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SUP-CHARACTERIZATION OF STRATIFIABLE SPACES

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We prove that a T_1-space (X, τ) is stratifiable if and only if, for each $U \in \tau$, one can find a continuous function f_U: $X \to I$ such that $f_U^{-1}(0) = X - U$ and, for each $\mathfrak{U} \subset \tau$, $\sup_{U \in \mathfrak{U}} f_U$ is continuous. This result is closely related to characterizations of metrizable and paracompact spaces, by J. Nagata, and J. Guthrie and M. Henry.

1. Introduction. J. Nagata [**4**, Theorem 5] and J. Guthrie and M. Henry [**2**, Theorem 2] have characterized metrizable spaces in terms of collections of real-valued functions with continuous "sups" and "infs". Nagata's theorem can be reformulated as follows:

THEOREM (Nagata). *A T_1-space X is metrizable if and only if there is a family $\mathfrak{F}$ of functions from X into* $[0, 1]$ *such that*
(a) *for each $\mathfrak{F}' \subset \mathfrak{F}$*, $\sup \mathfrak{F}'$ *and* $\inf \mathfrak{F}'$ *are continuous*;
(b) $\{f^{-1}([\varepsilon, 1])$: $\varepsilon > 0, f \in \mathfrak{F}\}$ *is a base for X.*

In the paper of Guthrie and Henry, it is shown that the Sorgenfrey line admits a collection $\mathfrak{F}$ of functions satisfying (b) such that "infs" from $\mathfrak{F}$ are continuous. One might, therefore, expect that nothing interesting happens if just sups are required to be continuous. In this paper we show instead that a characterization of stratifiable spaces is obtained. Our main result is the following:

THEOREM 1. *The following are equivalent for a T_1-space (X, τ):*
(a) *X is stratifiable*;
(b) *There exists a family $\mathfrak{F} = \{f_U$: $U \in \tau\}$ of functions from X into* $[0, 1]$ *such that*
(i) *for each $\mathfrak{U} \subset \tau$*, $\sup_{U \in \mathfrak{U}} f_U$ *is continuous*;
(ii) *for each open set $U, f_U^{-1}(0) = X - U$.*
(c) *There exists a collection $\mathfrak{F} = \bigcup_{n \in \omega} \mathfrak{F}_n$ such that*
(i) *for each $n \in \omega$ and $\mathfrak{F}' \subset \mathfrak{F}_n$*, $\sup \mathfrak{F}'$ *is continuous*;
(ii) $f^{-1}\{([\varepsilon, 1])$: $\varepsilon > 0, f \in \mathfrak{F}\}$ *is a base for X.*

Observe that the property that one obtains by just requiring sups to be continuous in Nagata's theorem is formally weaker than (b) and stronger than (c). However, by this theorem all these properties are equivalent.

2. Preliminaries. It turns out not to be difficult to prove that a space satisfying (c) is stratifiable. Proving (a) $\Rightarrow$ (b) is the hard part. It involves building a very strong type of "stratification", which is done in Lemma 2.3, the proof of which is the purpose of this section.

If (X, τ) is a stratifiable space, then for each $U \in \tau$ and $x \in U$, one can assign an open neighborhood U_x of x satisfying

$$U_x \cap V_y \neq \varnothing \Rightarrow x \in V \quad \text{or} \quad y \in U.$$

(cf. Lemma 4.2 of [**1**]). Let $U_{x^1} = U_x$ and $U_{x^n} = (U_{x^{n-1}})_x$, for $n = 2, 3, \ldots$.

A *neighbornet* R of a space X is a binary relation on X such that $R[x] = \{y\colon x\ R\ y\}$ is a neighborhood of x for each $x \in X$. If $\mathcal{Q}$ is a collection of subsets of X, let $\mathcal{Q}_x = \{A \in \mathcal{Q} \mid x \in A\}$. If $\mathcal{V}$ is a cover of X, let $\mathcal{V}(x) = \cap \mathcal{V}_x$. For any point finite open cover $\mathcal{V}$ of X and $k = 2, 3, \ldots$, let $N^k(\mathcal{V})$ be the neighbornet defined by $N^k(\mathcal{V})[x] = (\mathcal{V}(x))_{x^k}$. Let $N(\mathcal{V})$ be the neighbornet defined by $N(\mathcal{V})[x] = \mathcal{V}(x)$. By Corollary 4.7 of [**3**], there exists a point-finite open cover $\mathcal{V}'$ of X such that $N(\mathcal{V}') \subset (N^3(\mathcal{V}))^3$ (recall that, for relations $R \subset X \times X$, $R^n = R^{n-1} \circ R$).

LEMMA 1.1. *$(N^3(\mathcal{V}))^3$ and $N(\mathcal{V}')$ satisfy the following*:

(a) $y \in N(\mathcal{V}')[x] \Rightarrow N(\mathcal{V}')[y] \subset N(\mathcal{V}')[x]$,

(b) *each* $N(\mathcal{V}')[x] \subset \mathcal{V}(x)$,

(c) *for each* $0 \in \tau$ *and* $y \in 0$, $O_{y^3} \cap N(\mathcal{V}')[x] \neq \varnothing \Rightarrow x \in 0$ *or* $y \in \mathcal{V}(x)$.

Proof. Part (a). $y \in N(\mathcal{V}')[x] = \mathcal{V}'(x)$ implies that $N(\mathcal{V}')[y] = \mathcal{V}'(y) \subset \mathcal{V}'(x) = N(\mathcal{V}')[x]$.

Part (b). Note that each $N(\mathcal{V}')[x] \subset (N^3(\mathcal{V}))^3[x]$. So it suffices to show that $(N^3(\mathcal{V}))^3[x] \subset \mathcal{V}(x)$. Clearly $N^3(\mathcal{V})[x] = (\mathcal{V}(x))_{x^3} \subset \mathcal{V}(x)$. Therefore $y \in N^3(\mathcal{V})[x] \Rightarrow y \in \mathcal{V}(x) \Rightarrow \mathcal{V}(y) \subset \mathcal{V}(x) \Rightarrow N^3(\mathcal{V})[y] \subset \mathcal{V}(y) \subset \mathcal{V}(x)$. Consequently, $(N^3(\mathcal{V}))^2[x] \subset \mathcal{V}(x)$. Similarly, $z \in (N^3(\mathcal{V}))^2 \Rightarrow \mathcal{V}(z) \subset \mathcal{V}(x) \Rightarrow (N^3(\mathcal{V}))[z] \subset \mathcal{V}(z) \subset \mathcal{V}(x)$. Consequently, $(N^3(\mathcal{V}))^3[x] \subset \mathcal{V}(x)$, as desired.

Part (c). $O_{y^3} \cap N(\mathcal{V}')[x] \neq \varnothing \Rightarrow O_{y^3} \cap (N^3(\mathcal{V}))^3[x] \neq \varnothing$. Thus there exists p, $w \in X$ such that $O_{y^3} \cap \mathcal{V}(p)_{p^3} \neq \varnothing$, with $p \in (\mathcal{V}(w))_{w^3}$, $w \in (\mathcal{V}(x))_{x^3}$; hence, $y \in (\mathcal{V}(p))_{p^2}$ or $p \in O_{y^2}$. If $y \in (\mathcal{V}(p))_{p^2}$ then $y \in \mathcal{V}(p) \subset \mathcal{V}(w) \subset \mathcal{V}(x)$. If $p \in O_{y^2}$ then $O_{y^2} \cap (\mathcal{V}(w))_{w^3} \neq \varnothing$; hence, $y \in \mathcal{V}(w)_{w^2} \subset \mathcal{V}(w) \subset \mathcal{V}(x)$ or $w \in O_y$. But $w \in O_y$ implies that $O_y \cap (\mathcal{V}(x))_{x^3} \neq \varnothing$ which implies that $y \in (\mathcal{V}(x))_{x^2} \subset \mathcal{V}(x)$ or $x \in 0$. This completes the proof.

LEMMA 1.2. *To each $x \in (X, \tau)$ one can assign a sequence $\{h_n(x)\}$ of open neighborhoods of x such that*

(i) $h_0(x) \supset h_1(x) \supset \ldots,$

(ii) $y \in h_n(x) \Rightarrow h_n(y) \subset h_n(x),$

(iii) $y \in U \in \tau \Rightarrow$ *there exists n such that* $y \notin (\cup\{h_n(x) \mid x \notin U\})^-$,

(iv) *For* $n > 0$ *and* $y \in U \in \tau$, $U_{y^3} \cap h_n(x) \neq \varnothing \Rightarrow x \in U$ *or* $y \in h_{n-1}(x)$.

Proof. From Lemma 4.2 and Theorem 4.17 of [3], we can find a sequence $\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots$ of point finite open covers of X which satisfy the following condition:

($*$) For each $y \in U \in \tau$, there exists n such that

$$y \notin (\cup \{\mathcal{V}_n(x) \mid x \notin U\})^- .$$

Let $\mathcal{W}_0 = \mathcal{V}_0$ and $\mathcal{W}_n = (\mathcal{W}_{n-1} \cup \mathcal{V}_n)'$ for $n = 1, 2, \ldots$. (Recall that $N(\mathcal{W}_n) \subset (N^3(\mathcal{W}_{n-1} \cup \mathcal{V}_n))^3$.) For each $x \in X$ and $n = 0, 1, 2, \ldots,$ let $h_n(x) = N(\mathcal{W}_n)[x] = \mathcal{W}_n(x)$. Let us check that the $h_n(x)$ satisfy conditions (i)–(iv) above.

(i) $h_n(x) = N(\mathcal{W}_n)[x] \subset (\mathcal{W}_{n-1} \cup \mathcal{V}_n)(x) \subset \mathcal{W}_{n-1}(x) = h_{n-1}(x)$, where the first containment follows from Lemma 1.1(b).

(ii) $y \in h_n(x) \Rightarrow h_n(y) = \mathcal{W}_n(y) \subset \mathcal{W}_n(x) = h_n(x)$.

(iii) From (i) we get that $h_n(x) \subset (\mathcal{W}_{n-1} \cup \mathcal{V}_n)(x) \subset \mathcal{V}_n(x)$. Since the $\mathcal{V}_n(x)$ satisfy (*) then so do the $h_n(x)$.

(iv) $U_{y^3} \cap h_n(x) \neq \varnothing \Leftrightarrow U_{y^3} \cap N((\mathcal{W}_{n-1} \cup \mathcal{V}_n)')[x] \neq \varnothing \Rightarrow x \in U$ or $y \in (\mathcal{W}_{n-1} \cup \mathcal{V}_n)(x) \subset \mathcal{W}_{n-1}(x) = h_{n-1}(x)$. This completes the proof.

Let Q_0 denote the set of rational numbers in $]0, 1]$.

LEMMA 1.3. *To each $U \in \tau$ and $r \in Q_0$, one can assign a closed $U_r \subset X$ such that*

(1) $s < r \Rightarrow U_r \subset U_s^0$,

(2) $U = \cup\{U_r \mid r \in Q_0\}$,

(3) *for each* $r \in Q_0$, $\{U_r \mid U \in \tau\}$ *is closure-preserving.*

Proof. Let $\{0 = r_0, r_1, \ldots\}$ be an enumeration of the rationals in $[0, 1]$. Let $U_{r_0} = U$. Suppose U_{r_k} has been defined for $k < n$. Define U_{r_n} as follows: Choose $k(n) < n$ such that $r_{k(n)} < r_n$ and $r_{k(n)} = \max\{r_j \mid j < n \text{ and } r_j < r_n\}$. Let $U_{r_n} = X - \cup\{h_n(x) \mid x \notin U^0_{r_{k(n)}}\}$ and let us verify that the U_r satisfy all requirements.

(3) From Lemma 2.2(ii) we get that, for each n, $\{h_n(x) \mid x \in X\}$ is an interior-preserving open cover of X. Therefore $\{\cup_{x \in A} h_n(x) \mid A \subset X\}$ is

also interior-preserving or, equivalently, $\{X - \cup_{x \in A} h_n(x) \mid A \subset X\}$ is closure-preserving. This shows that, for each r, $\{U_r \mid U \in \tau\}$ is closure-preserving.

(2) Let $y \in U \in \tau$. From Lemma 1.2(iii), there exists n such that $y \notin (\cup\{h_n(x) \mid x \notin U\})^-$. From Lemma 1.2(i),

$$y \nabla \left(\cup \{h_j(x) \mid x \notin U\}\right)^-,$$

for $j \geq n$. Find r_m with $m \geq n$ such that $r_m < r_k$ for each $0 < k < m$ (if no such r_m exists, then $r_m \geq \min\{r_1, \ldots, r_n\} \neq 0$, for $m \geq n$, a contradiction). Then $y \in U_{r_m} = X - \cup\{h_m(x) \mid x \notin U\}$.

(1) Suppose $r_m < r_n$ and let us show that $U_{r_n} \subset U^0_{r_m}$. We consider two cases.

Case 1. $m < n$. Then

$$U_{r_n} = X - \bigcup \left\{h_n(x) \mid x \notin U^0_{r_{k(n)}}\right\} \subset X - \bigcup \left\{h_n(x) \mid x \notin U^0_{r_m}\right\},$$

because $r_m \leq r_{k(n)}$. So $y \notin U^0_{r_m}$ implies that $h_n(y) \cap U_{r_n} = \varnothing$ which implies that $y \notin U_{r_n}$. Hence $U_{r_n} \subset U^0_{r_m}$.

Case 2. $m > n$. By induction, let us assume that $U_{r_j} \subset U^0_{r_k}$ for $r_k < r_j$ and $k + j < m + n$. Let $r_t = \min\{r_j \mid r_j > r_m \text{ and } j < m\}$. Then $r_m < r_t \leq r_n$ and it suffices to show that $U_{r_t} \subset U^0_{r_m}$: Suppose not. Then there exists $y \in U_{r_t} - U^0_{r_m}$. Let $k^0(m) = m$ and $k^j(m) = k(k^{j-1}(m))$ for $j = 1, 2, \ldots$. We will prove that $y \in U_{r_t} - U^0_{r_m}$ implies the following:

(**) For each $j \geq 0$, $k^j(m) > t$ and $y \in h_{k^j(m)}(x_j)$ for some $x_j \in U^0_{r_{k^{j+1}(m)}} - U^0_{r_{k^j(m)}}$.

(Proof by induction.) Since $y \in U_{r_t}$ and $r_{k(m)} < r_t$, we get that $y \in U^0_{r_{k(m)}}$ (note that $k(m) + t < m + n$; so $U_{r_t} \subset U^0_{r_{k(m)}}$). Letting $x_0 = y$ we get that (**) is valid for $j = 0$.

Suppose (**) is valid for $j \leq i$ and let us show its validity for $i + 1$. Since

$$x_i \in U^0_{r_{k^{i+1}(m)}} - U^0_{r_{k^i(m)}} \quad \text{and}$$

$$U^0_{r_{k^i(m)}} = X - \left(\bigcup \left\{h_{k^i(m)}(x) \mid x \notin U^0_{r_{k^{i+1}(m)}}\right\}\right)^-,$$

then $(U^0_{r_{k^{i+1}(m)}})_{x_i^3} \cap h_{k^i(m)}(x_{i+1}) \neq \varnothing$, for some $x_{i+1} \notin U^0_{r_{k^{i+1}(m)}}$. Therefore, by Lemma 1.2(iv), $x_i \in h_{k^i(m)-1}(x_{i+1}) \subset h_{k^{i+1}(m)}(x_{i+1})$ which implies that $y \in h_{k^i(m)}(x_i) \subset h_{k^{i+1}(m)}(x_i) \subset h_{k^{i+1}(m)}(x_{i+1})$, because of Lemma 1.2(ii).

Also $x_{i+1} \in U^0_{r_{k^{i+2}(m)}}$: Suppose not. Since

$$U_{r_{k^{i+1}(m)}} = X - \bigcup \left\{h_{k^{i+1}(m)}(x) \mid x \notin U^0_{r_{k^{i+2}(m)}}\right\},$$

we get that $h_{k^{i+1}(m)}(x_{i+1}) \cap U_{r_{k^{i+1}(m)}} = \varnothing$. So $y \notin U_{r_{k^{i+1}(m)}}$ (because $y \in h_{k_{i+1}(m)}(x_{i+1})$) which contradicts $y \in U_{r_t} \subset U_{r_{k^{i+1}(m)}}$ (note that $k^{i+1}(m) + t < m + n$).

Finally $k^{i+1}(m) > t$: Suppose $k^{i+1}(m) \leq t$. Then $r_{k^{i+1}(m)} \leq r_{k(t)}$. Since $x_{i+1} \notin U^0_{r_{k^{i+1}(m)}}$ then $x_{i+1} \notin U^0_{r_{k(t)}}$ (again $k^{i+1}(m) + k(t) < m + n$). Since $x_i \in h_{k^i(m)-1}(x_{i+1})$ and $y \in h_{k^i(m)}(x_i)$, by Lemma 1.2 (ii; i), we get that $y \in h_{k^i(m)}(x_i) \subset h_{k^i(m)}(x_{i+1}) \subset h_{k^i(m)-1}(x_{i+1}) \subset h_t(x_{i+1})$ (because $k^i(m) > t$, by induction hypothesis). Since $h_t(x_{i+1}) \cap U_{r_t} = \varnothing$ (because $x_{i+1} \notin U^0_{r_{k(t)}}$) and $y \in U_{r_t}$, we get a contradiction. This completes the proof of (**).

Since $m > k(m) > k^2(m) > \cdots$, (**) yields a contradiction (a strictly decreasing sequence of positive integers!). Consequently, $U_{r_t} \subset U^0_{r_m}$, which completes the proof.

3. Proof of main result. That (b) $\Rightarrow$ (c) is clear. To get (c) $\Rightarrow$ (a), we prove that any space satisfying (c) has a σ-cushioned pair-base. To this end, let $\{(q_n, r_n): n \in \omega\}$ enumerate all pairs (q, r) of rationals in $]0, 1]$ such that $q < r$. Let $\mathfrak{U}_{n,m} = \{f^{-1}(]r_n, 1]), f^{-1}(]q_n, 1])): f \in \mathfrak{F}_m\}$, where $\mathfrak{F} = \bigcup_{m\in\omega} \mathfrak{F}_m$ satisfies (c). Then $\mathfrak{U} = \bigcup_{n,m\in\omega} \mathfrak{U}_{n,m}$ is easily seen to be a pair-base. To see that $\mathfrak{U}_{n,m}$ is cushioned, first note that any subset of $\mathfrak{U}_{n,m}$ has the form

$$\left\{\left(f^{-1}(]r_n, 1]), f^{-1}(]q_n, 1])\right): f \in \mathfrak{F}'_m\right\} \quad \text{where } \mathfrak{F}'_m \subset \mathfrak{F}_m.$$

Then

$$\bigcup_{f\in\mathfrak{F}'_m} f^{-1}(]r_n, 1]) \subset \left(\sup \mathfrak{F}'_m\right)^{-1}(]r_n, 1])$$
$$\subset \left(\sup \mathfrak{F}'_m\right)^{-1}[r_n, 1] \subset \bigcup_{f\in\mathfrak{F}'_m} f^{-1}(]q_n, 1]).$$

It remains to prove (a) $\Rightarrow$ (b). By Lemma 2.3, for each $U \in \tau$, define $f_U: X \to I$ by

$$f_U(x) = \begin{cases} 1, & \text{if } x \in U_1, \\ \inf\{r \in D \mid x \notin U_r\}, & \text{otherwise.} \end{cases}$$

Clearly $f_U^{-1}(0) = X - U$. Also each f_U is continuous (note that $f_U^{-1}([t,1]) = \bigcap\{U_s \mid s < t,\ s \in D\}$ and $f_U^{-1}(]t,1]) = \bigcup\{U_s \mid s > t,\ s \in D\} = \bigcup\{U_s^0 \mid s > t,\ s \in D\}$, for each $t \in D$, and $\{]t,1] \mid t \in D\} \cup \{[0,t[\mid t \in D\}$ is a subbasis for I).

Now let $\mathcal{U} \subset \tau$ and let us show that $\sup_{U \in \mathcal{U}} f_U$ is continuous: First, note that, for each $r \in D$,

$$\left(\sup_{U \in \mathcal{U}} f_U\right)^{-1}(]r,1]) = \bigcup_{U \in \mathcal{U}} f_U^{-1}], r, 1].$$

To complete the proof, we need to show that $(\sup_{U \in \mathcal{U}} f_U)^{-1}([r,1])$ is closed. Suppose there exists

$$p \in \left[\left(\sup_{U \in \mathcal{U}} f_U\right)^{-1}([r,1])\right]^{-} - \left(\sup_{U \in \mathcal{U}} f_U\right)^{-1}([r,1]).$$

Then $(\sup_{U \in \mathcal{U}} f_U)(p) = \delta < r$. Then, for $\delta < s < t < r$, $s, t \in D$,

$$p \notin \bigcup_{U \in \mathcal{U}} f_U^{-1}([s,1]) \supset \bigcup_{U \in \mathcal{U}} U_s = \left(\bigcup_{U \in \mathcal{U}} U_s\right)^{-} \supset \left(\bigcup_{U \in \mathcal{U}} f_U^{-1}([t,1])\right)^{-},$$

by Lemma 1.3(3). So there exists a neighborhood 0 of p such that $0 \cap (\bigcup_{U \in \mathcal{U}} f_U^{-1}([t,1])) = \varnothing$, which implies that $f_U(0) \subset [0,t[$, for each $U \in \mathcal{U}$; therefore, $(\sup_{U \in \mathcal{U}} f_U)^{-1}([r,1]) \cap 0 = \varnothing$, a contradiction.

From the proof of the "if" part of Theorem 1 and Proposition 2 of [4], one easily gleans the following result.

THEOREM 2. *A T_1-space Y is paracompact if and only if, for each open cover $\mathcal{V}$ of Y, there exists a family $\{f_\alpha\colon Y \to I\}_{\alpha \in \Lambda}$ of continuous functions such that*

(i) $\sup_{\alpha \in \Gamma} f_\alpha$ *is continuous, for each* $\Gamma \subset \Lambda$,
(ii) $\{f_\alpha^{-1}(]0,1])\}_{\alpha \in \Lambda}$ *refines* $\mathcal{V}$.

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