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PETTIS INTEGRATION VIA THE STONIAN TRANSFORM

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Let (Ω, Σ, μ) be a finite measure space, and let f be a bounded weakly measurable function from Ω into a Banach space X. Let S be the Stone space of the measure algebra $\Sigma/\mu^{-1}(0)$. Then f induces a continuous map $\hat{f}$: $S \to (X^{**}, \text{weak*})$ in a natural way. Criteria for Pettis integrability of f are investigated in the context of this "Stonian transform" $\hat{f}$ of f. In particular, some insight is achieved as to how f can be Pettis integrable without being weakly equivalent to any strongly measurable function. The fine structure of $(X^{**}, \text{weak*})$ is also examined in this setting.

Let (Ω, Σ, μ) be a complete probability space, X a real Banach space, f: $\Omega \to X$ a function with bounded range. Then f is said to be strongly measurable if there is a sequence (f_n) of measurable X-valued simple functions on Ω such that $\| f_n(w) - f(w)\| \to 0$ a.e. (μ). In this case, the Bochner integral of f over $E \in \Sigma$ is given by $\int_E f\, d\mu = \lim \int_E f_n\, d\mu$, where the integral of a simple function is defined in the obvious way. This integral has been extensively studied; the principal results are recorded in the monograph of Diestel and Uhl [**3**].

There is another concept of measurability and integrability which is less well understood. A bounded function f: $\Omega \to X$ is *weakly measurable* if $x^* \circ f$ is a measurable scalar-valued function on Ω for each $x^* \in X^*$, the dual space of X. If $E \in \Sigma$, then the *Dunford integral* of f over E is the member x_E^{**} of X^{**} defined by $x_E^{**}(x^*) = \int_E x^* \circ f\, d\mu$. If x_E^{**} is a member of X, for each $E \in \Sigma$, then f is said to be *Pettis integrable*, and we write $x_E = (P)\int_E f\, d\mu$ for the integral. This notion was originally studied by Pettis [**18**]; again the classical results are recorded in [**3**]. Recently there has been considerable progress in the study of the Pettis integral [**6**, **7**, **8**, **9**, **10**, **24**, **27**, **28**].

In this work we analyze the nature of Pettis integrability by converting measurability into continuity. Let S be the Stone representation space of the measure algebra $\Sigma/\mu^{-1}(0)$, and let $h \to \hat{h}$ be the usual isometry between $L^\infty(\Omega, \Sigma, \mu)$ and $C(S)$. Then the *Stonian transform* of f: $\Omega \to X$ is the function $\hat{f}$: $S \to X^{**}$ defined by $\langle \hat{f}(s), x^*\rangle = \widehat{x^* \circ f}(s)$, for all $x^* \in X^*$ [**26**]. Clearly $\hat{f}$ is continuous when X^{**} is given the weak*-topology. We consider the questions: (a) Can $\hat{f}$ actually be computed in a meaningful way? (b) What does the structure and position of the image set $\hat{f}(S)$ in X^{**} tell us about integrability of f? (c) Does $\hat{f}$ furnish any insights about the problem of Pettis integration which could not equally well be deduced from the original f? The major results include:

(1) For the standard examples of weakly measurable functions, Pettis integrable or not, $\hat{f}$ can be computed in a concrete way. This is carried out in some detail in §2. The continuous function $\hat{f}$ captures, in some sense, the essence of the measurable function f. Each point value $\hat{f}(s)$ depends on all of X^* (being a linear functional), but is also determined locally at s as a limit of integral averages. Indeed **[24]** $\int_A f\,d\mu = \int_a \hat{f}\,d\mu$ for any measurable set A and corresponding clopen set $a \subset S$, even though the ranges of f and $\hat{f}$ are often disjoint subsets of X^{**}.

(2) It is shown in **[27]** that for perfect measure spaces (Ω, Σ, μ), Pettis integrability can be described in terms of $\overline{\text{co}}\,\hat{f}(a)$, the weak* closed convex hull of $\hat{f}(a)$, as a runs over all non-empty clopen subsets of S. Indeed $f\colon \Omega \to X$ is Pettis integrable if and only if $\overline{\text{co}}\,\hat{f}(a) \cap X$ is non-empty and separable for each a. Talagrand **[28]** has recently given a striking improvement (Theorem 3.2), removing both the need for perfect measure spaces and the need to mention separability in the equivalence. This has considerably broadened and simplified the present paper.

This characterization theorem does not necessarily yield a method for computing the Pettis integral of a given f. In particular, it does not produce an approximating sequence of X-valued simple functions which yield the integral in the manner described by Geitz **[9]**. We describe (Theorem 3.5) a strategy for doing this, and we indicate (Theorem 3.8) a setting where this strategy may always be applied.

(3) A function $f\colon \Omega \to X$ can be Pettis integrable without being weakly equivalent to any strongly measurable function. Sentilles **[24]** has introduced the notion of a purely weakly measurable function f (i.e., such that $\hat{f}(S)$ is essentially contained in $X^{**}\setminus X$) as a way of isolating this phenomenon. We study this extreme but fundamental case in detail, and obtain results on integrability and non-integrability of such f. For example, if f is purely weakly measurable, and $H = \{x^*\hat{f}\colon \|x^*\| \le 1\}$ is $\sigma(C(S), M(S))$-compact, then f *cannot* be Pettis integrable (Theorem 4.1).

(4) Pettis integrability of f is related to two intrinsically defined subspaces of X^{**}. Corson **[2]** showed that the Hewitt realcompactification νX of (X, weak) is a norm-closed subspace of X^{**}. Grothendieck **[12]** introduced $Ba^1(X)$, the set of points in X^{**} which are weak*-limits of sequences from X. McWilliams **[16]** proved that $Ba^1(X)$ is also a norm-closed subspace of X^{**}. We show (Theorem 4.5) that $\nu X \cap Ba^1(X) = X$

for every X. This splitting of X^{**} seems to be critical for f: if $\hat{f}(S) \subset \nu X - X$, then f *cannot* be Pettis integrable (Theorem 4.3). But if $\hat{f}(S) \subset Ba^1(X)$, then f must be Pettis integrable, even if $\hat{f}(S)$ is not weak*-metrizable (Theorem 4.6, and Example 2.3).

While we are far from a complete understanding of this situation, it appears that $\hat{f}$ offers a way of exploring X^{**} which the original f fails to do. Note also that f: $\Omega \to X$ induces a Baire measure μf^{-1} on (X, weak) and a regular Borel measure ν on (X^{**}, weak^*). Edgar [**4**, **5**] has demonstrated the importance of these image measures in analyzing the behavior of f. The relevant fact is that the support of ν in (X^{**}, weak^*) is precisely $\hat{f}(S)$ [**24**], and this permits an analysis of $\operatorname{spt} \nu$ that might otherwise be difficult to achieve.

(5) The study of $\hat{f}$ yields an integral-free characterization of Pettis integrability expressed in terms of f and Ω alone (Theorem 5.3). It also leads to conditions for Pettis integrability expressed in terms of X^* (Theorem 5.2) and X^{***} (Theorems 5.6 and 5.7).

1. Preliminaries. Throughout, S denotes the Stone space of the complete Boolean algebra $\Sigma/\mu^{-1}(0)$. Thus S is compact, hyperstonian, and has the countable chain condition on open sets. The measure μ induces a regular Borel measure (also written μ) on S, and a subset of S is nowhere dense iff it has μ^*-measure 0. The natural isometries between $L^\infty(\Omega, \Sigma, \mu)$ and $C(S)$, and $L^1(\Omega, \mu)$ and $L^1(S, \mu)$, are denoted by $h \to \hat{h}$, for a scalar function h. If $A \subset \Sigma$, then $[A]$ denotes its μ-equivalence class, and a is the corresponding clopen subset of S. Note that $\int_A h\, d\mu = \int_a \hat{h}\, d\mu$ for all A and h.

The Stonian transform $\hat{f}$ of a vector function f: $\Omega \to X$ has been studied by Sentilles [**24**, **25**, **26**, **27**] and Kuo [**15**]. The defining relation $\langle \hat{f}(s), x^* \rangle = \widehat{x^* \circ f}(s)$ will be abbreviated as simply $x^*\hat{f} = \widehat{x^*f}$. It is immediate that $\hat{f}$ is continuous from S to (X^{**}, weak^*); hence $\hat{f}(S)$ is a weak* compact subset of X^{**}. Two functions f, g: $\Omega \to X$ have the same transform if and only if they are weakly equivalent ($x^*f = x^*g$ a.e. $(\mu) \forall x^* \in X^*$).

The following result of Kuo and Sentilles will be used repeatedly.

THEOREM 1.1 [**15**, **26**]. *A continuous function η: $S \to (X^{**}, \text{weak}^*)$ is the Stonian transform of a strongly measurable function (i.e., $\eta = \hat{f}$ for some f) if and only if $\eta^{-1}(X^{**} \setminus X)$ is nowhere dense in S. In this case:* (a) *$\eta(S) \cap X$ is norm separable; and* (b) *η is continuous into (X, norm) except on some closed nowhere dense set $M \subset S$. (Thus $\{x^* \circ \eta: \|x^*\| \leq 1\}$ is an equicontinuous subset of $C(S - M)$.)*

Consequently, if ξ: $S \to (X^{**}, \text{weak})$ is continuous, then (since $\xi^{-1}(X^{****} \setminus X^{**})$ is empty), ξ is norm continuous on the complement of a nowhere dense set in S. We use this to obtain non-integrability results in §4.

DEFINITION 1.2 [**24**]. The function f: $\Omega \to X$ is *purely weakly measurable* if $\hat{f}^{-1}(X \setminus \{0\})$ is nowhere dense in S.

THEOREM 1.3 [**24**]. *Every f: $\Omega \to X$ can be written as $g + h$, where g is strongly measurable and h is purely weakly measurable. Moreover, $x^*g(w) \cdot x^*h(w) = 0$ a.e. (μ) for each $x^* \in X^*$, and g and h are unique up to strong equivalence under this condition.*

Note that the function h can have $\hat{h} \equiv 0$. We say that h: $\Omega \to X$ is purely weakly measurable and *essentially non-zero* if $h(s) \in X^{**} \setminus X$ a.e. (μ). The problem of Pettis integrability can be reduced to this situation. In this extreme yet critical case, Pettis integrability of f can only obtain if the (X^*, μ)-integral averages of the point values $\hat{f}(s)$ in $X^{**} \setminus X$ somehow end up in X. As we shall see, the behavior of the weak*-closed convex hulls of sets $\hat{f}(a)$, a clopen in S, is the determining factor [**27**, **28**].

For the convenience of the reader, we record several recent results about the Pettis integral, achieved under the assumption that (Ω, Σ, μ) is a perfect measure space [**14**, **22**]. With this hypothesis one has:

FREMLIN'S THEOREM [**6**]. *Let (h_n) be a sequence of measurable scalar-valued functions on Ω. Then either (h_n) has a subsequence with no measurable cluster point (for the topology of pointwise convergence on Ω), or (h_n) has a subsequence which converges pointwise a.e. (μ).*

As Edgar [**5**] has pointed out, the study of Pettis integration is basically the study of the set $H = \{x^*f\colon \|x^*\| \leq 1\}$. Fremlin's deep result shows that H is norm-compact in $L^1(\mu)$ when μ is perfect. This leads to

STEGALL'S THEOREM [**7**]. *The range of the Dunford integral, $\{x_E^{**}\colon E \in \Sigma\}$ is relatively norm-compact in X^{**}. Hence if f is Pettis integrable, then the vector measure $F(E) = (P)\int_E f\,d\mu$ has relatively norm-compact range in X.*

R. F. Geitz [**8**, **9**, **10**] has combined these ideas with earlier results of R. C. James to achieve several striking descriptions of Pettis integrability.

In particular, and still in the setting of a perfect measure space, we have

GEITZ'S THEOREM [9]. *The function f is Pettis integrable if and only if there is a bounded sequence (f_n) of X-valued simple functions such that* $\lim_n x^*f_n(w) = x^*f(w)$ *a.e.* (μ) *for each* $x^* \in X^*$.

Note that the exceptional set may vary with x^*; if the same set works for all x^*, the condition characterizes strong measurability of f.

The paper is organized as follows: in §2, we compute $\hat{f}$ for three important examples of weakly measurable functions. §3 discusses the general problem of computing $\hat{f}$ via a sequential process. §4 takes up the relation of $\hat{f}(S)$ to the subspaces νX and $Ba^1(X)$ of X^{**}, and shows how $\hat{f}$ can determine non-integrability. §5 contains (Ω, f) and X^{***} characterizations of Pettis integrability. We conclude in §6 with a list of open questions.

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2. Computing the Stonian transform.

To illustrate the techniques involved, we calculate $\hat{f}$ for three standard examples of bounded weakly measurable functions f. In each case, $\Omega = [0, 1]$, $\Sigma =$ Lebesgue measurable sets, and $\mu =$ Lebesgue measure.

EXAMPLE 2.1. $f\colon \Omega \to l^\infty$ is defined by $f(t) = (\chi_{A_n}(t))$, where $A_1 = [0, 1]$, $A_2 = [0, \frac{1}{2}]$, $A_3 = [\frac{1}{2}, 1]$, $A_4 = [0, \frac{1}{4}], \ldots$. This example is due to Hagler and is discussed in [3, pp. 43–44]. Here f is Pettis integrable, with $x_E = (\mu(E \cap A_n)) \in c_0$ for each $E \in \Sigma$. However, f is not weakly equivalent to any strongly measurable function, since l^1 is separable and f is not essentially separably valued.

For each $t \in \Omega$, let $N_t = \{n \in N\colon t \in A_n\}$. If $t \neq t'$, then $N_t \cap N_{t'}$ is finite. Thus $M_t = \mathrm{cl}_{\beta N} N_t - N_t$ defines a pairwise disjoint family of non-empty clopen subsets of $\beta N - N$ [11, 6S]. The sets M_t yield a simple proof that f is weakly measurable (different from the one given in [3]). Since $l^\infty = C(\beta N)$, $(l^\infty)^* = M(\beta N)$, the space of bounded regular Borel measures on βN, and any $\lambda \in (l^\infty)^*$ can be represented as $\lambda_1 + \lambda_2$, where $\lambda_1 \in l^1$ and λ_2 lives on $\beta N - N$. Since the sets M_t are pairwise disjoint, $\langle \lambda_2, f(t) \rangle = 0$ a.e. (μ). But $\langle \lambda_1, f(t) \rangle$ is clearly measurable, so f is weakly measurable.

Now $\hat{f}(s)$ can be computed as follows. At each "level", $\bigcup\{A_k\colon 2^n \leq k < 2^{n+1}\} = \Omega$, and so $S = \bigcup\{a_k\colon 2^n \leq k < 2^{n+1}\}$, where the corresponding clopen sets a_k are pairwise disjoint. For each $s \in S$, let $N_s = \{n \in N\colon s \in a_n\}$, so that $N_s \cap \{2^k, \ldots, 2^{k+1} - 1\}$ has exactly one member for each k. Define $\phi_s \in (l^\infty)^{**}\setminus l^\infty$ as a function on βN: $\phi_s(p) = 1$ if $p \in N_s$ and 0 otherwise. Thus ϕ_s vanishes on $\beta N - N$, and on a subset of N of density 1. If $\lambda \in (l^\infty)^* = M(\beta N)$, then $\langle \phi_s, \lambda \rangle = \int_{\beta N} \phi_s(p)\, d\lambda(p) = \lambda(N_s)$.

Now $\langle \hat{f}(s), \lambda_2 \rangle = \widehat{\lambda_2 f}(s) = 0$ for all s, since $\langle \lambda_2, f(t) \rangle = 0$ a.e. (μ), and

$$\begin{aligned}\langle \hat{f}(s), \lambda_1 \rangle = \widehat{\lambda_1 f}(s) &= \left(\sum_1^\infty \lambda_1(\{n\}) \cdot \chi_{A_n} \right)\hat{}(s) \\ &= \sum_1^\infty \lambda_1(\{n\}) \cdot \chi_{a_n}(s) \\ &= \sum_{n \in N_s} \lambda_1(\{n\}) = \lambda_1(N_s) = \lambda(N_s).\end{aligned}$$

Thus $\hat{f}(s) = \phi_s$, since they have the same effect on each member of $(l^\infty)^*$.

Since $\hat{f}(S) \subset X^{**}\setminus X$, f is purely weakly measurable. We point out several additional facts about the structure of $\hat{f}(S)$ which will be needed later:

(a) $\overline{\text{co}}\, \hat{f}(S)$, the weak*-closed convex hull of $\hat{f}(S)$ in X^{**}, meets $c_0 \subset X$, even though $\hat{f}(S)$ does not. Consider $x_\Omega = (P)\int_\Omega f\, d\mu = (\mu(A_n))$. Since $M(\beta N - N)$ annihilates both c_0 and $\hat{f}(S)$, it suffices to show that given $\lambda \in l^1$ and $\varepsilon > 0$, $\exists x^{**} \in \text{co}(\hat{f}(S))$ with $|\langle x_\Omega - x^{**}, \lambda \rangle| < \varepsilon$. Choose n such that $\Sigma\{|\lambda(k)| : k \geq 2^{n+1}\} < \varepsilon/2$, and choose $s_i \in a_{2^n+i}$ for $0 \leq i < 2^n$. Then $x^{**} = \Sigma\{\mu(a_{2^n+i})\hat{f}(s_i) : 0 \leq i < 2^n\}$ is in the convex hull of $\hat{f}(S)$, and $x^{**}(k) = \Sigma\{\mu(a_{2^n+i}) : a_{2^n+i} \subset a_k\} = \mu(a_k) = x_\Omega(k)$ for $1 \leq k < 2^{n+1}$. Then $|\langle x_\Omega - x^{**}, \lambda \rangle| < \varepsilon$ as desired.

(b) For each $s \in S$, let $C(s) = \bigcap\{a_n : n \in N_s\}$. Then each $C(s)$ is a non-empty closed nowhere dense subset of S, and $\hat{f} \mid C(s)$ is constant. (We use this idea in §3 to find an approximating sequence of simple functions.)

(c) Let $P = \hat{f}(S)^\perp \subset X^*$. Then P properly contains $M(\beta N - N)$; for example, $\lambda = \delta(1) - \delta(2) - \delta(3)$ is in P. Thus X^*/P is a quotient space of l^1 and therefore separable.

(d) Each $\phi_s = \hat{f}(s)$ is a Baire class 1 function on βN.

(e) $\hat{f}(S)$ is weak*-homeomorphic to the Cantor set. For $\hat{f}(S)$ is weak*-compact, but the topology of pointwise convergence on N is coarser (and T_2). Thus $(\hat{f}(S), \text{weak*})$ can be viewed as a subset of $\{0, 1\}^N$. This means that $\hat{f}(S)$ is a compact, zero-dimensional metric space; it is also dense in itself, so it is homeomorphic to 2^N.

(f) $\hat{f}(S)$ is weakly closed and discrete in X^{**}. Since $\hat{f}(S)$ is weak*-compact, the first assertion is immediate. For the second, let $Y = \{g: \beta N \to R \mid g \text{ is bounded and } g \mid \beta N - N \equiv 0\}$. Then Y is a norm-closed subspace of $(l^\infty)^{**}$ and contains $\hat{f}(S)$. Since there is an obvious isometry $T: Y \to l^\infty$, $Y^* \simeq M(\beta N)$. Fix $s_0 \in S$, and choose $p \in \mathrm{cl}_{\beta N} N_{s_0} \setminus N_{s_0} \subset \beta N - N$. If $\hat{f}(s) \neq \hat{f}(s_0)$, then $N_s \cap N_{s_0}$ is finite, so $p \notin \mathrm{cl}_{\beta N} N_s$. Thus $\langle \delta(p), T\hat{f}(s_0) \rangle = 1$, but $\langle \delta(p), T\hat{f}(s) \rangle = 0$. This shows that $\hat{f}(S)$ is weakly discrete.

EXAMPLE 2.2. Assume the Continuum Hypothesis, and let $t \to \alpha_t$ be a bijection of $[0, 1]$ onto $[0, \omega_1)$, where ω_1 is the first uncountable ordinal. Let $X = C[0, \omega_1]$, so that $X^* = M[0, \omega_1] = l^1[0, \omega_1]$ and $X^{**} = l^\infty[0, \omega_1]$. Define $f: \Omega \to X$ by $f(t) = \chi_{[0, \alpha_t]}$. Note that $f = 1 - g$, where g is the original example, due to Phillips [**19**], of a non-Pettis integrable function. In the Phillips example, which also assume CH, the range is $l^\infty[0, 1] = l^\infty[0, \omega_1)$, and the function values are the members $\chi_{(\alpha_t, \omega_1)}$ of $C[0, \omega_1)$.

Now f is weakly measurable. Indeed if $\lambda = c_0\delta(\omega_1) + \sum_{n=1}^\infty c_n \delta(\alpha_n) \in X^*$, let $\alpha_0 = \sup \alpha_n$ and $A = \{t \in \Omega: \alpha_t \leq \alpha_0\}$. Then A is countable, and if $t \notin A$, $\lambda f(t) = \sum_{n=1}^\infty c_n = \lambda([0, \omega_1))$. Thus λf is constant a.e. (μ).

Clearly then f is weakly equivalent to $h(t) = \chi_{[0, \omega_1)}$ for all t. Equivalently, $\hat{f}(s) = \chi_{[0, \omega_1)} \in X^{**} \setminus X$, so f is purely weakly measurable, and $\hat{f}$ has one point range. Note that the single member of $\hat{f}(S)$ is Borel but not Baire measurable on $[0, \omega_1)$, and as an element of X^{**} belongs to $\nu X \setminus X$. Of course f is not Pettis integrable, since $\int_E f = \mu(E) \cdot \chi_{[0, \omega_1)}$.

REMARK. Again assuming CH, a similar example can be constructed for the space $X = l^\infty / c_0 = C(\beta N - N)$. Following [**11**, 6V], let p be a P-point of $\beta N - N$, and let $(V_\alpha)_{\alpha < \omega_1}$ be a decreasing family of clopen subsets of $\beta N - N$ with $\bigcap_{\alpha < \omega_1} V_\alpha = \{p\}$. Given a bijection $t \to \alpha_t$ of $[0, 1]$ onto $[0, \omega_1)$, the map $f: \Omega \to X$ given by $f(t) = \chi(V_{\alpha_t})$ has properties analogous to those of 2.2.

EXAMPLE 2.3. $f: \Omega \to L^\infty[0, 1]$ is defined by $f(t) = \chi_{[0, t)}$. This example has been considered by Edgar [**4**], who shows that f is weakly measurable, but not weakly equivalent to any strongly measurable function. Here f is Pettis integrable, with $x_E(t) = \mu(E \cap [t, 1]) \in C[0, 1] \subset L^\infty$. As noted by Edgar, the range of f is contained in $D[0, 1] = \{h: [0, 1] \to R \mid h$ is right-continuous and has left-hand limits$\}$. In turn, $D[0, 1]$ is isometrically isomorphic to $C(K)$, where K is the "double arrow space" $[0, 1] \times \{0, 1\}$, endowed with the relative topology of the lexicographic unit square. Thus neighborhoods of points $(t,0)$ (resp., $(t,1)$) look to the left (resp., right). K is compact T_2, first countable, separable, and perfectly normal, but not metrizable (see [20] for additional information). If $0 \leq t \leq 1$, let $A_t = \{(p, j): p < t\}$, $B_t = A_t \cup \{(t,0)\}$, $C_t = B_t \cup \{(t,1)\}$. Then A_t is a cozero set of K, B_t is clopen, and C_t is a zero set. Now f can be conveniently reformulated as the map from Ω to $X = C(K)$ defined by $f(t) = \chi_{B_t}$. We use this representation to calculate $\hat{f}(s)$.

Fix $s \in S$, and note that if $A \in \Sigma$, and the corresponding clopen set a contains s, then

$$(*) \qquad \operatorname*{ess\,inf}_{t\in A} h(t) \leq \hat{h}(s) \leq \operatorname*{ess\,sup}_{t\in A} h(t) \quad \text{for every } h \in L^\infty(\Omega).$$

Let $\mathcal{F}_s$ be the family of all closed intervals A in $[0, 1]$ such that $s \in a$. Then $\bigcap \mathcal{F}_s$ is a singleton $\{t_0\}$ of $[0, 1]$. Either $\mathcal{F}_s$ contains all intervals $[r, t_0]$ with $0 \leq r < t_0$, or $\mathcal{F}_s$ contains all intervals $[t_0, r]$ with $t_0 < r \leq 1$; these possibilities are mutually exclusive.

Suppose the first alternative obtains (i.e., $\mathcal{F}_s$ looks to the left at t_0). Let (p, j) be a fixed point of K, with corresponding point mass $\delta(p, j) \in C(K)^* = M(K)$. Then $\langle \hat{f}(s), \delta(p, j)\rangle = (\delta(p, j) \circ f)\hat{}(s)$. Now the scalar function $h(t) = \delta(p, j) \circ f(t) = \chi_{B_t}(p, j)$ has the value 1, if $p < t$ or $p = t, j = 0$; and 0 if $p > t$ or $p = t, j = 1$. A short computation using $(*)$ now shows that $\hat{h}(s) = 1$ if $p < t_0$, and 0 if $p \geq t_0$. Thus $\langle \hat{f}(s), \delta(p, j)\rangle = \chi_{A_{t_0}}(p, j)$.

To prove that $\hat{f}(s)$ can be identified with the function $\chi_{A_{t_0}}$, we still need to show that $\langle \hat{f}(s), \lambda \rangle = \lambda(A_{t_0})$ for every $\lambda \in M(K)$. This works for linear combinations of point masses, by the preceding argument. Now suppose that $\lambda \in M^+(K)$ is purely non-atomic. Given $\varepsilon > 0$, $\exists \delta > 0$ such that if $t_0 - \delta < t \leq t_0$, then $\lambda(A_{t_0}) - \varepsilon < \lambda(A_t) = \lambda(B_t) \leq \lambda(B_{t_0}) = \lambda(A_{t_0})$. An application of $(*)$, with $h = \lambda \circ f$ and $A = [t_0 - \delta, t_0]$, yields $\lambda(A_{t_0}) - \varepsilon \leq (\widehat{\lambda \circ f})(s) \leq \lambda(A_{t_0})$. Hence $|\langle \hat{f}(s), \lambda \rangle - \langle \chi_{A_{t_0}}, \lambda \rangle| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\langle \hat{f}(s), \lambda \rangle = \lambda(A_{t_0})$, as desired.

If the second alternative holds (i.e., $\mathcal{F}_s$ looks to the right at t_0), then similar arguments show that $\hat{f}(s)$ can be identified with the function $\chi_{C_{t_0}}$ on K.

(a) The members of $\hat{f}(S) = \{\chi_{A_t}: 0 < t \leq 1\} \cup \{\chi_{C_t}: 0 \leq t < 1\}$ are all Baire-1 functions on K. Also $\hat{f}(S) \subset X^{**} \setminus X$, so f is purely weakly measurable.

(b) Let $P = \hat{f}(S)^\perp \subset X^* = M(K)$. Any member of P is purely atomic, and satisfies $\mu(t,0) = -\mu(t,1)$ for $0 < t < 1$. It can be shown that X^*/P contains a copy of $M[0,1]$. Thus, in contrast to 2.1(c), X^*/P is not separable and indeed not weakly compactly generated. The significance of this is indicated by Theorem 3.8.

(c) $\hat{f}(S)$ is weak*-homeomorphic to $K_0 = K - \{(0,0) \cup (1,1)\}$. To see this, define ϕ: $K_0 \to \hat{f}(S) \subset (C(K)^{**}, \text{weak*})$ by $\phi(t,0) = \chi_{A_t}$, $\phi(t,1) = \chi_{C_t}$. ϕ is clearly one-to-one and onto. If $(t_\alpha, j_\alpha) \to (t_0, 0)$ in K_0, then $(\chi(A_{t_\alpha})$ or $\chi(C_{t_\alpha})) \to \chi(A_{t_0})$ pointwise on K. Similarly, if $(t_\alpha, j_\alpha) \to (t_0, 1)$ in K_0, then $(\chi(A_{t_\alpha})$ or $\chi(C_{t_\alpha})) \to \chi(C_{t_0})$ pointwise on K. But the weak* topology on $\hat{f}(S)$ coincides with the (coarser, Hausdorff) topology of pointwise convergence on K, so ϕ is continuous and therefore a homeomorphism.

Thus $(\hat{f}(S), \text{weak*})$ is separable, first countable, and perfectly normal, but not metrizable. Hence $(\hat{f}(S), \text{weak*})$ need not be an Eberlein compact, even for Pettis integrable f.

(d) It can be shown that $\hat{f}(S)$ is weakly closed and discrete in X^{**}.

3. Integrability and approximating sequences for f via $\hat{f}$. Talagrand's dramatic, and simple, extension [**28**] of Sentilles' integrability characterization [**27**] has definitively settled the geometric conjecture of Geitz regarding Pettis integrability. With Professor Talagrand's kind permission, we include a proof of his result. With this as background, we investigate the existence of an approximating sequence of simple functions for f which does not depend on *a priori* knowledge of the integral of f.

LEMMA 3.1 (TALAGRAND [**28**]). *Let f: $\Omega \to X$ be bounded and weakly measurable, and let $\eta > 0$. If for each finite subset A of X and each $\varepsilon > 0$, the set $H(A, \varepsilon) = \{x^*f\colon \|x^*\| \le 1, |x^*(x)| \le \varepsilon \, \forall x \in A, \text{ and } \int_\Omega x^*f \ge \eta\}$ is non-empty, then*

(a) *there exists y^*, $\|y^*\| \le 1$, such that $y^*\hat{f} \in \bigcap_{A,\varepsilon>0} \widehat{\overline{H(A, \varepsilon)}}$; and*

(b) *for each countable subset C of X, there exists $z^* \in X^*$ such that $z^* \mid C \equiv 0$ and $z^*\hat{f} = y^*\hat{f}$.*

Proof. (a) If $A \supset B$ and $\varepsilon < \varepsilon'$, then $H(A, \varepsilon) \subset H(B, \varepsilon')$. Thus $\{H(A, \varepsilon)\colon A \text{ finite in } X, \varepsilon > 0\}$ has the finite intersection property. Since f is bounded, each $\overline{H(A, \varepsilon)}$ is weakly compact in either L^1 or L^2.

Choose $g \in \bigcap_{A,\varepsilon>0} \overline{H(A, \varepsilon)}$. Fix A and ε. Choose a sequence (x_n^*) with $\|x_n^*\| \le 1$, $x_n^*f \to g$ a.e., and $|x_n^*(x)| \le \varepsilon \, \forall x \in A$. Let $y_{A,\varepsilon}^*$ be a weak* cluster point of (x_n^*). Then $y_{A,\varepsilon}^*f = g$ a.e., so $\int y_{A,\varepsilon}^*f = \int g \ge \eta$, and $y_{A,\varepsilon}^*f \in H(A, \varepsilon)$. Hence $\hat{g} = y_{A,\varepsilon}^*\hat{f} \in \widehat{\overline{H(A, \varepsilon)}}$, for any A and $\varepsilon > 0$. Let y^* be any particular $y_{A,\varepsilon}^*$, and (a) follows.

(b) Let $C = \bigcup_{n=1}^\infty C_n$, an increasing union of finite subsets of X. Then $g \in \bigcap_{n=1}^\infty \overline{H(C_n, 1/n)}$. For each n, choose $y_n^*f \in H(C_n, 1/n)$ so that $\|g - y_n^*f\| < 1/n$, where the norm is in L^1 or L^2. Let $(y_{n_k}^*)$ be a subsequence such that $y^*_{n_k} f \to g$ a.e. Let z^* be a weak* cluster point of $(y^*_{n_k})$. Then $z^*f = g = y^*f$ a.e., so $z^*\hat{f} = y^*\hat{f}$. But $|y^*_n(x)| < 1/n\ \forall x \in C_n$, so $z^* \mid C \equiv 0$.

THEOREM 3.2 (TALAGRAND). *Let $\tilde{X} = \{x^{**}$: x^{**} is in the weak*-closure of a countable subset of $X\}$. Then f is Pettis integrable iff $(\overline{\text{co}}\ \hat{f}(a)) \cap \tilde{X} \neq \varnothing$ for every non-empty clopen subset a of S.*

Proof. The Hahn-Banach Theorem shows that $1/\mu(A) \cdot x^{**}_A \in \overline{\text{co}}\ \hat{f}(a)$; hence integrability yields a non-empty intersection. Conversely, suppose f is not Pettis integrable over Ω. Then the Dunford integral x^{**}_Ω is not weak*-continuous on $\{x^*: \|x^*\| \leq 1\}$. Hence $\exists \eta > 0$ such that each $H(A, \varepsilon)$ in 3.1 is non-empty. Choosing y^* as in 3.1, $\int y^*f \geq \eta > 0$. Since $y^*\hat{f}$ is continuous, $\exists$ a non-empty clopen set a in S with $y^*\hat{f}(s) > \eta/2\ \forall s \in a$. Choose $\tilde{x} \in (\overline{\text{co}}\ \hat{f}(a)) \cap \tilde{X}$, so that $y^*(\tilde{x}) \geq \eta/2$. Let $C = \{x_n\}$ be a sequence in X, containing $\tilde{x}$ in its weak*-closure. Use 3.1(b) to choose z^*. Then $z^*(\tilde{x}) = 0$ (since $z^* \mid C \equiv 0$), and $z^*(\tilde{x}) \geq \eta/2$ (since $z^*\hat{f} = y^*\hat{f}$), a contradiction. A similar proof holds if f is not Pettis integrable over some measurable subset E of Ω.

In [8] Geitz defines the core of f over a measurable set A to be $\text{cor}_f(A) = \bigcap_{\mu(B)=0} \overline{\text{co}}\ f(A - B)$, the closure taken weakly in X. It is a routine (but useful) exercise to show that $\text{cor}_f(A) = (\overline{\text{co}}\ \hat{f}(A)) \cap X$. Thus 3.2 settles Geitz's conjecture that f is Pettis integrable iff $\text{cor}_f(A) \neq \varnothing$ for all A. We remark that

$$\overline{\text{co}}\hat{f}(a) = \overline{\text{co}}\{1/\mu(B)x^{**}_B : B \subset A, \mu(B) > 0\},$$

whether f is Pettis integrable or not.

The method of 3.2 also furnishes a natural setting for Geitz's description [9] of Pettis integrability in terms of simple functions. Note that, as in 3.2, the measure space need not be perfect.

COROLLARY 3.3 (TALAGRAND). *Let f: $\Omega \to X$ be bounded and weakly measurable. These are equivalent*: (a) *f is Pettis integrable, and $\{x^*f$: $\|x^*\| \leq 1\}$ is separable in $L^2(\Omega)$*; (b) *there is a sequence (f_n) of X-valued simple functions such that $x^*f_n \to x^*f$ a.e. for all $x^* \in X^*$.*

Proof. (a) $\Rightarrow$ (b): This follows easily from the Martingale Convergence Theorem. (b) $\Rightarrow$ (a): The separability is straightforward. Now repeat the argument of 3.2, with $C = \bigcup \text{range}(f_n)$. Then $z^*f_n = 0$ a.e. for all n, so $z^*f = 0$ a.e., by hypothesis. This is impossible, since $y^*f = z^*f$ a.e.

A little more can be added to 3.2 if (Ω, Σ, μ) is perfect.

PROPOSITION 3.4 **[27]**. *If* (Ω, Σ, μ) *is perfect, then* f *is Pettis integrable iff* $(\overline{\mathrm{co}}\, \hat{f}(a)) \cap X$ *is non-empty and separable for all clopen* a.

Proof. (necessity) The operator $T\colon X^* \to L^1(\mu)$ given by $T(x^*) = x^* \circ f$ is compact, using Fremlin's Theorem. As in the proof of Stegall's Theorem, the adjoint T^* is compact, and maps $L^\infty(\mu)$ into X via the formula $T^*(g) = (P)\int_\Omega gf\,d\mu$. Thus $\overline{T^*(L^\infty(\mu))} = T^{-1}(0)^\perp$ is a separable subspace of X. But $\overline{\mathrm{co}}\, \hat{f}(S) \cap X \subset T^{-1}(0)^\perp$: if $T(x^*) = x^*f = 0$ in $L^1(\mu)$, then $x^*\hat{f} = \widehat{x^*f} \equiv 0$, so that $x^* \mid \overline{\mathrm{co}}\, \hat{f}(S) \equiv 0$. Hence all $(\overline{\mathrm{co}}\, \hat{f}(a)) \cap X$ must be separable.

Theorem 3.2 is quite powerful, but not necessarily useful for computing the Pettis integral in specific cases. However, the sets $\overline{\mathrm{co}}\, \hat{f}(a) \cap X$ suggest a way to construct an approximating *sequence* (f_n) of X-valued simple functions for which $\int_\Omega |x^*f_n - x^*f|\, d\mu \to 0$, thus obtaining an explicit formula for $(P)\int f\,d\mu$ via 3.3. We now give the details of this procedure.

Note first that there are several formal, not very useful, ways of computing $\hat{f}$. Each member s of S is an ultrafilter of equivalence classes of measurable subsets of Ω. The notation $a \downarrow s$ symbolizes convergence of the corresponding family of clopen subsets of S which contain s. Then if $x^* \in X^*$, continuity of $\hat{f}$ yields:

$$x^*\hat{f}(s) = \lim_{a \downarrow s} \frac{1}{\mu(A)} \int_A x^*f\,d\mu \tag{3.1}$$

and

$$x^*\hat{f}(s) = \lim_{a\downarrow s}\ \operatorname*{ess\,sup}_{w \in A \in a} x^*f(w) = \lim_{a\downarrow s}\ \operatorname*{ess\,inf}_{w\in A\in a} x^*f(w). \tag{3.2}$$

In point of fact, the values of $\hat{f}$ in §2 were calculated using at most a *sequence* of a's, rather than the full complexity of the ultrafilter. We crystallize this notion as follows: Let $\mathcal{P} = (\pi_n)$ be a sequence of (finite, measurable) partitions of Ω such that π_{n+1} refines π_n for each n. There is a corresponding sequence (also denoted (π_n)) of clopen partitions of S. If $s \in S$, let $a_n(s)$ be the unique member of π_n which contains s, and let $A_n(s)$ be the corresponding member of the partition of Ω. The set $C(s) = \bigcap_{n=1}^\infty a_n(s)$ is a non-empty zero set of S, but $\bigcap_{n=1}^\infty A_n(s)$ may be empty.

Now given x^*, s, and a fixed sequence $\mathcal{P}$ of partitions, consider the conditions

$$\pi(s, x^*): x^*\hat{f}(s) = \lim_n \frac{1}{\mu(a_n(s))} \int_{a_n(s)} x^*\hat{f}\, d\mu$$

$$\left(= \lim_n \frac{1}{\mu(A_n(s))} \int_{A_n(s)} x^* f\, d\mu\right)$$

and

$$\pi^*(s, x^*): x^*\hat{f}(s) = \lim_n \operatorname*{ess\,sup}_{w \in A_n(s)} x^* f(w).$$

These limits (when they exist) are constant on any $C(s)$.

The limit in $\pi^*(s, x^*)$ always exists, whether or not it equals $x^*\hat{f}(s)$. The limit in $\pi(s, x^*)$ need not exist (Example 3.9). However, for fixed x^*, Stegall's Theorem, in conjunction with the usual conditional expectation operators, yields a sequence $\mathcal{P}$ such that $\pi(s, x^*)$ does hold a.e. (μ). The condition $\pi^*(s, -x^*)$ is simply $x^*\hat{f}(s) = \lim_n \operatorname{ess\,inf}_{w \in A_n(s)} x^* f(w)$, so that

$$\pi^*(s, x^*) \text{ and } \pi^*(s, -x^*) \text{ together imply } \pi(s, x^*).$$

The significance of these ideas lies in the following method of sequential approximation of f, $\hat{f}$, and $(P)\int f\, d\mu$.

THEOREM 3.5. *Let (Ω, Σ, μ) be a finite measure space, $f\colon \Omega \to X$ bounded and weakly measurable. Assume that* (1) *there is a sequence $\mathcal{P} = (\pi_n)$ of partitions such that $\pi^*(s, x^*)$ holds a.e. on S for any $x^* \in X^*$; and* (2) *each $\overline{\text{co}}\, \hat{f}(a)$ meets X. For each $a_{pn} \in \pi_n$, choose any $x_{pn} \in \overline{\text{co}}\, \hat{f}(a_{pn}) \cap X$, and let $f_n = \Sigma_p \chi(A_{pn}) \cdot x_{pn}$. Then $(P)\int_A f\, d\mu = \lim_n \int_A f_n\, d\mu$ (i.e., the limit in the weak topology of X).*

Proof. Fix x^*, and let $M = \{s \in S: \pi^*(s, x^*) \text{ or } \pi^*(s, -x^*) \text{ fails}\}$. We show that $x^*\hat{f}_n \to x^*\hat{f}$ pointwise on $S - M$. Since $\mu(M) = 0$, the Dominated Convergence Theorem then yields $x_A^{**} = w^*\text{-}\lim \int_A f_n$. Thus $1/\mu(A) \cdot x_A^{**} \in (\overline{\text{co}}\, \hat{f}(a)) \cap \tilde{X}$, and an application of 3.2 completes the proof.

Fix $s \in S - M$, $\varepsilon > 0$. Choose a positive integer n_0 such that $n \geq n_0$ gives both

$$\left| x^*\hat{f}(s) - \operatorname*{ess\,sup}_{w \in A_n(s)} x^* f(w) \right| \quad \text{and} \quad \left| x^*\hat{f}(s) - \operatorname*{ess\,inf}_{w \in A_n(s)} x^* f(w) \right|$$

less than $\varepsilon/3$. Fix n for the moment, $n \geq n_0$. Now $s \in a_n(s) = a_{pn}$ for some p. Since $x_{pn} \in \overline{\mathrm{co}}\, \hat{f}(a_{pn})$, there exist non-negative scalars $\alpha_1, \ldots, \alpha_q$, $\sum_{i=1}^q \alpha_i = 1$, and $s_1, \ldots, s_q \in a_{pn}$ so that $|x^*(x_{pn} - \sum_{i=1}^q \alpha_i \hat{f}(s_i))| < \varepsilon/3$. Thus

$$\begin{aligned}|x^*(\hat{f}_n(s) - \hat{f}(s))| &= |x^*(x_{pn} - \hat{f}(s))| \\ &\leq \left| x^*\left(x_{pn} - \sum_{i=1}^q \alpha_i \hat{f}(s_i) \right) \right| + \sum_{i=1}^q \alpha_i \cdot |x^*\hat{f}(s_i) - x^*\hat{f}(s)|.\end{aligned}$$

But

$$\operatorname*{ess\,inf}_{w \in A_n(s)} x^*f(w) \leq x^*\hat{f}(t) \leq \operatorname*{ess\,sup}_{w \in A_n(s)} x^*f(w),$$

where t is any of $s, s_1, \ldots, s_q$, and it follows that the final term is less than $2\varepsilon/3$. Thus $|x^*\hat{f}_n(s) - x^*\hat{f}(s)| < \varepsilon$ for all $n \geq n_0$, and the proof is complete.

Now we give conditions under which (1) of Theorem 3.5 is satisfied. Again let $\mathcal{P} = (\pi_n)$ be an increasing sequence of partitions of Ω.

LEMMA 3.6. *Let h: $\Omega \to R$ be bounded and measurable. Suppose $\hat{h} \mid C(s)$ is constant, for some $s \in S$. Then*

$$\hat{h}(t) = \lim_n \operatorname*{ess\,sup}_{w \in A_n(s)} h(w) = \lim_n \operatorname*{ess\,inf}_{w \in A_n(s)} h(w) \quad \textit{for all } t \in C(s).$$

Proof. Let L be the constant value of $\hat{h}$ on $C(s)$. Given $\varepsilon > 0$, choose a clopen neighborhood b of $C(s)$ so that $|\hat{h}(t) - L| < \varepsilon/2 \,\forall t \in b$. Now by compactness, there is an n_0 such that $C(s) \subset a_{n_0}(s) \subset b$. Then

$$\begin{aligned}\lim_n \operatorname*{ess\,sup}_{w \in A_n(s)} h(w) - \lim_n \operatorname*{ess\,inf}_{w \in A_n(s)} H(w) &\leq \operatorname*{ess\,sup}_{w \in A_{n_0}(s)} h(w) - \operatorname*{ess\,inf}_{w \in A_{n_0}(s)} h(w) \\ &= \sup_{t \in A_{n_0}(s)} \hat{h}(t) - \inf_{t \in A_{n_0}(s)} \hat{h}(t) < \varepsilon.\end{aligned}$$

Since L lies between the two limits, the result follows.

Now if $h = x^*\hat{f}$, then the conclusion of 3.6 tells us that $\pi^*(s, x^*)$ holds "locally uniformly": i.e., given $\varepsilon > 0$, there is an integer n_0 such that $|x^*\hat{f}(t) - \operatorname{ess\,sup}_{w \in A_{n_0}(t)} x^*f(w)| < \varepsilon$ for every $t \in a_{n_0}(s)$. This is a strong condition, and does not hold for every $\mathcal{P}$ and f: it is pointed out in 3.9 that constancy of $\hat{f}$ on sets $C(s)$, and local uniformity of convergence, both fail in Example 2.3. Example 2.1 satisfies both constancy and local uniformity of convergence (see Theorem 3.8).

PROPOSITION 3.7. *Let f: $\Omega \to X$ be a bounded weakly measurable function. If $\pi(s, x^*)$ holds for every $s \in S$ and $x^* \in X^*$, then so does $\pi^*(s, x^*)$.*

Proof. If $t \in C(s)$, then $a_n(t) = a_n(s)$ for every n. Hence the limit on the right side of $\pi(s, x^*)$, if it exists, is the same for all $t \in C(s)$. Apply Lemma 3.6, with $h = x^*f$, to complete the proof.

Thus the validity of $\pi(s, x^*)$ for all s and x^* insures that the sequential approximation of $(P)\int f\,d\mu$ given in Theorem 3.5 is available. Since the condition $\pi(s, x^*)$ arises naturally in connection with the usual conditional expectation operators E_{π_n}: $L^1(\mu) \to L^1(\mu)$, it is sometimes easier to verify than $\pi^*(s, x^*)$.

THEOREM 3.8. *Let (Ω, Σ, μ) be a finite measure space, f: $\Omega \to X$ a bounded weakly measurable function. Let $P = \hat{f}(s)^{\perp} \subset X^*$. If X^*/P is weakly compactly generated, then there is a sequence of partitions (π_n) such that $\pi(s, x^*)$ and $\pi^*(s, x^*)$ hold for all s and x^*.*

Proof. Let H be a weakly compact subset of X^*/P with dense linear span. Now $\hat{f}(S) \subset (X^*/P)^* = P^{\perp} \subset X^{**}$; it follows from Eberlein's Theorem and the Lebesgue Convergence Theorem that $\{x^*\hat{f}\colon x^* + P \in H\}$ is a norm-compact subset of $L^1(S, \mu)$. Let $\{x_n^*\hat{f}\}$ be a countable dense subset (for the L^1-norm), and form a sequence of partitions (π_n) such that $x_1^*\hat{f}, \ldots, x_n^*\hat{f}$ have oscillation less than $1/n$ on each clopen set in π_n. Then each $x_n^*\hat{f}$ is trivially constant on each $C(s)$.

Now let $x^* + P$ be an arbitrary member of H. Choose a subsequence $(x_{n_k}^*)$ such that $x_{n_k}^*\hat{f} \to x^*\hat{f}$ pointwise on S, using Eberlein's Theorem again. Then $x^*\hat{f}$ is constant on each $C(s)$. Since the linear span of H is norm-dense in X^*/P, $\widehat{x^*f} \mid C(s)$ is constant for every $x^* \in X^*$ and $s \in S$. Now Lemma 3.6 yields $\pi^*(s, x^*)$ for all s and x^*, and so $\pi(s, x^*)$ also holds for all s and x^*.

EXAMPLE 3.9. Consider the usual sequence of dyadic partitions of $[0, 1]$: $\pi_1 = \{[0, 1]\}$, $\pi_2 = \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}, \ldots$. In Examples 2.1 and 2.2, $\pi(s, x^*)$ and $\pi^*(s, x^*)$ do hold for all s and x^*. This may be verified directly, or by using Theorem 3.8 (X^*/P is one-dimensional in 2.2, and separable in 2.1).

The situation in Example 2.3 is more complicated (here X^*/P is not WCG). If $\lambda \in M(K) = X^*$ is purely non-atomic, or if the atomic part includes only dyadic points, then $\pi(s, \lambda)$ and $\pi^*(s, \lambda)$ hold for every s. If

$\lambda = \delta(t_0)$, where t_0 is not a dyadic point, then the limit in $\pi(s, \lambda)$ fails to exist for every s such that $\bigcap \mathfrak{F}_s = \{t_0\}$ (there may or may not be a subsequence of the sequence in $\pi(s, \lambda)$ which converges to $\lambda \hat{f}(s)$). The limit in $\pi^*(s, \lambda)$ exists for all s; however, if s is a "left" ultrafilter at t_0, the limit is not equal to $\lambda \hat{f}(s)$. For any $\lambda \in M(K)$, $\pi(s, \lambda)$ and $\pi^*(s, \lambda)$ hold except for at most a countable union of nowhere dense zero-sets of S. Thus condition (1) of Theorem 3.5 is satisfied, so that the dyadic partition can be used to calculate $(P)\int f\, d\mu$.

We do not know if (1) of Theorem 3.5 is valid for arbitrary f. Even for real-valued f, the natural (dyadic) partitions of $[0, 1]$ need not yield $\pi^*(s, x^*)$ a.e. Consider, for example, the characteristic function of A, where A is nowhere dense in $[0, 1]$, and meets each dyadic interval in a set of positive measure.

4. Integrability of weakly measurable functions, and subspaces of X^{**}. In this section we study the compactness of $H = \{x^*\hat{f}: \|x^*\| \leq 1\}$ in $(L^1(S, \mu), \|\ \ \|_1)$ and $(C(S), \sigma(C(S), M(S)))$. Since $(S, \text{Borel}(S), \mu)$ is a perfect measure space, Fremlin's Theorem implies that the first of these always holds. However, the second requirement is to strong: we show it forces non-integrability for purely weakly measurable functions.

The section also brings out the significance of two subspaces of X^{**}, νX and $Ba^1(X)$, for Pettis integrability. Theorems 4.3, 4.5, and 4.6 make this precise.

THEOREM 4.1. *If f is purely weakly measurable, and essentially non-zero, and H is $\sigma(C(S), M(S))$-compact, then f cannot be Pettis integrable.*

Proof. Define T: $X^* \to C(S)$ by $T(x^*) = x^*\hat{f}$. Then T is weakly compact, so T^{**} maps X^{***} into $C(S)$. We claim that $T^{**}(x^{***}) = x^{***}\hat{f}$ as functions on S, for any x^{***} in the third dual of X.

Let $s \in S$. Then $\langle \delta(s), T^{**}(x^{***})\rangle = \langle x^{***}, T^*(\delta(s))\rangle$. But $T^*(\delta(s)) = \hat{f}(s)$, for if $x^* \in X^*$, then $\langle T^*(\delta(s)), x^*\rangle = \langle \delta(s), T(x^*)\rangle = x^*\hat{f}(s) = \langle \hat{f}(s), x^*\rangle$. Thus $\langle \delta(s), T^{**}(x^{***})\rangle = \langle x^{***}, \hat{f}(s)\rangle = x^{***}\hat{f}(s)$, as desired.

Now if $s_\alpha \to s_0$ in S, then $\langle \delta(s_\alpha), T^{**}(x^{***})\rangle \to \langle \delta(s_0), T^{**}(x^{***})\rangle$, and so $x^{***}\hat{f}(s_\alpha) \to x^{***}\hat{f}(s_0)$. This says that $\hat{f}$ is a continuous map from S into (X^{**}, weak).

The remark following Theorem 1.1 shows that $\hat{f}$ is norm continuous on $S - M$, where M is closed and nowhere dense in S. Choose $s_0 \in S - M$ such that $\hat{f}(s_0) \in X^{**} - X$. Let B be a norm-closed ball in X^{**}, centered at $\hat{f}(s_0)$ and missing X. By continuity there is a clopen subset a of S, $s_0 \in a$, such that $\hat{f}(a) \subset B$. Now the weak closed convex hull of $\hat{f}(a)$ is weakly compact, hence weak*-compact, and so $\overline{\text{co}}\, \hat{f}(a) \subset B$. But then $1/\mu(A) \cdot x_A^{**} \in B \subset X^{**} \setminus X$, and so f is not Pettis integrable over A.

The space X^{***} plays a special role in §5. The point here is that continuity of $x^{***}\hat{f}$ is much stronger than the (automatic) continuity of $x^*\hat{f}$.

COROLLARY 4.2. *If f is purely weakly measurable, essentially non-zero, and Pettis integrable, then $\hat{f}(a)$ cannot be $\sigma(X^{**}, X^{***})$-compact for any non-empty clopen a.*

Proof. If $\hat{f}(a)$ is weakly compact, then the weak and weak* topologies on $\hat{f}(a)$ coincide, so $\hat{f}$ is weakly continuous on a. Then the argument of 4.1 goes through.

Note that for Examples 2.1 and 2.3, which are Pettis integrable with $\hat{f}(S) \subset X^{**} \setminus X$, each $\hat{f}(a)$ is weakly closed and discrete. Example 2.2 is a special case of the next result.

Corson [**2**] showed that the Hewitt realcompactification νX of (X, weak) is $\{x^{**} \in X^{**}$: for each countable subset C of X^*, $\exists x \in X$ such that $x^{**} \mid C = x \mid C\}$, with the relative weak*-topology. See [**29**, Th. 5.2] for an alternate proof.

THEOREM 4.3. *If $\hat{f}(S) \subset \nu X - X$, then f cannot be Pettis integrable.*

Proof. We show that $H = \{x^*\hat{f}\colon \|x^*\| \le 1\}$ is $\sigma(C(S), M(S))$-compact, and apply Theorem 4.1. Since H is uniformly bounded, it is enough [**12**] to show that every sequence $(x_n^*\hat{f})$ in H has a cluster point for the topology of pointwise convergence on S. Let x_0^* be a weak* cluster point of (x_n^*). Let $s_1, \ldots, s_n \in S$. For each i, choose $x_i \in X$ such that $\hat{f}(s_i)$ and x_i agree on x_0^* and (x_n^*). It follows easily that $(x_n^*\hat{f})$ clusters pointwise to $x_0^*\hat{f}$.

It suffices in the above to have $\hat{f}(S - M) \subset \nu X$, or even $\hat{f}(a - M) \subset \nu X$, where a is clopen and M is nowhere dense. Note further that H is pointwise compact, convex, and satisfies (by virtue of continuity of $x^*\hat{f}$) the separation property of A. Bellow [**1**]. Thus the pointwise (on S), $\sigma(C(S), M(S))$ and L^1-norm topologies must agree on H. On the other hand, the argument in 4.1 tells us that $\hat{f}$ is continuous from S to (X^{**}, weak). Theorem 1.1 then says that for any clopen a, there is a nowhere dense set M such that the norm, weak and weak*-topologies agree on $\hat{f}(a - M) \subset X^{**}$. Finally, we observe that by virtue of [**1**, Theorem 3], $\{x^*f\colon \|x^*\| \le 1\}$ fails the separation property on every set of positive measure when $\hat{f}(S) \subset X^{**}\setminus X$.

Theorem 4.3 implies that if $\hat{f}(S - M) \subset \nu X\setminus X$, where M is nowhere dense, then $\{x^*f\colon \|x^*\| \le 1\}$ is norm-compact in $L^1(\Omega, \mu)$. Moreover, using 1.1(b), or the existence of a Bochner measurable $g\colon \Omega \to X^{**}$ which is weakly equivalent to $\hat{f}$, there is an increasing sequence $(B_n) \subset \Sigma$, $\mu(B_n) \to \mu(\Omega)$, such that $\{(x^*f)\cdot\chi_{B_n}\colon \|x^*\| \le 1\}$ is $\|\ \|_\infty$-compact in L^∞ for all n.

These ideas, along with the decomposition theorem (1.3), lead to a proof that if $\nu X = X^{**}$, then X has the Radon-Nikodym Property. However this is true by more direct reasoning: if $\nu X = X^{**}$, then X is reflexive. The argument here was kindly provided by W. B. Johnson and G. Edgar. If X^* is non-reflexive, a well-known result of R. C. James yields an x^{**} which does not attain its norm on the ball of X^*. Choose a sequence (x_n^*) in B_{X^*} with $x^{**}(x_n^*) \to \|x^{**}\|$, and let x_0^* be a weak* cluster point of (x_n^*). Then no member of X can match x^{**} on $\{x_n^*\} \cup \{x_0^*\}$, so $x^{**} \notin \nu X$.

Now we turn to the space $Ba^1(X) = \{x^{**}\colon \exists$ a sequence (x_n) in X, weak*-convergent to $x^{**}\}$. Odell and Rosenthal [**17**] give many interesting results about $Ba^1(X)$. Geitz and Uhl [**10**] show that if Ω is a compact T_2 space, μ is a regular Borel measure, and $f\colon \Omega \to X$ is a bounded function such that x^*f is a Baire class 1 function on Ω for each $x^* \in X^*$, then f is Pettis integrable.

We note first that $Ba^1(X)$ and νX have only X in common.

LEMMA 4.4. *Let T be a countably compact space. Let $f\colon T \to R$ satisfy: (1) for each countable subset A of T, $\exists g \in C(T)$ such that $g\,|\,A = f\,|\,A$; and (2) f is Baire class 1 on T. Then $f \in C(T)$.*

Proof. Suppose f is not continuous at t_0. Then $\exists \varepsilon > 0$ and $B \subset T$ such that $t_0 \in \bar{B}$ and $|f(t_0) - f(t)| \ge \varepsilon$ $\forall t \in B$. Choose a sequence (f_n) in $C(T)$ which converges pointwise to f. Now $Z = \{t \in T\colon f_n(t) = f_n(t_0)\ \forall n\}$ is a zero-set of T, and $f\,|\,Z \equiv f(t_0)$. Hence $Z \cap B = \emptyset$, and so $B \subset U = T - Z$. Note that U is a cozero set, but not clopen, since $t_0 \in \bar{B} \subset \bar{U}$. Let $U = \bigcup_1^\infty F_n$, where each F_n is closed, and $F_n \subsetneq \operatorname{int} F_{n+1}\ \forall n$. Then B meets infinitely many $F_n\setminus F_{n-1}$, since $t_0 \in \bar{B}$. Find a sequence $t_{n_k} \in B \cap (F_{n_k}\setminus F_{n_{k-1}})\ \forall k$. Since T is countably compact, (t_{n_k}) clusters to some point w, necessarily in $T - U = Z$, and $|f(t_{n_k}) - f(w)| = |f(t_{n_k}) - f(t_0)| \ge \varepsilon\ \forall k$. Thus $f|\,\{t_{n_k}\} \cup \{w\}$ is not continuous, a contradiction. Hence $f \in C(T)$.

THEOREM 4.5. *If X is a Banach space, then $Ba^1(X) \cap \nu X = X$.*

Proof. Let $T = (\mathrm{Ball}(X^*), \text{weak}^*)$. If $\phi \in Ba^1(X) \cap \nu X$, then $\phi \mid T$ satisfies (1) and (2) of Lemma 4.5. Thus ϕ is weak*-continuous on $\mathrm{Ball}(X^*)$, and so ϕ is weak*-continuous on X^*, i.e., $\phi \in X$.

THEOREM 4.6. *If $\hat{f}(S) \subset Ba^1(X)$, then f is Pettis integrable.*

Proof. This follows immediately from 3.2.

Examples 2.1 and 2.3 are special cases of this result.

EXAMPLE 4.7. A Pettis integrable function f such that $\hat{f}(S) \not\subset Ba^1(X)$. Let JH denote the James Hagler space [**13**, p. 215], and let $X = JH^*$. Then X is weakly sequentially complete, so $Ba^1(X) = X$. Also X has the weak RNP (since $JH \not\supset l^1$), but fails the RNP. Hence some Pettis integrable function g into X is not weakly equivalent to any strongly measurable function. Let f be an essentially non-zero, purely weakly measurable part of g. Then f is Pettis integrable, and $\hat{f}(s) \in X^{**} \setminus Ba^1(X)$ on a set of positive measure.

5. Other descriptions of Pettis integrability. In this section we use our results on $\hat{f}$ to derive a condition for Pettis integrability expressed in terms of f and Ω alone. It is convenient to use the notion of polar developed in Schaefer [**23**]: if $\langle F, G\rangle$ is a duality, and $H \subset F$, then $H^{\circ} = \{y \in G\colon \langle x, y\rangle \leq 1\, \forall x \in H\}$. Let $^{\cdot}$ and $^{\circ}$ denote polarity with respect to $\langle X, X^*\rangle$ and $\langle X^*, X^{**}\rangle$.

LEMMA 5.1. *Let $f\colon \Omega \to X$ be bounded and weakly measurable. If a is a non-empty clopen subset of S, then $\hat{f}(a)$ and $\{1/\mu(B)x_B^{**}\colon B \subset A, \mu B > 0\}$ have the same polar in X^*.*

Proof. If $x^*\hat{f}(s) \leq 1\, \forall s \in a$, then

$$x^*\left(\frac{1}{\mu B}x_B^{**}\right) = \frac{1}{\mu B}\int_B x^* f\, d\mu$$
$$= \frac{1}{\mu(b)}\int_b x^*\hat{f}\, d\mu \leq 1.$$

Conversely, if $x^*((1/\mu B)x_B^{**}) \leq 1$ for all $B \subset A$, then for a fixed

$$s \in a, x^*\hat{f}(s) = \lim_{b \downarrow s, b \subset a} \frac{1}{\mu(b)} \int_b x^*\hat{f}\, d\mu$$

$$= \lim_{b \downarrow s, b \subset a} \frac{1}{\mu B} \int_B x^* f\, d\mu \leq 1.$$

Thus $x^* \in \hat{f}(A)^{\circ}$.

THEOREM 5.2. *Let f be bounded and weakly measurable. These are equivalent*: (a) *f is Pettis integrable*; (b) *$\hat{f}(a)^{\circ}$ is w^*-closed in X^* for all clopen a*; (c) *$\hat{f}(a)^{\perp}$ is w^*-closed in X^* for all clopen a.*

Proof. Clearly (a) implies (b) and (c), using 5.1 and the obvious adaptation for $\hat{f}(a)^{\perp}$. (b) $\Rightarrow$ (a): assume that $C = \hat{f}(a)^{\circ}$ is weak*-closed in X^*. Let $D = C^{\cdot}$. Then $D = C^{\circ} \cap X = \hat{f}(a)^{\circ\circ} \cap X = \overline{\text{co}}(\hat{f}(a) \cup \{0\}) \cap X$, by the Bipolar Theorem [23]; here $\overline{\text{co}}$ denotes the $\sigma(X^{**}, X^*)$-closed convex hull. Since $\hat{f}(a)$ is $\sigma(X^{**}, X^*)$-compact, $\overline{\text{co}}(\hat{f}(a) \cup \{0\}) = \{\lambda x^{**}: 0 \leq \lambda \leq 1, x^{**} \in \overline{\text{co}}\, \hat{f}(a)\}$.

If $D = \{0\}$, then $X^* = D^{\cdot} = C^{\cdot\cdot} = C$, since C is weak* closed, and so $\hat{f}(a) \subset \hat{f}(a)^{\circ\circ} = C^{\circ} = \{0\}$. Thus $\overline{\text{co}}\, \hat{f}(a)$ meets X in this case. If D properly contains 0, then $\{\lambda x^{**}: 0 < \lambda \leq 1, x^{**} \in \overline{\text{co}}\, \hat{f}(a)\}$ meets X, and so $\overline{\text{co}}\, \hat{f}(a)$ meets X. Now an appeal to Theorem 3.2 suffices.

To obtain (c) $\Rightarrow$ (a), assume non-integrability, and choose y^* as in 3.1(a). Then $T = (y^* + \hat{f}(S)^{\perp}) \cap \{x^*: \|x^*\| \leq 1\}$ is w^*-closed in X^*. Using 3.1(b), $z^* - y^* \in \hat{f}(S)^{\perp}$, so $z^* \in T$. Note that z^* can be chosen so as to vanish on any prescribed countable set in X. Since T is w^*-closed, this implies that $0 \in T$. This is a contradiction, since $y^*\hat{f}$ is not identically zero (3.1(a)).

We thank M. Talagrand for pointing out the proof of (c) $\Rightarrow$ (a) to us.
Theorem 5.2 now yields an (Ω, f, X) characterization of integrability which is also integral free!

THEOREM 5.3. *Let f be bounded and weakly measurable. These are equivalent*: (a) *f is Pettis integrable*; (b) *if a bounded net (x_α^*) is w^*-convergent to x^*, and $x_\alpha^* f \leq 1$ a.e. on $A \in \Sigma$ for each α, then $x^* f \leq 1$ a.e. on A*; (c) *the same as* (b), *with equality replacing inequality and* 0 *replacing* 1.

Proof. It is apparent that if f is Pettis integrable, both (b) and (c) hold. Conversely, (b) and (c) of 5.3 imply (b) and (c) of 5.2, using the Krein-Smulian Theorem. Hence the proof.

The theorem remains valid if "bounded net" is replaced by "net".

Theorem 5.3 implies a sufficient condition for Pettis integrability involving X^{***}. If x^{***} belongs to the third dual, then $x^{***}f$ is a measurable function on Ω, so that $\widehat{x^{***}f} \in C(S)$. On the other hand, $x^{***}\hat{f}$ is a bounded real-valued function on S, but it is not obvious that it is even μ-measurable. Also, if $\hat{f}(s_0) \in X^{**} \setminus X$, and one chooses x^{***} with value 1 at $\hat{f}(s_0)$ and vanishing on X, then $x^{***}\hat{f}(s_0) \neq \widehat{x^{***}f}(s_0) = 0$. In Example 2.2, $x^{***}\hat{f} \equiv 1$, $\widehat{x^{***}f} \equiv 0$ can occur for suitable x^{***}.

THEOREM 5.4. *The following are equivalent*: (1) *f is weakly equivalent to a strongly measurable function*; (2) *there is a subset M of S, $\mu(M) = 0$, such that $x^{***}\hat{f}(s) = \widehat{x^{***}f}(s)$ for all x^{***} and all $s \in S - M$.*

Proof. (1) $\Rightarrow$ (2): By Theorem 1.1, there is a subset M of S, $\mu(M) = 0$, such that $\hat{f}(S - M) \subset X$. Fix x^{***} and $s \in S - M$. Let $x^* = x^{***} \mid X$. Then $x^{***}\hat{f}(s) = x^*\hat{f}(s) = \widehat{x^*f}(s) = \widehat{x^{***}f}(s)$. (In particular, $x^{***}\hat{f}$ is a μ-measurable function on S.)

(2) $\Rightarrow$ (1): X^{***} is the direct sum of X^* and $X^{\perp}$. If $x^{***} \in X^{\perp}$, then $x^{***}f(w) = 0\, \forall w \in \Omega$, so $\widehat{x^{***}f}(s) = 0\, \forall s \in S$. Then $x^{***}(\hat{f}(s)) = 0\, \forall s \in S - M$, and so $\hat{f}(S - M) \subset X^{\perp\perp} = X$. Now (1) follows from Theorem 1.1.

If one allows the exceptional set M to vary with x^{***}, as is appropriate for weak measurability, then, even without assuming that (Ω, Σ, μ) is perfect, one has:

PROPOSITION 5.5. *Each of the following implies the next*:

(a) $x^{***}\hat{f} = \widehat{x^{***}f}$ *on* $S - M_{x^{***}}$, $\mu(M_{x^{***}}) = 0$.

(b) $x^{***}\hat{f}$ *is μ-measurable on S for all* x^{***}.

(c) $\{x^{***}\hat{f}\colon \|x^{***}\| \leq 1\}$ *is* $\| \ \|_1$-*compact in* $L^1(S, \mu)$ (*hence* $\{x^*f\colon \|x^*\| \leq 1\}$ *is* $\| \ \|_1$-*compact in* $L^1(\Omega, \mu)$).

Proof. (a) $\Rightarrow$ (b) is clear. Since μ is a Radon measure on S, hence perfect, Fremlin's Theorem shows that (b) $\Rightarrow$ (c).

Note that in Example 2.2, where $\hat{f}$ is constant on S, both (b) and (c) hold, but (a) does not (see the remark preceding 5.4). However, one can prove the following.

THEOREM 5.6. *The following are equivalent*: (a) *for each* x^{***}, $x^{***}\hat{f} = \widehat{x^{***}f}$ *a.e. on S*; (b) $f\colon \Omega \to X$ *and* $\hat{f}\colon S \to X^{**}$ *are both Pettis integrable. In this case,* $(P)\int_A f\, d\mu = (P)\int_a \hat{f}\, d\mu$ *for each* $A \in \Sigma$.

Proof. (a) ⇒ (b): First we appeal to 5.3 to obtain integrability of f. Let (x_α^*) be a bounded net in X^*. If $x_\alpha^* f \leq 1$ a.e. on Ω and (x_α^*) is $\sigma(X^*, X)$ convergent to x^*, then each $x_\alpha^* \hat{f} \leq 1$ on S. Let x^{***} be a $\sigma(X^{***}, X^{**})$-cluster point of (x_α^*). Then $x^{***}\hat{f} \leq 1$ on S, and so $\widehat{x^{***}f} \leq 1$ a.e. on S, hence everywhere on S by continuity. Now $x^{***} \mid X = x^*$, so we get $x^* f \leq 1$ a.e. on Ω. Thus f is integrable, by 5.3. Since

$$\int_a x^{***}\hat{f}\, d\mu = \int_a \widehat{x^{***}f}\, d\mu = \int_A x^{***} f\, d\mu = \int_A x^* f\, d\mu$$

it now follows that $\hat{f}$ is Pettis integrable, and the integrals of f and $\hat{f}$ coincide.

(b) ⇒ (a): If both f and $\hat{f}$ are integrable (implying that $\hat{f}$ is X^{***}-measurable into X^{**}), then the integrals must coincide. If not, choosing an x^* which separates $(P)\int_A f\, d\mu$ and $(P)\int_a \hat{f}\, d\mu$ quickly leads to a contradiction. Thus, fixing x^{***} and letting a be clopen in S,

$$\int_a x^{***}\hat{f} = x^{***}\left(\int_a \hat{f}\right) = x^{***}\left(\int_A f\right) = \int_A x^{***} f = \int_a \widehat{x^{***}f},$$

whence $x^{***}\hat{f} = \widehat{x^{***}f}$ a.e. on S.

Fremlin and Talagrand [7] give an example of a Pettis integrable f: $\Omega \to l^\infty$ such that the range of the integral is not norm relatively compact. It can be shown that there is $x^{***} \in (l^\infty)^{***}$ with $x^{***}\hat{f}$ not measurable. Hence integrability of f alone is weaker than 5.6(b).

The next result characterizes integrability of f alone. Consider T: $X^* \to L^1(\Omega)$, $T(x^*) = x^* f$, and $\hat{T}$: $X^* \to L^1(S)$, $\hat{T}(x^*) = x^*\hat{f}$. Thus $\widehat{T(x^*)} = \hat{T}(x^*)$. Also, $\langle \hat{T}^*(\chi_a), x^* \rangle = \int_a x^*\hat{f} = \int_A x^* f = \langle T^*(\chi_A), x^* \rangle$ for all x^*. Thus $\hat{T}^*(\chi_a) = T^*(\chi_A) = x_A^{**}$, the Dunford integral of f over A.

Let $X^\perp = \{x^{***}: x^{***} \mid X \equiv 0\}$, and recall that $X^{***} = X^* \oplus X^\perp$.

PROPOSITION 5.7. *The following are equivalent*:

(a) *f is Pettis integrable*; (b) $T^{**}(X^\perp) = 0$; (c) $\hat{T}^{**}(x^{***}) = \widehat{x^{***}f}$ *a.e. on S.*

Proof. (a) ⇒ (b): T is a weakly compact operator [**5**, 4.1], so the range of T^{**} is in $L^1(\Omega)$. Then if $A \in \Sigma$ and $x^{***} \in X^\perp$, $\langle T^{**}(x^{***}), \chi_A \rangle = \langle x^{***}, T^*(\chi_A) \rangle = \langle x^{***}, (P)\int_A f\, d\mu \rangle = 0$.

(b) ⇒ (c): If $x^{***} \in X^\perp$, then $\langle \hat{T}^{**}(x^{***}), \chi_a \rangle = \langle x^{***}, \hat{T}^*(\chi_a) \rangle = \langle x^{***}, T^*(\chi_A) \rangle = \langle T^{**}(x^{***}), \chi_A \rangle = \langle 0, \chi_A \rangle = 0$. Now if x^{***} is an arbitrary member of the third dual, let $x^{***} = x^* + x^\perp$ be the canonical decomposition. Then $\hat{T}^{**}(x^{***}) = \hat{T}^{**}(x^*)$, by the preceding. Hence

$\langle \hat{T}^{**}(x^{***}), \chi_a \rangle = \langle x^*, \hat{T}^*(\chi_a) \rangle = \langle \hat{T}(x^*), \chi_a \rangle = \langle \widehat{x^*f}, \chi_a \rangle = \langle \widehat{x^{***}f}, \chi_a \rangle$ for all clopen a, so (c) holds.

(c) $\Rightarrow$ (a): If $x_A^{**} \notin X$, choose $x^{***} \in X^\perp$, $x^{***}(x_A^{**}) = 1$. Then $\hat{T}^{**}(x^{***}) = \widehat{x^{***}f} = 0$ a.e. on S. Hence, $1 = \langle x^{***}, x_A^{**} \rangle = \langle x^{***}, \hat{T}^*(\chi_a) \rangle = \langle \hat{T}^{**}(x^{***}), \chi_a \rangle = 0$ This contradiction completes the proof.

EXAMPLE 5.8. Suppose (Ω, Σ, μ) is a $\{0,1\}$-valued measure space. Then the measure algebra $\Sigma/\mu^{-1}(0) = \{[\varnothing], [\Omega]\}$, and the Stone space S consists of a single point $s_0 = \{[\Omega]\}$. A scalar function on Ω is measurable iff it is constant a.e. A bounded function $f\colon \Omega \to X$ is strongly measurable iff it is constant a.e., and weakly measurable iff $x^* \circ f$ has a constant value $c(x^*)$ a.e., for each $x^* \in X^*$. The range of the Dunford integral contains (at most) 2 points, x_Ω^{**} and 0. Also, $\hat{f}(s_0)(x^*) = \widehat{x^*f}(s_0) = \operatorname{ess\,sup}_{w \in \Omega} x^*f(w) = c(x^*) = \int_\Omega x^*f\,d\mu = x_\Omega^{**}(x^*)$. Thus $\hat{f}(S)$ is the singleton $\{x_\Omega^{**}\}$.

THEOREM 5.9. *Let $f\colon \Omega \to X$ be a bounded weakly measurable function on a $\{0,1\}$-measure space. Then the following conditions are equivalent:*

(a) *f is weakly equivalent to a strongly measurable function;*

(b) *$x^{***}\hat{f}(s) = \widehat{x^{***}f}(s)$ a.e. (μ) for each x^{***};*

(c) *f is Pettis integrable;* (d) *$\hat{f}(S)^\perp$ is weak*-closed in X^*.*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) hold for any finite measure space, by the preceding results. (d) $\Rightarrow$ (a): $x_\Omega^{**\,-1}(0)$ is weak*-closed, so x_Ω^{**} is weak*-continuous on X^*. Thus f is weakly equivalent to a constant function.

Note that $x^{***} \circ \hat{f}$ is always measurable on S in this setting, whether or not it agrees with $\widehat{x^{***}f}$.

6. Open questions. 1. What is a characterization of pure weak measurability in terms of f and Ω alone?

Here is a measure theoretic description: if $\lambda = \mu f^{-1}$ is the Baire measure on (X, weak), induced by f, then $\lambda^*(K) = 0$ for every norm-compact subset K of X with $0 \notin K$.

2. Let $f\colon \Omega \to X$ be a bounded weakly measurable function. Does there always exist an increasing sequence (π_n) of partitions of Ω such that $\pi^*(S, x^*)$ holds a.e. on S for each x^*?

3. Which functions $\eta\colon S \to X^{**}$ are Stonian transforms of bounded weakly measurable functions $f\colon \Omega \to X$? When is η the transform of a

Pettis integrable function? If f: $\Omega \to X$ is Pettis integrable, but not weakly equivalent to any strongly measurable function, let Z be the weak*-closed linear span of $\hat{f}(S)$ in X^{**}, and let $Y = Z \cap X$. Then $Y^* = X^*/\hat{f}(S)^{\perp}$, and $Y^{**} = Z$. Then [**1**, Theorem 3] shows that η: $S \to Y^{**}$ defined by $\eta(s)([x^*]) = \langle \hat{f}(s), x^* \rangle$ cannot be the Stonian transform of any weakly measurable g: $\Omega \to Y$.

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