smooth leaves and is a smooth foliation off l_0. Let $\mathscr{F}$ be the smooth foliation of $\boldsymbol{R}^2$ which is identical to $\mathscr{L}$ except $\mathscr{F} = \tilde{g}_k A_k \mathscr{G}_0$ on each S_k. That $\mathscr{F}$ is smooth follows from condition (3) on $\{l_k\}$. Because $\mathscr{G}$ is an isotope of $\mathscr{G}_0$ on S, $\mathscr{L} \cap S_k$ is an isotope of $\mathscr{F} \cap S_k$. Thus, there is an isotopy, h_t on $\boldsymbol{R}^2\backslash\{l_0\}$, which carries $\mathscr{L}\backslash\{l_0\}$ to $\mathscr{F}\backslash\{l_0\}$ and fixes $(\boldsymbol{R}^2\backslash\{l_0\})\backslash \cup S_k$. As a fact, h_t may not be smoothly extended onto l_0 (Theorem 1), but may be continuously extended to the identity on l_0 if the C^0-size of h_t and h_t^{-1} on each S_k approaches 0 as $k \to \infty$. This follows from the fact that h, A_k, and B_k have C^0-sizes proportionate to the strip widths. So $\mathscr{L}$ is an erratic lamination.

The composition, $\tilde{g}_k A_k B_k$ carries reversing points to reversing points since each of its Jacobian matrices has (1, 0) as an eigenvector. Due to the cubic shearing of the S_k near (0, 0), the funnel widths of $g_k A_k \mathscr{L}$ on S_k enlarge proportionate to the strip widths as $k \to \infty$. Briefly, this cubic shearing is suppressed by the sextic compression of the B_k. In fact, as $k \to \infty$, these funnel widths decrease quadratically relative to the strip widths: Consequently, every nonreversing topological pathway through S_k contains a point, q_k, such that $|y(q_k)| < 1/10(2^{-2(k+2)})$.

Let γ be a topological pathway which crosses l_0. For some

$N > 0$, γ crosses all strips S_k, $k > N$. If eventually each such crossing contains a reversing point, then γ contains a sequence of such points converging on l_0. This contradicts that γ is differentiable and unit speed. Thus for some larger $N > 0$, γ contains a sequence $\{q_k \in S_k\}$ as above which limits on $q_0 \in l_0$. Since $y(q_k) \to 0$, $x(q_0) = 0$. So γ must cross l_0 at $(0, 0)$. But further, since $y(q_k) \to 0$ quadratically, $\gamma'(q_k) \cdot (\partial/\partial y) \to 0$, implying that γ is tangent to l_0. Since all smoothly imbedded 1-disks in $\boldsymbol{R}^2$ may be parametrized by a unit speed arc, the desired result is attained.

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Received June 4, 1979.

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