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# A COMMUTATIVITY THEOREM FOR RINGS WITH DERIVATIONS

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**Let $R$ be a prime ring with no nonzero nil ideals and suppose that $d$ is a derivation of $R$ such that $d(x^n) = 0$, $n = n(x) \geqq 1$, for all $x \in R$. It is shown that either $d = 0$ or $R$ is an infinite commutative domain of characteristic $p \neq 0$ and $p \backslash n(x)$ if $d(x) \neq 0$.**

Let $R$ be an associative ring. Recall that an additive mapping $d$ of $R$ into itself is a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$.

In [2] it was shown that if $R$ is a prime ring and $d$ is a derivation of $R$ such that $d(x^n) = 0$ for all $x \in R$, where $n \geqq 1$ is a fixed integer, then either $d = 0$ or $R$ is an infinite commutative domain of characteristic $p \neq 0$ where $p \backslash n$. Moreover, the following question was raised:

If $R$ is a ring with no nonzero nil ideals and $d$ is a derivation of $R$ such that $d(x^n) = 0$, $n = n(x) \geqq 1$, for all $x \in R$, can we conclude that $R$ must be rather special or $d = 0$?

If $d$ is an inner derivation (i.e., if there exists an element $a \in R$ such that $d(x) = ax - xa$) Herstein's hypercenter theorem [3] asserts that under the above conditions $d$ must be zero. This is not always the case for arbitrary derivations. Take for instance a commutative domain $A$ of characteristic $p \neq 0$ and let $d$ be the usual derivation on the polynomial ring $A[X]$; then $d(f^p) = 0$ for all $f \in A[X]$, but $d \neq 0$.

We shall prove the following

THEOREM. *Let $R$ be a prime ring with no nonzero nil ideals and let $d$ be a derivation of $R$ such that*

$$d(x^n) = 0 , \qquad n = n(x) \geqq 1 , \qquad \text{for all} \quad x \in R .$$

*Then either $d = 0$ or $R$ is an infinite commutative domain of characteristic $p \neq 0$ and $p \backslash n(x)$ if $d(x) \neq 0$.*

For primitive rings the above theorem was proved in [2]; however the proof we give here is independent.

Notice that the conclusion of the theorem is false if one removes the assumption of primeness. In fact, let $R = A[X] \oplus M_2(A)$ where $A$ is a commutative domain of characteristic $p \neq 0$ and $M_2(A)$ is the ring of $2 \times 2$ matrices over $A$. Let $d$ be the derivation of $R$ defined as follows: $d$ is the usual derivation on the polynomial ring $A[X]$ and $d = 0$ on $M_2(A)$. Then $R$ has no nil ideals, $d(r^p) = 0$ for all $r \in R$, but $d \neq 0$ and $R$ is not commutative.

We begin with a slight generalization of a result of Posner [4, Lemma 3].

LEMMA. *Let $R$ be a prime ring with a derivation $d \neq 0$ and let $U$ be a nonzero ideal of $R$. If $d(u)u = ud(u)$, for all $u \in U$, then $R$ is commutative.*

*Proof.* Let $u, v \in U$; since $d(u)u = ud(u)$, $d(v)v = vd(v)$ *and* $d(u + v)(u + v) = (u + v)d(u + v)$ we get

$$d(u)v + d(v)u = ud(v) + vd(u) . \tag{1}$$

Thus, since $u$ and $uv$ lie in $U$, arguing as above we have that

$$d(u)uv + d(uv)u = ud(uv) + uvd(u) = ud(u)v + u(ud(v) + vd(u)) .$$

Hence, from (1) and the fact that $d(u)u = ud(u)$ it follows that $d(uv)u = u(d(u)v + d(v)u)$. In other words, $d(u)(vu - uv) = 0$ for all $u, v \in U$. From this we obtain

$$0 = d(u)(vxu - uvx) = d(u)v(xu - ux)$$

for all $u, v \in U$ and $x \in R$. Since $R$ is a prime ring we conclude that $U = Z(U) \cap K$ where $Z(U)$ is the center of $U$ and $K = \{u \in U / d(u) = 0\}$. If $U = K$ then the primeness of $R$ forces $d = 0$, a contradiction; hence $U = Z(U)$ is commutative and, so, $R$ is commutative.

We now prove the theorem stated above

*Proof of the Theorem.* To prove the theorem it is enough to show that if $d \neq 0$, then $R$ is commutative. In fact, if this is the case, then $nx^{n-1}d(x) = d(x^n) = 0$, $n = n(x) \geqq 1$, for all $x \in R$. Since $d \neq 0$ it follows that $R$ is of characteristic $p \neq 0$ (and $p \backslash n(x)$ if $d(x) \neq 0$); thus, $d(x^p) = px^{p-1}d(x) = 0$ for all $x \in R$. If $R$ is finite then $R$ is a field and all its elements are $p$th powers, forcing $d = 0$; hence $R$ is infinite.

We also note that given $x, y \in R$, there exists $k \geqq 1$ such that $d(x^k) = d(y^k) = 0$. In fact it is enough to consider $k = nm$ where $d(x^n) = 0$ and $d(y^m) = 0$.

*Henceforth we assume $d \neq 0$.* Our object is to show that $R$ is commutative.

Let $J$ be the Jacobson radical of $R$. Suppose first that $J \neq 0$. We shall prove that $d(x)x = xd(x)$, for all $x \in J$, by Lemma 1 the result will follow.

Let $x \in J$ and $y \in R$; let $n \geqq 1$ be such that

$$d((1+x)^{-1}y^n(1+x)) = d(y^n) = 0 .$$

Then,

$$d((1+x)(1+x)^{-1}y^n(1+x)) = d(y^n + y^n x) = y^n d(x) .$$

On the other hand,

$$\begin{aligned} &d((1+x)(1+x)^{-1}y^n(1+x)) \\ &\quad = d((1+x)^{-1}y^n(1+x) + x(1+x)^{-1}y^n(1+x)) \\ &\quad = d(x)(1+x)^{-1}y^n(1+x) . \end{aligned}$$

Therefore,

$$d(x)(1+x)^{-1}y^n(1+x) = y^n d(x) .$$

Multiplying this last equality from the right by $(1+x)^{-1}$, we get

$$d(x)(1+x)^{-1}y^n = y^n d(x)(1+x)^{-1} .$$

Thus $d(x)(1+x)^{-1}$ commutes with some power of every element in $R$ and so $d(x)(1+x)^{-1}$ is in the hypercenter of $R$. By [3], since $R$ has no nil ideals, the hypercenter of $R$ coincides with the center of $R$. Hence $d(x)(1+x)^{-1}$ is central and so, on commuting it with $x$, we obtain $d(x)x = xd(x)$. This establishes the theorem when $J \neq 0$.

*Thus we may assume, henceforth,* that $R$ is a semisimple ring.

We claim that $R$ has no zero-divisors. In fact, let $a \neq 0$ in $R$ and let $\lambda = \{y \in R / ya = 0\}$. If $y \in \lambda$ and $x \in R$, there exists $n \geqq 1$ such that

$$d((ax + axy)^n) = d((ax)^n) = 0 .$$

Since $ya = (axy)^2 = 0$ it follows that

$$(ax)^n d(y) = d((ax)^n y) = 0 .$$

This says that $d(y)$ annihilates on the right a suitable power of every element in the right ideal $aR$. By [1], since $R$ is semisimple, we have $aRd(y) = 0$. Hence, since $R$ is prime and $a \neq 0$, we conclude that $d(y) = 0$. In other words, $d$ vanishes on $\lambda$, a left ideal of $R$. By the primeness of $R$, it is easy to check that this forces $d = 0$, unless $\lambda = 0$. Thus, $R$ has no zero-divisors.

We go on with the final steps of the proof by showing that if $R$ is a domain then $R$ is commutative. As before it is enough to show that $d(x)x = xd(x)$ for all $x \in R$.

Let $x \neq 0$ in $R$ and let $A = C_R(x^n)$ be the centralizer of $x^n$ in $R$, where $n \geqq 1$ is such that $d(x^n) = 0$. If $a \in A$, then

$$0 = d(ax^n - x^n a) = d(a)x^n - x^n d(a) ;$$

that is, $A$ is invariant under $d$ and we may consider $d$ as a derivation on $A$.

Now, $A$ is a domain whose center, $Z(A)$, is nonzero for $0 \neq x^n \in Z(A)$. By localizing $A$ at $Z(A)\backslash\{0\}$ we obtain a domain $Q \supset A$ whose center is a field containing $x^n$; in particular, $x$ is invertible in $Q$. As it is well known, $d$ extends uniquely to a derivation on $Q$ (which we shall also denote by $d$) as follows:

$$d(az^{-1}) = d(a)z^{-1} + ad(z)z^{-2} , \qquad a \in A , \qquad z \in Z(A)\backslash\{0\} .$$

Moreover, by our basic hypothesis on $d$, we have that $d(q^m) = 0$, $m = m(q) \geqq 1$, for all $q \in Q$.

Let $q \in Q$ and let $m \geqq 1$ be such that

$$d(q^m) = d(x^{-1}q^m x) = 0 .$$

Then,

$$d(x)x^{-1}q^m x = d(x(x^{-1}q^m x)) = d(q^m x) = q^m d(x) .$$

Multiplying this equality from the right by $x^{-1}$, we obtain

$$d(x)x^{-1}q^m = q^m d(x)x^{-1} .$$

In other words, $d(x)x^{-1}$ lies in the hypercenter of $Q$. As before, by [3], it follows that $d(x)x^{-1}$ lies in the center of $Q$ and so, we conclude that $d(x)x = xd(x)$. This completes the proof of the theorem.

We finish with the following

COROLLARY. *Let $R$ be a prime ring with no nonzero nil ideals. If $d$ is a derivation of $R$ such that $d(u^n) = 0$, $n = n(u) \geqq 1$, for all $u \in U$, where $U$ is a nonzero ideal of $R$, then either $d = 0$ or $R$ is an infinite commutative domain of characteristic $p \neq 0$ and $p \backslash n(u)$ if $d(u) \neq 0$.*

*Proof.* Suppose $d \neq 0$. Let

$$\delta(U) = \{u \in U / d^i(u) \in U, \text{ for all } i \geqq 1\} .$$

Then, $\delta(U)$ is an ideal of $R$ invariant under $d$. Moreover, by hypothesis, some power of every element in $U$ lies in $\delta(U)$. Since $R$ has no nonzero nil ideals, we must have $\delta(U) \neq 0$.

Now, as an ideal of $R$, $\delta(U)$ is also a prime ring with no nonzero nil ideals. By the above theorem, the conclusion holds in $\delta(U)$. Since $R$ is prime, the result follows.

## REFERENCES

1. B. Felzenszwalb, *On a result of Levitzki,* Canad. Math. Bull., **21** (2) (1978), 241–242.
2. ———, *Derivations in prime rings,* Proc. Amer. Math. Soc., **84** (1982), 16–20.
3. I. N. Herstein, *On the hypercenter of a ring,* J. Algebra, **36** (1975), 151–157.
4. E. C. Posner, *Derivations in prime rings,* Proc. Amer. Math. Soc., **8** (1957), 1093–1100.

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