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ON THE TRANSFORMATION OF FOURIER COEFFICIENTS OF CERTAIN CLASSES OF FUNCTIONS

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Suppose $f(x) \in L^1(0, \pi)$ and let $a=\{a_\nu\}(b=\{b_\nu\})$ denote the Fourier cosine (sine) coefficients of f extended to $(-\pi, \pi)$ as an even (odd) function, that is

$$a_0=\frac{2}{\pi}\int_0^\pi f(x)dx\ ,\quad a_\nu=\frac{2}{\pi}\int_0^\pi f(x)\cos \nu x dx\ ,\qquad \nu=1,2,\cdots$$

$$b_\nu=\frac{2}{\pi}\int_0^\pi f(x)\sin \nu x dx\ .$$

The sequence transformations T and T' are defined by

$$(Ta)_0=a_0,\ (Ta)_\nu=\frac{1}{\nu}\sum_{j=1}^{\nu} a_j,\quad (T'a)_\nu=\sum_{j=\nu}^{\infty}(a_j/j),\quad \nu=1,2,\cdots .$$

The purpose of this note is to characterize those rearrangement invariant function spaces $L^\sigma(0, \pi)$ which are left invariant by the operators T and T' acting on Fourier coefficients of functions in these spaces. Our results include and improve some results of Hardy, Bellman and Alshynbaeva.

G. H. Hardy [5] proved that if $f \in L^p(0, \pi)$ for some p, $1 \leqq p < \infty$, then $Ta = \{(Ta)_\nu\}$ is the sequence of Fourier cosine coefficients of a function also in $L^p(0, \pi)$; R. Bellman [2] proved the analogous theorem for T' except that now $1 < p \leqq \infty$. Recently E. Alshynbaeva [1] gave necessary and sufficient conditions on an Orlicz space $L_{M\Phi}$ in order that $L_{M\Phi}$ may replace the L^p space in the results of Hardy and Bellman, thus answering a question of P. L. Ul'yanov. The analogues for the sequences $\{b_\nu\}$ were also studied.

We denote by f^* the nonnegative, nonincreasing function on $(0, \pi)$ which is equi-measurable with f, that is, for all $\lambda > 0$

$$|\{x \in (0, \pi): |f(x)| > \lambda\}| = |\{x \in (0, \pi): f^*(x) > \lambda\}|\ .$$

We suppose throughout that σ is a function norm defined on the measurable functions on $(0, \pi)$ which is rearrangement invariant in the sense that $\sigma(f) = \sigma(f^*)$. The associate of σ, denoted σ', is then also rearrangement invariant and is given by

$$\begin{aligned}\sigma'(f) &= \sup\left\{\left|\int_0^\pi f(x)g(x)dx\right|: \sigma(g) \leqq 1\right\} \\ &= \sup\left\{\int_0^\pi f^*(x)g^*(x)dx: \sigma(g) \leqq 1\right\}\ .\end{aligned} \tag{1}$$

The upper and lower Boyd indices α, β of the Banach space $L^{\sigma}(0, \pi) = \{f: \sigma(f) < \infty\}$ are defined in [4] and satisfy $0 \leqq \beta \leqq \alpha \leqq 1$. For the Lorentz spaces $L^{p,q}(0, \pi)$ and in particular for the Lebesgue spaces $L^p(0, \pi)$, the indices α, β are both equal to p^{-1}. Indices for the Orlicz spaces are computed in [3]. It is well known that $L^{\infty}(0, \pi) \subseteqq L^{\sigma}(0, \pi) \subseteqq L^1(0, \pi)$ for every σ and it is not difficult to see that $L^p(0, \pi) \subseteqq L^{\sigma}(0, \pi) \subseteqq L^q(0, \pi)$ whenever $p^{-1} < \beta$, $\alpha < q^{-1}$.

We shall state and prove our theorems only for the case of cosine coefficients a; for the case of sine coefficients b the statements of the theorems are the same with b replacing a and sine, replacing cosine throughout while the proofs are similar.

Concerning the sequence $\{a_\nu\}$ and the transformations T and T' we have the following theorems.

THEOREM 1. *The following statements are equivalent.*

(a) *For every $f \in L^{\sigma}(0, \pi)$ with Fourier cosine coefficients $a = \{a_\nu\}$, Ta is the sequence of Fourier cosine coefficients of a function in $L^{\sigma}(0, \pi)$.*

(b) *The lower index β of $L^{\sigma}(0, \pi)$ satisfies $\beta > 0$.*

THEOREM 2. *The following statements are equivalent.*

(a) *For every $f \in L^{\sigma}(0, \pi)$ with Fourier cosine coefficients $a = \{a_\nu\}$, $T'a$ is the sequence of Fourier cosine coefficients of a function in $L^{\sigma}(0, \pi)$.*

(b) *The upper index α of $L^{\sigma}(0, \pi)$ satisfies $\alpha < 1$.*

Since $\alpha = \beta = p^{-1}$ for the space L^p, Theorems 1 and 2 yield the results of Hardy and Bellman cited above. It is well known, and in any event follows easily from the formulae for α, β in [3], that for the Orlicz space $L_{M\Phi}$, the lower index β satisfies $\beta > 0$ if and only if Φ satisfies the Δ_2 condition, i.e., $\Phi(2t) \leqq M\Phi(t)$, $t \geqq t_0$; the upper index α satisfies $\alpha < 1$ if and only if the Young's function Ψ complementary to Φ satisfies the Δ_2 condition. Hence Theorems 1 and 2 yield Alshynbaeva's Theorems 1 and 2 with a sharpening of the necessity part of his Theorem 2 in that we do not have to assume $|t \log t| \leqq c\Phi(t)$, $t \geqq t_0 > 0$.

We shall require the following lemma relating to the operators P and P' defined for $0 < x < \pi$ by

$$(Pf)(x) = \cot(x/2)\int_0^x f(t)dt, \quad (P'f)(x) = \int_x^{\pi} f(t)\cot(t/2)dt .$$

LEMMA 1. *The following are equivalent.*

(a) *$Pf \in L^{\sigma}(0, \pi)$ for every $f \in L^{\sigma}(0, \pi)$.*

(b) *There is a constant c such that $\sigma(Pf) \leqq c\sigma(f)$, for all $f \in L^{\sigma}(0, \pi)$.*

(c) *The upper index α of $L^{\sigma}(0, \pi)$ satisfies $\alpha < 1$.*

(d) *The lower index β' of $L^{\sigma'}(0, \pi)$ satisfies $\beta' > 0$.*

(e) *There is a constant c such that $\sigma'(P'f) \leqq c\sigma'(f)$, for all $f \in L^{\sigma'}(0, \pi)$.*

(f) *$P'f \in L^{\sigma'}(0, \pi)$ for every $f \in L^{\sigma'}(0, \pi)$.*

Proof of Lemma 1. Let $(P_1 f)(x) = \left(\int_0^x f(t)dt\right)\Big/x$. There are positive constants c, c_1, c_2 such that for all $f \geqq 0$

$$c_1(Pf)(x) \leqq (P_1 f)(x) \leqq c_2\left((Pf)(x) + \int_0^{\pi} f(t)dt\right) \leqq c_2((Pf)(x) + c\sigma(f))$$

and since $f \in L^{\sigma}(0, \pi)$ if and only if $|f| \in L^{\sigma}(0, \pi)$ it follows that (a) is equivalent to the corresponding statement with P replaced by P_1; similarly P_1 may replace P in (b). Analogously, $(P_1'f)(x) = \int_x^{\pi}(f(t)/t)dt$ may replace P' in statements (e) and (f). Thus, it suffices to prove the lemma with P replaced by P_1 and P' replaced by P_1' throughout. For this, the chain of implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) follows in turn from Lorentz [7, p. 486], Boyd [4, p. 1253], Boyd [4, Lemma 5] and Boyd [4, p. 1253]; (e) clearly implies (f), while if (f) holds and $f \in L^{\sigma}(0, \pi)$, $g \in L^{\sigma'}(0, \pi)$ with $f \geqq 0$, $g \geqq 0$ then Fubini's theorem shows that

$$\int_0^{\pi} g(x)(P_1 f)(x)dx = \int_0^{\pi} f(t)(P_1'g)(t)dt \leqq \sigma(f)\sigma(P_1'g) < \infty$$

so $P_1 f \in L^{\sigma}$ (see Lorentz [7, p. 484]) and (a) holds. This proves the lemma.

LEMMA 2. *If $a = \{a_\nu\}$ is the sequence of Fourier cosine coefficients of $f \in L^{\sigma}(0, \pi)$ then $c = \{c_\nu\}$, $c_0 = 0$, $c_\nu = a_\nu/\nu$, $\nu = 1, 2, \cdots$ is the sequence of Fourier cosine coefficients of a function $F \in L^{\sigma}(0, \pi)$.*

Proof of Lemma 2. Let $K(t) = -\log|2\sin(t/2)|$, $|t| < \pi$. According to [8, p. 180], c is the sequence of Fourier cosine coefficients of

$$F(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)K(t)dt, \ 0 < x < \pi .$$

Now for any t, $|t| < \pi$ we set $f_t(x) = f(x+t)$ and observe that since f is even on $(-\pi, \pi)$, for all $\lambda > 0$

$$\begin{aligned}|\{x \in (0, \pi): |f(x)| > \lambda\}| &= \frac{1}{2}|\{x \in (-\pi, \pi): |f(x)| > \lambda\}| \\ &= \frac{1}{2}|\{x \in (-\pi, \pi): |f_t(x)| > \lambda\}| \\ &\geqq \frac{1}{2}|\{x \in (0, \pi): |f_t(x)| > \lambda\}|\end{aligned}$$

so that $f_t(x)$ considered as a function on $0<x<\pi$ satisfies $(f_t)^*(x) \leqq f^*(x/2)$ and it then follows from (1) that $\sigma(f_t) \leqq 2\sigma(f)$. Hence, if $g \in L^{\sigma'}(0, \pi)$ with $g \geqq 0$

$$\begin{aligned}\int_0^\pi |F(x)|g(x)dx &\leqq \int_{-\pi}^\pi |K(t)|dt\int_0^\pi |f(x+t)|g(x)dx \\ &\leqq \int_{-\pi}^\pi |K(t)|\sigma(f_t)\sigma'(g)dt \leqq 2\sigma(f)\sigma'(g)\int_{-\pi}^\pi |K(t)|dt\end{aligned}$$

so that upon taking the supremum over $g \in L^{\sigma'}(0, \pi)$ with $g(x) \geqq 0$, $\sigma'(g) \leqq 1$ it follows that $\sigma(F) \leqq 2\sigma(f)\int_{-\pi}^\pi |K(t)|dt < \infty$. Thus $F \in L^\sigma(0, \pi)$ and the lemma is proved.

Since $L^\sigma(0, \pi)$ contains all the constant functions, we may assume without loss of generality that $a_0=0$ in the proofs of Theorem 1 and 2.

Proof of Theorem 1. As Hardy [5] has shown, Ta is the sequence of Fourier cosine coefficients of $g(x) = (P'f(x) + F(x))/2$, where F is given by Lemma 2. Thus, if (a) holds, Lemma 2 shows that we must have $P'f \in L^\sigma(0, \pi)$ whenever $f \in L^\sigma(0, \pi)$ and then Lemma 1 shows that $\beta > 0$ so (b) holds. Conversely, if (b) holds, Lemma 1 shows that $P'f \in L^\sigma(0, \pi)$ while Lemma 2 shows that $F \in L^\sigma(0, \pi)$ so that $g \in L^\sigma(0, \pi)$ and (a) holds. This proves the theorem.

Proof of Theorem 2. Suppose first that (a) holds and $f \in L^\sigma(0, \pi)$. Let δ be such that $\int_0^\delta f^*(x)dx = \frac{1}{2}\int_0^\pi |f(x)|dx$, $0 < \delta < \pi$, and set

$$g(x) = \begin{cases} f^*(x) & \text{if} \quad 0 < x < \delta \\ -f^*(x) & \text{if} \quad \delta < x < \pi . \end{cases}$$

Clearly $g \in L^\sigma(0, \pi)$, $g(x)$ is nonnegative, nonincreasing on $(0, \delta)$, and $\int_0^\pi g(x)dx = 0$. Let $a^\# = \{a_\nu^\#\}$ denote the Fourier cosine coefficients of $g(x)$. Since $g \in L^\sigma(0, \pi)$ and (a) holds, it follows that $(T'a^\#)_1 = \sum_{j=1}^\infty (a_j^\#)/j$ converges, and according to Loo [6, p. 273]

$$(T'a^\#)_1 = \lim_{N\to\infty} \frac{1}{\pi}\int_0^\pi (1-\cos Nx)(Pg)(x)dx .$$

But then since $(Pg)(x)$ is integrable on (δ, π) the Riemann Lebesgue lemma guarantees the existence of

$$\lim_{N\to\infty}\int_0^{\delta}(1-\cos Nx)(Pg)(x)dx\ .$$

Now, $(Pg)(x)$ nonincreasing on $(0, \delta)$ shows

$$\begin{aligned}\lim_{N\to\infty}\int_0^{\delta}(1-\cos Nx)(Pg)(x)dx &\geqq \lim_{N\to\infty}\sum_{k=1}^{\left[\frac{N\delta}{2\pi}\right]}(Pg)\left(\frac{2k\pi}{N}\right)\frac{1}{\pi}\int_{2(k-1)\pi/N}^{2k\pi/N}(1-\cos Nx)dx\\ &= \frac{1}{\pi}\lim_{N\to\infty}\frac{2\pi}{N}\sum_{k=1}^{\left[\frac{N\delta}{2\pi}\right]}(Pg)\left(\frac{2k\pi}{N}\right)\\ &= \frac{1}{\pi}\int_0^{\delta}(Pg)(x)dx\\ &= \frac{2}{\pi}\int_0^{\delta}g(t)\log\left|\frac{\sin(\delta/2)}{\sin(t/2)}\right|dt\ .\end{aligned}$$

It follows that $|g(t)|\log^+(1/t)$ is integrable on $(0, \pi)$ and hence [6, p. 273] $T'a^{\#}$ is the sequence of Fourier cosine coefficients of $H(x) = ((Pg)(x) + G(x))/2$ where G is the function associated by Lemma 2 to the sequence $a^{\#}$. Since $G \in L^{\sigma}(0, \pi)$ for any σ, and $H \in L^{\sigma}(0, \pi)$ by hypothesis, it follows that $Pg \in L^{\sigma}(0, \pi)$. Now, since $|(Pf)(x)| \leqq (P|g|)(x)$ it follows that $Pf \in L^{\sigma}(0, \pi)$ whenever $f \in L^{\sigma}(0, \pi)$ so then Lemma 1 shows $\alpha < 1$. Thus (a) implies (b).

Conversely, suppose (b) holds. There is a number $p > 1$ such that $\alpha < p^{-1}$ so $L^{\sigma}(0, \pi) \subset L^{p}(0, \pi)$ and hence if $f \in L^{\sigma}(0, \pi)$ Hölder's inequality shows $\int_0^{\pi}|f(t)|\log^+(1/t)dt < \infty$. According to Loo [6, p. 273–274] $T'a$ is then the sequence of Fourier cosine coefficients of $h(x) = (Pf(x) + F(x))/2$ where F is the function of Lemma 2. Now Lemma 1 shows that $Pf \in L^{\sigma}(0, \pi)$ and hence $h \in L^{\sigma}(0, \pi)$ so (a) holds. The theorem is proved.

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