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SOME POINCARÉ SERIES RELATED TO IDENTITIES OF 2×2 MATRICES

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A partial solution to a problem of Procesi has recently been given by Formanek, Halpin, Li by determining the Poincaré series of the ideal of two variable identities of $M_2(k)$. Two related results are obtained in this article.

A weak identity of $M_n(k)$ is a polynomial which vanishes identically on sl_n, the subspace of $M_n(k)$ of matrices of trace zero. We show that the Poincaré series of the ideal of two variable weak identities of $M_2(k)$ is rational. In addition it is shown that the ideal of identities of upper triangular 2×2 matrices in an arbitrary finite number of variables has a rational Poincaré series. As an application we are able to determine this ideal precisely.

Introduction. Let $S = K\langle x_1,\ldots,x_n\rangle$ be the free associative algebra over k where k is any field of characteristic zero. S is naturally graded by giving x_1 degree $(1,0,\ldots,0)$, x_2 degree $(0,1,\ldots,0)$, etc. Denote by $S_{(i_1,\ldots,i_n)}$ the subspace of S generated by monomials of degree $(i_1,\ldots,i_n)$. If A is a homogeneously generated ideal of S then we associate a series to A, called the Poincaré series of A, via

$$P(A) = \sum_{i_1,\ldots,i_n \geq 0} a(i_1,\ldots,i_n) s_1^{i_1} s_2^{i_2} \cdots s_n^{i_n}$$

where $a(i_1,\ldots,i_n) = \dim_k(A \cap S_{(i_1,\ldots,i_n)})$. In **[1]** Formanek, Halpin, Li showed that the Poincaré series of the ideal of two variables identities of $M_2(k)$ is a rational function in s_1 and s_2. In this article we obtain two related results.

A weak identity of $M_n(k)$ is a polynomial which vanishes upon substitution of elements of $\mathrm{sl}_n(k)$, where $\mathrm{sl}_n(k)$ denotes the subspace of $M_n(k)$ of matrices of trace zero. The notion of a weak identity was introduced by Razmyslov **[2]** in connection with the study of central polynomials. Let $T_2^W(x_1, x_2)$ denote the ideal of $k\langle x_1, x_2\rangle$ of weak identities of $M_2(k)$. In Section 1 we determine $P(T_2^W(x_1, x_2))$ and find that it is again a rational function in s_1 and s_2.

In §2 we consider the identities of the subalgebra of $M_2(k)$ consisting of upper triangular matrices. By restricting to upper triangular matrices we are able to obtain results more complete than those obtained in **[1]**. We calculate the Poincaré series of the ideal of identities of upper triangular 2×2 matrices in an arbitrary finite number of variables. As an application the ideal of identities of upper triangular 2×2 matrices is determined explicitly.

1. Weak identities of $\mathrm{M}_2(\mathrm{k})$. Let $T_2^W(x_1, x_2)$ denote the collection of two variable weak identities of $M_2(k)$ where k is a field of characteristic zero. It is easy to see that $T_2^W(x_1, x_2)$ is an ideal of $k\langle x_1, x_2\rangle$, although it is not a T-ideal in the usual sense. As in the case of the identities of $M_n(k)$, the ideal of weak identities $M_n(k)$ is homogeneously generated. The goal of this section is to determine $P(T_2^W(x_1, x_2))$.

Let

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{22} & -X_{11} \end{pmatrix}, \qquad Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & -Y_{11} \end{pmatrix}$$

be 2×2 generic matrices of trace zero. The x_{ij}, y_{ij} are commuting indeterminates. Define $R = k[X, Y]$ as the algebra generated over k by X and Y. R may be graded by assigning X degree $(1, 0)$ and Y degree $(0, 1)$. Let $A = k[x_{ij}, y_{ij}]$ be the commutative polynomial ring generated over k by the six indeterminates x_{ij}, y_{ij}. A may be graded by assigning each x_{ij} degree $(1, 0)$ and each y_{ij} degree $(0, 1)$.

The following lemma, which is analogous to a well known result on identities of $M_n(k)$, is clear.

LEMMA 1. *The sequence*

$$0 \to T_2^W(x_1, x_2) \to k\langle x_1, x_2\rangle \xrightarrow{\pi} k[X, Y] \to 0,$$

where $\pi(x_1) = X$ *and* $\pi(x_2) = Y$, *is an exace sequence of graded k-modules.*

By D, T we denote determinant, trace respectively. We define

$$\begin{aligned} B &= k[D(X), D(Y), T(XY)] \\ &= k\left[x_{11}^2 + x_{12}x_{21}, y_{11}^2 + y_{12}y_{21}, x_{12}y_{21} + x_{21}y_{12} + 2x_{11}y_{11}\right] \end{aligned}$$

B inherits a grading as a homogeneously generated submodule of A.

LEMMA 2. *B is a commutative polynomial ring over k in $D(X)$, $D(Y)$, $T(XY)$.*

Proof. This is easily seen by specializing $x_{12} = x_{21} = 0$.

The proof of the following lemma is routine and is therefore omitted.

LEMMA 3. *I, X, Y, XY are linearly independent over A and so are linearly independent over B.*

THEOREM 4. *$R = BI \oplus BX \oplus BY \oplus BXY$, a direct sum of k-spaces.*

Proof. The following relations are easily verified and show that $BI \oplus BX \oplus BY \oplus BYX \subseteq R$:

$$X^2 = -D(X)I,$$
$$Y^2 = -D(Y)I,$$
$$XY + YX = T(XY)I.$$

For the other inclusion note that B is the ring generated by $D(X)$, $D(Y)$, $T(XY)$. Therefore the three relations above show that $BI \oplus BX \oplus BY \oplus BXY$ is a ring containing X, Y and hence $R \subseteq BI \oplus BX \oplus BY \oplus BXY$.

The following easy lemma, used in [**1**], will be used extensively in the article.

LEMMA 5. *Let M and N be homogeneous k-submodules of $M_2(k[x_{ij}, y_{ij}])$.*

(1) *If $M \oplus N$ is a direct sum then $P(M \oplus N) = P(M) + P(N)$.*

(2) *If $U \in M_2(k[x_{ij}, y_{ij}])$ is a homogeneous nonzero divisor of degree (p, q) then $P(MU) = s_1^p s_2^q P(M)$.*

THEOREM 6. *We have*

$$P(R) = \frac{1}{(1-s_1)(1-s_2)(1-s_1s_2)} \tag{1}$$

and

$$P\big(T_2^W(x_1, x_2)\big) = \frac{s_1s_2(s_1+s_2-s_1s_2)}{(1-s_1)(1-s_2)(1-s_2s_2)(1-s_1-ns_2)}. \tag{2}$$

Proof. By Lemma 2 B is a commutative polynomial ring in $D(X)$, $D(Y)$, $T(XY)$ of degrees (2, 0), (0, 2), (1, 1) respectively. Therefore

$$\begin{aligned} P(B) &= P(k[D(X), D(Y), T(XY)]) \\ &= \big(1 + s_1^2 + s_1^4 + \cdots\big)\big(1 + s_2^2 + s_2^4 + \cdots\big)\big(1 + s_1s_2 + s_1^2s_2^2 + \cdots\big) \\ &= \frac{1}{\big(1-s_1^2\big)\big(1-s_2^2\big)(1-s_1s_2)}. \end{aligned}$$

Therefore

$$\begin{aligned} P(R) &= P(BI \oplus BX \oplus BY \oplus BXY) \\ &= P(B) + P(BX) + P(BY) + P(BXY) = (1+s_1)(1+s_2)P(B) \\ &= \frac{1}{(1-s_1)(1-s_2)(1-s_1s_2)}. \end{aligned}$$

For (2) we note that by the exact sequence of Lemma 1

$$\begin{aligned} P(T_2^W(x_1, x_2)) &= P(k\langle x_1, x_2\rangle) - P(R) \\ &= \frac{1}{1-s_1-s_2} - \frac{1}{(1-s_1)(1-s_2)(1-s_1s_2)} \\ &= \frac{s_1s_2(s_1+s_2-s_1s_2)}{(1-s_1)(1-s_2)(1-s_1s_2)(1-s_1-s_2)}. \end{aligned}$$

2. Upper triangular matrices. The object of study in this section is the ideal of identities of upper triangular 2×2 matrices.

We first establish the notation that will be used in this section.Let $A = k[x_{ij}^{(k)};\ 1 \le i \le j \le 2, 1 \le k \le n]$ be the commutative polynomial ring generated over k by the $3n$ variables $x_{ij}^{(k)}$. By $T_2^U(x_1,\ldots,x_n)$ we mean the ideal of identities of upper triangular 2×2 matrices in $x_1,\ldots,x_n$ with coefficients in k. Now let $X_1,\ldots,X_n$ be upper triangular 2×2 generic matrics where

$$X_l = \begin{pmatrix} x_{11}^{(l)} & x_{12}^{(l)} \\ 0 & x_{22}^{(l)} \end{pmatrix}.$$

$R = k[X_1,\ldots,X_n]$ denotes the algebra generated over k by $X_1,\ldots,X_n$.

We begin with a version of the well known diagonalization technique.

LEMMA 7. *$R = k[X_1, X_2,\ldots,X_n]$ is isomorphic (as k-algebras) to $k[X, X_2,\ldots,X_n]$ where*

$$X = \begin{pmatrix} x_{11}^{(1)} & 0 \\ 0 & x_{22}^{(1)} \end{pmatrix}.$$

Proof. The matrix X_1 is diagonalizable by some matrix T which may be taken upper triangular. Then

$$R \cong T^{-1}RT = k\left[X, T^{-1}X_2T,\ldots,T^{-1}X_nT\right] \cong k\left[X, X_2,\ldots,X_n\right].$$

In view of Lemma 7 from now on we will take $R = k[X_1,\ldots,X_n]$ where $X_1 = X$.

We grade $k\langle x_1,\ldots,x_n\rangle$ as in the previous section. Similarly $A = k[x_{ij}^{(k)}; 1 \le i \le j \le 2, 1 \le k \le n]$ and $B = k[x_{ii}^{(k)}; i = 1,2, 1 \le k \le n]$ are graded by giving each $x_{ij}^{(1)}$ degree $(1,0,\ldots,0)$, each $x_{ij}^{(2)}$ degree $(0,1,\ldots,0)$, etc. Also R is graded by assigning X_1 degree $(1,0,\ldots,0)$, X_2 degree $(0,1,\ldots,0)$, etc.

With these gradings we state an obvious lemma which is analogous to Lemma 1.

LEMMA 8. *The sequence below, with the obvious maps, is an exact sequence of graded k-modules*:

$$0 \to T_2^U(x_1,\ldots,x_n) \to k\langle x_1,\ldots,x_n\rangle \to R \to 0.$$

The main theorem of this section is the evaluation of $P(T_2^U(x_1,\ldots,x_n))$ which will be proved by induction on n. In order to start the induction at $n = 2$ we first calculate $P(R_0)$ where $R_0 = k[X_1, X_2]$.

LEMMA 9. *The commutator ideal* $[R_0, R_0]$ *equals*

$$k\left[x_{11}^{(1)}, x_{22}^{(1)}, x_{11}^{(2)}, x_{22}^{(2)}\right]\cdot\left[X_1, X_2\right].$$

Proof. $[R_0, R_0]$ is the ideal of R_0 generated by

$$[X_1, X_2] = \begin{pmatrix} 0 & \left(x_{11}^{(1)} - x_{22}^{(1)}\right)x_{12}^{(2)} \\ 0 & 0 \end{pmatrix}.$$

Now notice that

$$X_i[X_1, X_2] = x_{11}^{(i)}[X_1, X_2]$$

and

$$[X_1, X_2]X_i = x_{22}^{(i)}[X_1, X_2].$$

Therefore

$$[R_0, R_0] \subseteq k\left[x_{11}^{(1)}, x_{22}^{(1)}, x_{11}^{(2)}, x_{22}^{(2)}\right]\cdot\left[X_1, X_2\right].$$

For the reverse inclusion if $(x_{11}^{(1)})^a(x_{22}^{(1)})^b(x_{11}^{(2)})^c(x_{22}^{(2)})^d$ is any monomial in $k[x_{11}^{(1)}, x_{22}^{(1)}, x_{11}^{(2)}, x_{22}^{(2)}]$ then one sees easily that

$$\begin{aligned}&\left(x_{11}^{(1)}\right)^a\left(x_{22}^{(1)}\right)^b\left(x_{11}^{(2)}\right)^c\left(x_{22}^{(2)}\right)^d[X_1, X_2]\\ &\qquad = X_1^aX_2^c[X_1, X_2]X_1^bX_2^d \in [R_0, R_0].\end{aligned}$$

LEMMA 10.
$$P([R_0, R_0]) = \frac{s_1 s_2}{(1-s_1)^2(1-s_2)^2}.$$

Proof. Since $x_{11}^{(1)}, x_{22}^{(1)}, x_{11}^{(2)}, x_{22}^{(2)}$ have degrees (1,0), (1,0), (0,1), (0,1) respectively, we have
$$\begin{aligned}
P([R_0, R_0]) &= P\left(k\left[x_{11}^{(1)}, x_{22}^{(1)}, x_{11}^{(2)}, x_{22}^{(2)}\right]\cdot[X_1, X_2]\right)\\
&= s_1 s_2 P\left(k\left[x_{11}^{(1)}, x_{22}^{(1)}, x_{11}^{(2)}, x_{22}^{(2)}\right]\right)\\
&= s_1 s_2 \left(1 + s_1 + s_1^2 + \cdots\right)^2 \left(1 + s_2 + s_2^2 + \cdots\right)^2\\
&= \frac{s_1 s_2}{(1-s_1)^2(1-s_2)^2}.
\end{aligned}$$

LEMMA 11.
$$P(R_0) = \frac{1 - s_1 - s_2 + 2s_1 s_2}{(1-s_1)^2(1-s_2)^2}.$$

Proof. Since $R_0/[R_0, R_0] \cong k[x_1, x_2]$, a commutative polynomial ring, it follows that as k-spaces
$$R_0 \cong_k [R_0, R_0] \oplus_k k[x_1, x_2].$$
Therefore
$$\begin{aligned}
P(R_0) &= P([R_0, R_0]) + P(k[x_1, x_2])\\
&= \frac{s_1 s_2}{(1-s_1)^2(1-s_2)^2} + \frac{1}{(1-s_1)(1-s_2)} = \frac{1 - s_1 - s_2 + 2s_1 s_2}{(1-s_1)^2(1-s_2)^2}.
\end{aligned}$$

In order to calculate $P(T_2^U(x_1,\ldots,x_n))$ it suffices to calculate $P(k[X_1,\ldots,X_n])$. We proceed by induction on n, having established the case $n = 2$. The following lemma will be used to execute the inductive step.

LEMMA 12. *The ideal* $[X_1, R]$ *of* R *equals* $[X_1, X_2]B \oplus_k [X_1, X_3]B \oplus_k \cdots \oplus_k [X_1, X_n]B$.

Proof. The ideal $[X_1, R]$ is the ideal of R generated by
$$[X_1, X_2] = \begin{pmatrix} 0 & \left(x_{11}^{(1)} - x_{22}^{(1)}\right)x_{12}^{(2)} \\ 0 & 0 \end{pmatrix}$$
$$\vdots$$
$$[X_1, X_n] = \begin{pmatrix} 0 & \left(x_{11}^{(1)} - x_{22}^{(1)}\right)x_{12}^{(n)} \\ 0 & 0 \end{pmatrix}.$$

Notice that

$$X_i[X_1, X_j] = x_{11}^{(i)}[X_1, X_j]$$

and

$$[X_1, X_j]X_i = x_{22}^{(i)}[X_1, X_j].$$

The lemma now follows easily as in Lemma 9. Of course the sum above is direct since the $x_{12}^{(k)}$, $1 \le k \le n$, are distinct indeterminates.

As an immediate consequence of Lemma 12 we may compute $P([X_1, R])$.

LEMMA 13.

$$P([X_1, R]) = \frac{s_1 s_2 + s_1 s_3 + \cdots + s_1 s_n}{(1-s_1)^2(1-s_2)^2 \cdots (1-s_n)^2}.$$

THEOREM 14.

$$P(R) = \frac{(2(1-s_1)\cdots(1-s_n)) + (s_1 + \cdots + s_n) - 1}{(1-s_1)^2(1-s_2)^2 \cdots (1-s_n)^2}.$$

Proof. We induct on n. The case $n = 2$ is Lemma 11 so we assume $n \ge 3$ and that the theorem is true for $n - 1$ variables.

R has the following decomposition as a k-space:

$$R \cong_k R/[X_1, R] \oplus_k [X_1, R] \cong_k \bigoplus_{i=0}^{\infty} X_1^i k[X_2,\ldots,X_n] \oplus_k [X_1, R].$$

Therefore,

$$\begin{aligned} P(R) &= P\left(\bigoplus_{i=0}^{\infty} X_1^i k[X_2,\ldots,X_n]\right) + P([X_1, R]) \\ &= (1 + s_1 + s_1^2 + \cdots)P(k[X_2,\ldots,X_n]) + P([X_1, R]). \end{aligned}$$

By the inductive hypothesis $P(k[X_2,\ldots,X_n])$ equals

$$\frac{(2(1-s_2)\cdots(1-s_n)) + (s_2 + \cdots + s_n) - 1}{(1-s_2)^2(1-s_3)^2 \cdots (1-s_n)^2},$$

and by Lemma 13 $P([X_1, R])$ equals

$$\frac{s_1 s_2 + \cdots + s_1 s_n}{(1-s_1)^2 \cdots (1-s_n)^2}.$$

Thus

$$P(R) = \frac{(2(1-s_2)\cdots(1-s_n)) + (s_2 + \cdots + s_n) - 1}{(1-s_1)(1-s_2)^2(1-s_3)^2\cdots(1-s_n)^2}$$
$$+ \frac{s_1 s_2 + \cdots + s_1 s_n}{(1-s_1)^2\cdots(1-s_n)^2}$$
$$= \frac{(2(1-s_1)\cdots(1-s_n)) + (s_1 + \cdots + s_n) - 1}{(1-s_1)^2(1-s_2)^2\cdots(1-s_n)^2}.$$

We now prove the main result of this section.

THEOREM 15.

$$P(T_2^U(x_1,\dots,x_n)) = \frac{((1-s_1)\cdots(1-s_n) - (1-s_1-\cdots-s_n))^2}{(1-s_1-\cdots-s_n)(1-s_1)^2\cdots(1-s_n)^2}.$$

Proof. By the exact sequence of Lemma 8 we have

$$P(T_2^U(x_1,\dots,x_n)) = P(k\langle x_1,\dots,x_n\rangle) - P(k[X_1,\dots,X_n])$$
$$= \frac{1}{1-s_1-\cdots-s_n} - \frac{2((1-s_1)\cdots(1-s_n)) + (s_1 + \cdots + s_n) - 1}{(1-s_1)^2\cdots(1-s_n)^2}$$
$$= \frac{((1-s_1)\cdots(1-s_n) - (1-s_1-\cdots-s_n))^2}{(1-s_1-\cdots-s_n)(1-s_1)^2\cdots(1-s_n)^2}.$$

As an application of Theorem 15 we now give a precise description of $T_2^U(x_1,\dots,x_n)$. Let $T_1(x_1,\dots,x_n)$ denote the commutator ideal of $k\langle x_1,\dots,x_n\rangle$. In other words, $T_1(x_1,\dots,x_n)$ is the ideal of $k\langle x_1,\dots,x_n\rangle$ such that the sequence

$$0 \to T_1(x_1,\dots,x_n) \to k\langle x_1,\dots,x_n\rangle \to k[x_1,\dots,x_n] \to 0$$

is exact. It follows that

$$P(T_1(x_1,\dots,x_n)) = \frac{1}{1-s_1-\cdots-s_n} - \frac{1}{(1-s_1)\cdots(1-s_n)}$$
$$= \frac{(1-s_1)\cdots(1-s_n) - (1-s_1-\cdots-s_n)}{(1-s_1-\cdots-s_n)(1-s_1)\cdots(1-s_n)}.$$

We will show that $T_2^U(x_1,\dots,x_n) = (T_1(x_1,\dots,x_n))^2$. To show one inclusion is very easy. It then suffices to show that both members have the same Poincaré series. To calculate the Poincaré series of $(T_1(x_1,\dots,x_n))^2$ we need to make use of a combinatorial lemma, due to Formanek. We sketch a proof of the lemma.

LEMMA 16. (*Formanek*) *Let I and J be homogeneously generated ideals of $k\langle x_1,\ldots,x_n\rangle$. Then $P(IJ) = P(I)P(J)(1 - s_1 - \cdots - s_n)$.*

Proof. One first shows, using only elementary arguments, that I and J are free as left ideals on homogeneous generators. Let $\alpha(i_1,\ldots,i_n)$ equal the number of free generators of I considered as a left ideal of degree $(i_1,\ldots,i_n)$. Define

$$G(I) = \sum_{i_1,\ldots,i_n\geq 0} \alpha(i_1,\ldots,i_n)s_1^{i_1}s_2^{i_2}\cdots s_n^{i_n}.$$

Similarly define $G(J)$ and $G(IJ)$. Then $G(IJ) = G(I)G(J)$ and $P(I) = G(I)/(1 - s_1 - \cdots - s_n)$. The lemma follows.

THEOREM 17. $T_2^U(x_1,\ldots,x_n) = (T_1(x_1,\ldots,x_n))^2$.

Proof. We first show that $(T_1(x_1,\ldots,x_n))^2 \subseteq T_2^U(x_1,\ldots,x_n)$. Any element of $(T_1(x_1,\ldots,x_n))^2$ is a sum of terms of the form $r_1[x_i, x_j]r_2[x_k, x_l]r_3$ where $1 \leq i, j, k, l \leq n$ and $r_1, r_2, r_3 \in k\langle x_1,\ldots,x_n\rangle$. The commutator of two upper triangular matrices is strictly upper triangular. Therefore each term of the form above is an identity for R since any finite product of upper triangular 2×2 matrices where at least two of the factors are strictly upper triangular is zero. Therefore $(T_1(x_1,\ldots,x_n))^2 \subseteq T_2^U(x_1,\ldots,x_n)$.

As mentioned above it now suffices to show that $(T_1(x_1,\ldots,x_n))^2$ and $T_2^U(x_1,\ldots,x_n)$ have the same Poincaré series. By Lemma 16

$$\begin{aligned}P\big((T_1(x_1,\ldots,x_n))^2\big) &= (1 - s_1 - \cdots - s_n)\big(P(T_1(x_1,\ldots,x_n))\big)^2 \\ &= (1 - s_1 - \cdots - s_n)\left(\frac{(1-s_1)\cdots(1-s_n) - (1 - s_1 - \cdots - s_n)}{(1 - s_1 - \cdots - s_n)(1-s_1)\cdots(1-s_n)}\right)^2 \\ &= P\big(T_2^U(x_1,\ldots,x_n)\big).\end{aligned}$$

REFERENCES

1. E. Formanek, P. Halpin, W.-C. Li, *The Poincaré series of the ring of* 2×2 *generic matrices*, J. Algebra, **69** (1981), 105–112.

2. Y. Razmyslov, *On a problem of Kaplansky*, Izv. A.N.S.S.S.R. Sec. Math., **37** (1973), 479–496 (Russian). Translation: Math. USSR-Izv., Vol. **7** (1973), No. 3.

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