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ON THE BOUNDARY CURVES OF INCOMPRESSIBLE SURFACES

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Let K be a knot in S^3, and consider incompressible (in the stronger sense of π_1-injective), ∂-incompressible surfaces S in the exterior of K. A question which has been around for some time is whether the boundary-slope function $S \mapsto m_S/\ell_S$, where m_S and ℓ_S are the numbers of times each circle of ∂S wraps around K meridionally and longitudinally, takes on only finitely many values (for fixed K). This is known to be true for certain knots: torus knots, the figure-eight knot [4], 2-bridge knots [2], and alternating knots [3]. In this paper an affirmative answer is given not just for knot exteriors, but for all compact orientable irreducible 3-manifolds M with ∂M a torus. Further, we give a natural generalization to the case when ∂M is a union of tori.

To state this more general result it is convenient to use the projective lamination space $\mathscr{PL}(\partial M)$, defined in [4]. If ∂M is the union of tori $T_1, \cdots, T_n$, then $\mathscr{PL}(\partial M)$ is the join $\mathscr{PL}(T_1) * \cdots * \mathscr{PL}(T_n) = \boldsymbol{RP}^1 * \cdots * \boldsymbol{RP}^1$, a sphere S^{2n-1}. More concretely, suppose coordinates are chosen for each T_i. Then isotopy classes of finite systems of disjoint noncontractible simple closed curves on T_i are parametrized by the set $\boldsymbol{Z}^2/\pm$ of pairs $(a, b) \in \boldsymbol{Z}^2$, where (a, b) is identified with $(-a, -b)$. So systems on ∂M are parametrized by $(\boldsymbol{Z}^2/\pm)^n$. Restricting to nonempty systems and projectivising by identifying a system with any number of parallel copies of itself, yields $(\boldsymbol{Z}^2/\pm)^n - \{0\}/(v \sim \lambda v)$. This is the same as $(\boldsymbol{Q}^2/\pm)^n - \{0\}/(v \sim \lambda v)$. The natural completion of this space is $\mathscr{PL}(\partial M) = (\boldsymbol{R}^2/\pm)^n - \{0\}/(v \sim \lambda v)$, clearly a sphere of dimension $2n-1$. (We shall not be concerned with the geometrical interpretation of the points added in forming this completion.) A change of coordinates for the T_i's produces a projective transformation of this S^{2n-1}, so $\mathscr{PL}(\partial M)$ has a natural projective structure. (For surfaces of higher genus, $\mathscr{PL}$ has only a natural piecewise projective structure.)

THEOREM. *Let M be orientable, compact, irreducible, with ∂M a union of n tori. Then the projective classes of curve systems in ∂M which bound incompressible, ∂-incompressible surfaces in M form a dense subset of a finite (projective) polyhedron in $\mathscr{PL}(\partial M) = S^{2n-1}$ of dimension less than n.*

COROLLARY. *If $\partial M = T^2$, there are just a finite number of slopes* *realized by boundary curves of incompressible, ∂-incompressible surfaces in M.*

Returning to the case of a knot $K \subset S^3$, this means that if the peripheral torus $\partial N(K)$ is the only closed incompressible surface in $S^3 - K$, up to isotopy, then all but a finite number of Dehn surgeries on K yield irreducible non-Haken manifolds. More generally, for links $L \subset S^3$ of n components such that $S^3 - L$ is irreducible (i.e., L is nonsplit) and contains no closed nonperipheral incompressible surfaces, the theorem implies that the coefficients $(p_1/q_1, \cdots, p_n/q_n) \in T^n$ of the Dehn surgeries on L which yield either nonirreducible or Haken manifolds lie in a piecewise smooth finite subcomplex of T^n of dimension less than n. Thus, "most" surgeries on L yield irreducible non-Haken manifolds.

One may compare the assertion of the theorem with the fact (a consequence of duality) that the image of the boundary map $H_2(M, \partial M) \to H_1(\partial M)$ has rank equal to half the rank of $H_1(\partial M)$. Thus, passing to the sphere S^{2n-1} of rays through the origin in $H_1(\partial M; \boldsymbol{R})$, the image of the homological boundary map is a sphere $S^{n-1} \subset S^{2n-1}$.

The proof of the theorem will follow fairly easily from a recent fundamental result of [1] about branched surfaces in 3-manifolds, i.e., closed subsets locally diffeomorphic to the model in Figure 1a.

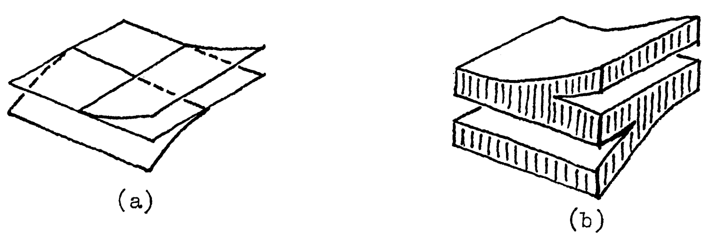

FIGURE 1

A branched surface B is said to *carry* a surface S if S lies in a fibered regular neighborhood $N(B)$ of B (indicated in Figure 1b) and is transverse to all the fibers of $N(B)$; S has *positive weights* if it meets all the fibers of $N(B)$. The result of [1] is that in a compact irreducible 3-manifold M with incompressible boundary there exist a finite number of branched surfaces $(B_i, \partial B_i) \subset (M, \partial M)$ such that the surfaces carried by these B_i's with positive weights are exactly all the incompressible, ∂-incompressible surfaces in M, up to isotopy.

Let $B \subset M$ be one of the branched surfaces of [1]. Then $\partial B = B \cap \partial M$ is a branched 1-manifold in ∂M with two key properties:

(1) There is no smooth disk $D \subset \partial M$ with $D \cap \partial B = \partial D$.

(2) There is no disk $D \subset \partial M$, smooth except for one outward cusp point in ∂D, with $D \cap \partial B = \partial D$.
Condition (2) is explicitly given in [1]. If condition (1) failed, then any surface carried by B with positive weights would have a boundary circle which was contractible in ∂M. This circle would then bound a disk component of the (incompressible) surface, contrary to the construction of B in [1].

(In the terminology of [4], ∂B would be a train track in ∂M, were it not for the fact that some components of $\partial M - \partial B$ can be digons.)

Let S be a surface carried by B with positive weights. No component of ∂S can be contractible in ∂M, since otherwise there would be a smooth disk $D \subset \partial M$ with $\partial D \subset \partial B$, and somewhere inside this disk condition (1) or (2) would be violated. Thus in each component T_i of ∂M which B meets, ∂S consists of a number of parallel nontrivial circles.

LEMMA. *There is an orientation $\mathcal{O}$ on ∂B with the property that all the circles of $T_i \cap \partial S$, with the orientations induced from $\mathcal{O}$, are homologous in T_i. Hence the $\mathcal{O}$-oriented class $[\partial S] \in H_1(\partial M)$ determines the class of ∂S in $\mathcal{PL}(\partial M)$, by simply forgetting the orientation and then projectivising.*

Proof. Let S_0 be a surface carried by B with positive weights. We can construct a fibered regular neighborhood $N(\partial S_0)$ of ∂S_0 in ∂M from a fibered regular neighborhood $N(\partial B)$ of ∂B in ∂M by slitting $N(\partial B)$ along certain circles and arcs in $N(\partial B)$ transverse to the fibers. Inverting this construction, we see that $N(\partial B)$ is obtainable from $N(\partial S_0)$ by adding certain fibered rectangles and annuli in $\partial M - N(\partial S_0)$, as shown in Figure 2. (Annuli would be necessary if ∂S_0 contained pairs of circles which were parallel in $N(\partial B)$.)

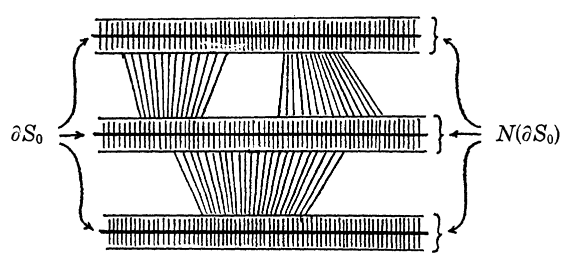

FIGURE 2

No fiber of a rectangle of $N(\partial B) - N(\partial S_0)$ can join a component of $\partial N(\partial S_0)$ to itself, since this would violate condition (2) above. Hence all the fibers of $N(\partial B)$ can be coherently oriented, as follows. On each component T_i of ∂M which ∂B meets, choose parallel orientations for all the circles of $\partial S_0 \cap T_i$; then an orientation for T_i determines an orientation for the fibers of $N(\partial S_0) \cap T_i$ which extends to $N(\partial B) \cap T_i$.

We may choose an oriented simple closed curve γ_i in T_i meeting $N(\partial B)$ in union of fibers of $N(\partial B)$, such that the orientation of γ_i agrees with the orientation of the fibers. To construct such a γ_i, start with any fiber of $N(\partial B)$, continue across a rectangle or annulus of $T_i - N(\partial B)$ to another fiber of $N(\partial B)$ on the opposite side of this rectangle or annulus, and so on. Eventually the curve so constructed must either close up or come arbitrarily close to closing up, in which case by rechoosing part of the curve in $T_i - N(\partial B)$ we can make it close up. From the existence of γ_i the first statement of the lemma clearly follows, since all the points of $\gamma_i \cap \partial S$ have intersection numbers of the same sign. The second statement of the lemma is then immediate. □

Proof of the Theorem. The surfaces carried by B are determined by assigning nonnegative integer weights to the components of $B - B'$, where B' is the branching locus of B. These weights a_i must satisfy certain equations of the form $a_i + a_j = a_k$, coming from the branching at B'. Thus the projective classes of surfaces carried by B correspond to the "rational" points of a convex polyhedral cell which is the intersection of the $(N-1)$-simplex $[0, \infty)^N - \{0\}/(v \sim \lambda v)$ with a linear subspace of $\boldsymbol{R}^N$, for some N. The same is true if one restricts to projective classes of boundary curve systems of surfaces carried by B: they are parametrized by the rational points of a convex polyhedral cell, c_B say. Restricting to surfaces of positive weights corresponds to taking c_B to be an open cell. Moreover, by the lemma, the open cell c_B maps into $\mathscr{PL}(\partial M)$ by a projective linear map. (Regarding ∂B as an oriented 1-complex, then a weighting of its edges by integers a_i satisfying the branching conditions $a_i + a_j = a_k$ determines a 1-cycle, which obviously depends linearly on the a_i's.)

It remains to see that $\dim c_B < n$. Let S_1 and S_2 be two surfaces carried by B with positive weights. If ∂S_1 and ∂S_2 are oriented by $\mathscr{O}$, then the intersection number $\partial S_1 \cdot \partial S_2$ is zero, provided ∂M is oriented as the boundary of M. To see this, perturb S_1 and S_2 slightly to be transverse (and still transverse to the fibers of $N(B)$). There are two possible configurations for the $\mathscr{O}$-orientations of ∂S_1 and ∂S_2 at the ends of an arc α of $S_1 \cap S_2$, as shown in Figure 3 below (where the fibers of $N(B)$ are in the vertical direction). In either case the two ends of α give points of $\partial S_1 \cap \partial S_2$ with opposite intersection numbers. So $\partial S_1 \cdot \partial S_2 = 0$.

Thus, the classes ∂S for surfaces S carried by B with positive weights belong to a self-annihilating subspace of $H_1(\partial M) \approx \boldsymbol{Z}^{2n}$. By linear algebra, this subspace has rank $\leqq n$, and the theorem follows. $\square$

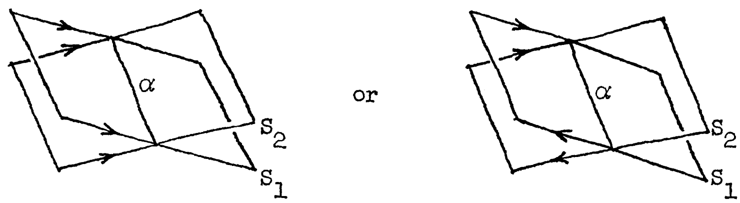

FIGURE 3

Questions.

(1) Is there a generalization of the theorem to 3-manifolds having boundary components of higher genus?

(2) For knot exteriors in S^3, must the boundary slopes of incompressible, ∂-incompressible surfaces always be integers?

(3) Are there nontrivial knots for which all the incompressible, ∂-incompressible surfaces have the same boundary slope?

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