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SPACES OF WEAKLY CONTINUOUS FUNCTIONS

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In this paper we study some properties about the space of weakly continuous functions on bounded sets of a Banach space E: $C_{wb}(E)$. We study the relation between $C_{wb}(E)$ and $C_{wbu}(E)$ (weakly uniformly continuous functions on bounded sets). And we give the following characterization: $C_{wbu}(E)$ is a barreled space if and only if E is reflexive.

0. **Notation and preliminaries.** Throughout this paper E will represent a real Banach space and B_n the closed ball of radius n. The basic definitions of locally convex spaces and their properties are explained in [6]. We will say that a Banach space is weakly compactly generated (WCG), when it has a weakly compact total subset. Both separable and reflexive spaces are particular cases of WCG spaces. For further information, see [2].

For the topological concepts that are used, we will follow [5]. We will say that a completely regular topological space is realcompact when each z-ultrafilter with the countable intersection property has an nonempty intersection. A subset of a topological space will be relatively pseudocompact when every real-valued continuous function defined on the space is bounded on the subset.

We will define the bw-topology on E as the finest which agrees with the weak topology on bounded subsets of E. A subset will be bw-closed (respectively bw-open) if and only if it is weakly closed (respectively relatively open) when it is restricted to each B_n.

If X is a topological space, $C(X)$ will represent the space of real-valued continuous functions on X. Except for when indicating the opposite, we will give $C(X)$ the compact-open topology, defined by the family of semi-norms

$$P_K(f) = \operatorname*{Sup}_{x \in K} |f(x)|$$

when K ranges over the compact subsets of X. $C_{wb}(E)$ will represent the space of real functions which are weakly continuous when restricted to the bounded subsets of E. $C_{wbu}(E)$ will be the space of real functions which are weakly uniformly continuous when restricted to the bounded subsets of E.

$C_{wbu}(E) \subset C_{wb}(E)$, if we give the topology of the uniform convergence on weakly compact subsets to both, we will have that $C_{wbu}(E)$ is a subspace of the locally convex space $C_{wb}(E)$.

1. **The space $C_{wb}(E)$.** The space E endowed with the bw-topo-

logy will be represent by X. It is evident that $C_{wb}(E)$ coincides with $C(X)$ as sets. On the other hand, the weak compacts of E and the compacts of X are the same (by being bounded). Therefore, both spaces are topologically isomorphic.

We are concerned with studying the properties of $C_{wb}(E)$, for which we need the following lemma:

LEMMA 1.1. *If E is a weakly normal space, then X is normal and hence completely regular.*

Proof. If E is a weakly normal space, then for every n, B_n endowed with the weak restricted topology is normal.

Let C and F be closed subsets of X, $C \cap F = \varnothing$.

$C_n = C \cap B_n$, $F_n = F \cap B_n$. C_1 and F_1 are weakly closed. By Urysohn's lemma, we have $f_1 \colon B_1 \to [0, 1]$ weakly continuous function such that:

$$f_1(C_1) = \{0\} \quad \text{and} \quad f_1(F_1) = \{1\}$$

will be $f_2^* \colon B_1 \cup C_2 \cup F_2 \to [0, 1]$ defined by

$$f_2^*|_{B_1} = f_1, \ f_2^*(C_2) = \{0\}, \ f_2^*(F_2) = \{1\} .$$

This function is weakly continuous and it is defined on a weakly closed subset of B_2, therefore, by Tietze's theorem, it can be extended to another function $f_2 \colon B_2 \to [0, 1]$ weakly continuous and such that

$$f_2(C_2) = \{0\} \quad \text{and} \quad f_2(F_2) = \{1\} .$$

We define by induction $f_n \colon B_n \to [0, 1]$ weakly continuous such that:

$$f_n(C_n) = \{0\}, \ f_n(F_n) = \{1\}, \text{ and } f_n|_{B_{n-1}} = f_{n-1} .$$

We define $f(x) = f_n(x)$ if $x \in B_n$.

We have that f is continuous on X, $f(C) = \{0\}$ and $f(F) = \{1\}$. Hence X is normal.

Unfortunately we have not a general result, eliminating the hypotheses of weak normality, which affirms that X is always a completely regular; which is necessary for the study of $C(X)$. Nevertheless, if E endowed with the coarser topology that makes the functions of $C_{wb}(E)$ continuous, is represented by $\tilde{X}$, we achieve that the above space is completely regular, and we have that the following inclusions are continuous:

$$X \longrightarrow \tilde{X} \longrightarrow (E, \sigma(E, E'))$$

giving equality to the first inclusion if and only if X is completely regular.

We proceed to study the properties of $C_{wb}(E)$. In the first place we will see when it is a bornological space. According to Nachbin-Shirota's theorem [7.8], this would be equivalent to X being realcompact. We have the following statement:

THEOREM 1.2. *If E is weakly normal, then $C_{wb}(E)$ is bornological if and only if E is weakly realcompact.*

Proof. Simply by noting the fact that the B_n balls with weak restricted topology are realcompacts, it follows that X (respectively E) is realcompact (respectively weakly realcompact), we will do the proof for E, but with light modifications serving for X.

Let $\{U_\alpha\}_{\alpha\in A}$ be a z-ultrafilter. Each $U_\alpha = f_\alpha^{-1}(0)$ with $f_\alpha: E \to R$ weakly continuous.

We have that for every index sequence $(\alpha_n) \in A$, $\bigcap_{n=1}^{\infty} U_{\alpha_n} \neq \emptyset$.

(1) There is n_0 such that $U_\alpha \cap B_{n_0} \neq \emptyset$ for every $\alpha \in A$.

If it were not like this, for each $n \in N$ we would have $\alpha_n \in A$ such that $U_{\alpha_n} \cap B_n = \emptyset$. With which we would have that $\bigcap_{n=1}^{\infty} U_{\alpha_n} = \emptyset$, failing the countable intersection property.

Therefore, for every $n \geqq n_0$ $\{U_\alpha \cap B_n\}_{\alpha\in A}$ is a filter basis in B_n.

(2) There exists $n_1 \geqq n_0$ in such a way that the filter basis $\{U_\alpha \cap B_{n_1}\}_{\alpha\in A}$ has the countable intersection property.

Supposing the above fails: for every $n \geqq n_0$ there would be $\{\alpha_{n,m}\}_{m\in N}$ index sequence in such a way that $\bigcap_{m=1}^{\infty} (U_{\alpha_{n,m}} \cap B_n) = \emptyset$. The countable family $\{U_{\alpha_{n,m}}\}_{n,m\in N, n\geqq n_0}$ has an empty intersection contrary to the countable intersection property.

(3) $\{U_\alpha \cap B_{n_1}\}_{\alpha\in A}$ is the basis of a z-filter with the countable intersection property in B_{n_1}, because $f_\alpha|_{B_{n_1}}$ is a continuous function on B_{n_1} endowed with the weak topology restricted, for every $\alpha \in A$.

(4) $\{U_\alpha \cap B_{n_1}\}_{\alpha\in A}$ is a basis for a z-ultrafilter.

If not, it would be $Z \in B_{n_1}$ zero in B_{n_1}, that is, $Z = f^{-1}(0)$ with f weakly continuous on B_{n_1}, in such a way that $Z \cap U_\alpha \cap B_{n_1} \neq \emptyset$ for every $\alpha \in A$, but Z not containing any $U_\alpha \cap B_{n_1}$. But since B_{n_1} is a weakly closed subset of E, by normality there exists $\tilde{f}: E \to R$ weakly continuous, such that $\tilde{f}|_{B_{n_1}} = f$. Furthermore, $\tilde{f}^{-1}(0) = \tilde{Z}$ will be a zero of E (with the weak topology), and $Z = \tilde{Z} \cap B_{n_1}$. But $\tilde{Z} \cap U_\alpha \neq \emptyset$ for every $\alpha \in A$, hence $\tilde{Z} = U_{\alpha_0}$ for some α_0 by being z-ultrafilter, then $Z = U_{\alpha_0} \cap B_{n_1}$.

Just as by hypothesis B_{n_1} is realcompact, it follows that $\bigcap_{\alpha\in A} U_\alpha \cap B_{n_1} \neq \emptyset$ and thus $\bigcap_{\alpha\in A} U_\alpha \neq \emptyset$. Hence E is weakly realcompact.

Since $\tilde{f}$ is also continuous on X, it can also be inferred that X

is realcompact.

We have then that $\tilde{X} = X$ is realcompact if and only if E is weakly realcompact, and that $C_{wb}(E)$ is bornological if and only if E is weakly realcompact.

Weakly realcompact Banach spaces are described in [1]. Nevertheless, there exists weakly realcompact spaces which are not weakly normal, as in the case of l^∞ [1, p. 12]. In any case, the class of weakly normal and weakly realcompact spaces is wide; particularly every *WCG* space is weakly Lindelöf [9] and thus is in our hypotheses.

The following statement gives a partial answer to the problem for the case of not necessarily normal spaces.

THEOREM 1.3. *If E is the dual of a separable space, then $C_{wb}(E)$ is bornological; in particular $C_{wb}(l^\infty)$ is bornological.*

Proof. Let $E = F'$ be. $\{x_n\}_{n\in N}$ a dense subset of F. We define:

$$\begin{aligned} f\colon \tilde{X} &\longrightarrow R^N \\ x' &\longrightarrow ((x_n, x'))_n\,. \end{aligned}$$

This map is one to one because $\{x_n\}_{n\in N}$ separates points of E by being dense in F. Furthermore, f is continuous by being $f = \tilde{f}\circ i$, being $i\colon \tilde{X}\to E$ continuous and $\tilde{f}\colon E\to R^N$ continuous, given that $\tilde{f}$ composed with $p_n\colon R^N\to R$ is x_n. Since R^N is realcompact and all its subsets are as well (by the points being G_δ-sets), we have, because of [4], that $\tilde{X}$ is realcompact and $C_{wb}(E)$ is bornological.

Finally we are going to see that $\tilde{X}$ is a *NS*-space. That is, every relatively pseudocompact and closed subset is compact. Through [7] we achieve that $C_{wb}(E)$ is always barreled.

PROPOSITION 1.4. *$C_{wb}(E)$ is barreled.*

Proof. Let $K\subset\tilde{X}$ be a relatively pseudocompact and closed subset. For all $x'\in E'$, $x'(K)$ is bounded, hence K is weakly bounded and thus bounded. Furthermore, $K\subset B_n$ for some n, from where it follows that K is weakly closed. As on the other hand each weakly continuous function on E is continuous on $\tilde{X}$, we have that K is weakly relatively pseudocompact. Also through [10] we achieve that K is weakly compact and thus compact in $\tilde{X}$.

2. The space $C_{wbu}(E)$. First of all, we will study the relationship between $C_{wbu}(E)$ and $C_{wb}(E)$.

PROPOSITION 2.1. *$C_{wbu}(E)$ is a dense subspace of $C_{wb}(E)$.*

Proof. Let f be a function of $C_{wb}(E)$. For every weakly compact subset K of E, $f|_K$ will be weakly continuous. Let $\tilde{f}_K$ be an extension to the Stone-Cech compactification of E endowed with the weak topology, $\beta(E)$, which will be uniformly continuous. Then $f_K = \tilde{f}_K|_E$ is weakly uniformly continuous on E. Then $f_K \in C_{wbu}(E)$ for every weakly compact subset K.

Obviously, $\{f_K\}_K$ is a net that converges uniformly on weakly compact subsets of E, to f.

PROPOSITION 2.2. *$C_{wbu}(E) = C_{wb}(E)$ if and only if E is reflexive.*

Proof. If E is reflexive, the equality holds because the balls are weakly compacts.

Conversely given f weakly continuous on E, we do have that $f \in C_{wb}(E) = C_{wbu}(E)$. Then f is weakly uniformly continuous on B_1 and consequently it is bounded on B_1, because it is totally bounded. Since f is weakly uniformly continuous on B_1, it follows that there exists a weak neighborhood of zero, V; such that $|f(x) - f(y)| < 1$ provided that $x - y \in V$ and $x, y \in B_1$. Since B_1 is weakly totally bounded, we infer that there exists $x_1, \cdots, x_{n_0} \in B_1$ such that

$$B_1 \subset \bigcup_{i=1}^{n_0} \{x_i + V\} .$$

Thus for every $x \in B_1$

$$|f(x)| \leqq \operatorname*{Max}_{i=1,\cdots,n_0} \{|f(x_i)| + 1\} .$$

This means that every weakly continuous function over E is bounded on B_1, hence B_1 is weakly relatively pseudocompact, and weakly closed. By [10] it follows that B_1 is weakly compact and thus E reflexive.

The proof of this proposition suggest that, if E is not reflexive, one method to find a function which belongs to $C_{wb}(E)$ and does not belong to $C_{wbu}(E)$, it would be to find a weakly continuous fuuction over E which is not bounded on B_1.

EXAMPLE 2.3. If E is a nonreflexive separable space, the James-Klee theorem [2, p. 7] states there exists $\phi \in E'$ which does not attain its norm.

We define the function

$$f_0 \colon B_1 \longrightarrow R \text{ by } f_0(x) = \frac{1}{\|\phi\| - \phi(x)} .$$

This function is weakly continuous on B_1, and is not bounded. Since E is a separable space, then it is *WCG* and therefore weakly normal.

Thus by Tietze's theorem there exists f weakly continuous on E, which extends f_0, and which is not bounded on the unit ball; therefore it can not belong to $C_{wbu}(E)$.

COROLLARY 2.4. *$C_{wbu}(E)$ is complete if and only if E is reflexive.*

Proof. If E is reflexive, $C_{wb}(E) = C_{wbu}(E)$ and also $C_{wb}(E)$ is complete by [3]. Thus $C_{wbu}(E)$ is complete.

Conversely, since $C_{wbu}(E)$ is dense in $C_{wb}(E)$, if it is complete, both spaces have to be the same and because of that it is reflexive.

THEOREM 2.5. *$C_{wbu}(E)$ is barrelled if and only if E is reflexive.*

Proof. If E is reflexive $C_{wbu}(E) = C_{wb}(E)$ and consequently barrelled. Conversely we consider the following diagram:

$$\begin{array}{ccccccc} B_1'' & \longrightarrow & B_2'' & \longrightarrow & \cdots & \longrightarrow & X'' \\ \uparrow{\scriptstyle i_1} & & \uparrow{\scriptstyle i_2} & & & & \uparrow{\scriptstyle i} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & \cdots & \longrightarrow & X\,. \end{array}$$

B_n'' are endowed with the weak star topology restricted. X'' will be the inductive limit of the spaces B_n''.

i is continuous because when composed with the inclusions $j_n: B_n \to X$ it follows that $i \circ j_n = j_n^* \circ i_n$, been $j_n^*: B_n'' \to X''$ the canonical inclusion in the inductive limit; obviously $j_n^* \circ i_n$ is continuous because i_n is also continuous. i is one to one and $i(X)$ is dense in X''.

Let us consider the map restriction $\phi: C(X'') \to C(X)$ $f \to f \circ i$.

(1) We have that $\phi(C(X'')) = C_{wbu}(E)$. Let us see it.

If $f \in C_{wbu}(E)$, it follows that $f_n = f|_{B_n}: B_n \to R$ is weakly uniformly continuous. Then, by density, it can be extended to $\tilde{f}_n$: $B_n'' \to R$ uniformly continuous, on the other hand,

$$\tilde{f}_n|_{B_{n-1}''} = \tilde{f}_{n-1} \text{ because } (\tilde{f}_n|_{B_{n-1}''})|_{B_{n-1}} = (\tilde{f}_n|_{B_n})|_{B_{n-1}} = f_n|_{B_{n-1}} = f_{n-1}$$

and

$$\tilde{f}_{n-1}|_{B_{n-1}} = f_{n-1}\,.$$

Then both functions are exactly the same over a dense part of B_{n-1}'' thus they are the same.

It can be defined $\tilde{f}: X'' \to R$ continuous by $\tilde{f}(x) = f_n(x)$ if $x \in B_n''$. Obviously $\tilde{f}|_X = f$, thus $f \in \phi(X''))$.

Conversely if $f \in \phi(C(X''))$ it follows that there exists $\tilde{f} \in C(X'')$ such that $f = \tilde{f}|_X$; but $\tilde{f}_n = \tilde{f}|_{B_n'}$ is continuous on B_n'' which is compact. Then $\tilde{f}_n$ is uniformly continuous. Therefore $f_n = f|_{B_n}$ is equal to $\tilde{f}_n|_{B_n}$ and because of that, weakly uniformly continuous.

(2) ϕ is linear.

(3) ϕ is one to one because $i(X)$ is dense in X''.

(4) ϕ is continuous because the continuity of $i: X \to X''$ implies that the compacts of X are compacts of X''.

On the other hand, the space X'' is a countable union of compacts and therefore the topology of $C(X'')$ is given by a countable family of semi-norms. Thus $C(X'')$ is metrizable. Because of the definition, X'' is a k-space and therefore $C(X'')$ is complete [11]; then $C(X'')$ is a Frechet space.

Since $C_{wbu}(E)$ is barreled it follows that f is a topological isomorphism, applying the Open Mapping Theorem. Then it can be inferred that $C_{wbu}(E)$ is complete and therefore E is reflexive.

COROLLARY 2.6. *If E is a Banach space, the following are equivalent:*

(i) *E is reflexive*

(ii) *$C_{wbu}(E)$ is a Frechet space*

(iii) *$C_{wbu}(E)$ is a Ptak space*

(iv) *$C_{wbu}(E)$ is complete*

(v) *$C_{wbu}(E)$ is barreled*

(vi) *$C_{wbu}(E) = C_{wb}(E)$.*

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