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L^p-BOUNDEDNESS OF THE MULTIPLE HILBERT TRANSFORM ALONG A SURFACE

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For an appropriate surface σ in R^n, we prove that the multiple Hilbert transform along σ is a bounded operator on $L^p(R^n)$, for p sufficiently close to 2. Our analysis of this singular integral operator proceeds via Fourier transform techniques—that is, on the "multiplier side"—with applications of Stein's analytic interpolation theorem and the Marcinkiewicz multiplier theorem. At the heart of our argument we have estimates of certain trigonometric integrals.

CONTENTS

I. Introduction. The present work continues that of Fabes, Nagel, Rivière, Stein, and Wainger on singular integral operators associated with curves or surfaces in R^n. For an appropriate curve γ: $R \to R^n$ we define the Hilbert transform H along γ by the principal value integral $Hf(x) = \text{p.v.}\int_{-\infty}^{\infty} f(x-\gamma(t))\,dt/t$, $x \in R^n$, $f \in C_c^\infty(R^n)$. In the papers [**1**] of Fabes; [**11**] and [**12**] of Stein and Wainger; [**2**] and [**3**] of Nagel, Rivière, and Wainger; and [**5**] of Nagel and Wainger, it has been shown that for a variety of curves γ, the operator H is bounded on $L^2(R^n)$, or on $L^p(R^n)$ for some or all p in the range $1 < p < \infty$; on the other hand, there are C^∞ curves γ for which H fails to be bounded even on $L^2(R^n)$.

Nagel and Wainger [**6**] have introduced the multiple Hilbert transform along σ, defined for $f \in C_c^\infty(R^n)$ and $x \in R^n$ by

$$(1)\quad Tf(x) = \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} T_{\varepsilon,N}f(x) = \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int\!\cdots\!\int_{\substack{\varepsilon<|t_i|<N\\ (1\leq i\leq k)}} f(x-\sigma(t))\frac{dt_1}{t_1}\cdots\frac{dt_k}{t_k}.$$

Here, σ is the k-surface in R^n given for $t = (t_1,\ldots,t_k) \in R^k$ by $\sigma(t) = (t_1,\ldots,t_k, \gamma_1(t),\ldots,\gamma_l(t))$ where $n = k + l$ and $\gamma_j(t) = \prod_{i=1}^k |t_i|^{\alpha_{i,j}}$. Nagel and Wainger showed that T is bounded on $L^2(R^n)$ if the exponents $\alpha_{i,j}$ are appropriately restricted. Our proof that T is bounded on $L^p(R^n)$ for p

sufficiently close to 2 proceeds under somewhat more stringent conditions on the exponents.

What is the interest in the operators H and T? They occur in the study of certain singular convolution operators $Kf = \mathcal{K} * f$. If the kernel $\mathcal{K}$ is odd and satisfies a one-parameter homogeneity condition—the simplest being $\mathcal{K}(tx) = t^{-n}\mathcal{K}(x)$ ($x \in R^n$, $t > 0$)—then H arises when one decomposes K by an appropriate variant of the Calderón-Zygmund "method of rotations", and one sees that L^p inequalities for H imply the same for K. In **[6]**, Nagel and Wainger impose a *multiple-parameter* homogeneity condition upon $\mathcal{K}$ and are led to T via the method of rotations. Again, bounds on T imply bounds on K. Moreover, in this case the kernel $\mathcal{K}$ may fail to be locally integrable at a set of points of positive dimension—e.g. along a line in R^n; this stands in contrast to previously studied singular convolution operators in which the kernel could be non-integrable only at the origin and at infinity. For a more detailed discussion, one should see **[6]** and Part I of **[12]**.

This paper incorporates substantially the author's Ph.D. thesis (1980, Wisconsin). The author wishes to express his deep appreciation to Professor Alexander Nagel, the thesis advisor, for his patient guidance in this work; and also to Professor Stephen Wainger, whose lectures in Fourier analysis initiated the author's interest in the subject.

II. Outline of the argument. Our first observation is that for each $f \in C_c^\infty(R^n)$, $\lim_{\varepsilon\to 0, N\to\infty} T_{\varepsilon,N}f(x)$ exists for every $x \in R^n$. Thus, the a priori inequality $\|Tf\|_p \le C_p \|f\|_p$ for $f \in C_c^\infty(R^n)$ will follow via Fatou's lemma from the same inequality for the truncated operators $T_{\varepsilon,N}$ provided the estimates are independent of ε and N. We therefore fix $\varepsilon > 0$ and $N > 0$ and study $T_{\varepsilon,N}$ for this and the following two sections.

Notice that $T_{\varepsilon,N}f$ is well-defined by (1) for a wide variety of functions f, for example $f \in \bigcup_{1\le p\le\infty} L^p(R^n)$, and that $T_{\varepsilon,N}$ is bounded on $L^p(R^n)$ for $1 \le p \le \infty$, but with a bound which could depend on ε and N. One sees easily also that for $f \in \bigcup_{1\le p\le 2} L^p(R^n)$, we have $(T_{\varepsilon,N}f)\hat{} = m\hat{f}$ where $\hat{}$ denotes the Fourier transform and m is given by

$$m = m_{\varepsilon,N}(x, y) = \int_{\substack{\varepsilon<|t_i|<N \\ (1\le i\le k)}} \cdots \int \exp i\left[\sum_{i=1}^{k} x_i t_i + \sum_{j=1}^{l} y_j \gamma_j(t)\right] \frac{dt_1}{t_1} \cdot \cdots \cdot \frac{dt_k}{t_k}$$

for $x \in R^k$ and $y \in R^l$. (We write $x = (x_1, \ldots, x_k)$; likewise for y.)

Our approach to the desired L^p estimates for $T_{\varepsilon,N}$ is by Stein's analytic interpolation theorem **[13**, pg. 205**]**. We thus define multipliers

$m_{\varepsilon,N,z}$ by

$$(2) \qquad m_{\varepsilon,N,z}(x, y) = \int_{\substack{\varepsilon<|t_i|<N \\ (1\leq i\leq k)}} \cdots \int \exp i\left[\sum_{i=1}^{k} x_i t_i + \sum_{j=1}^{l} y_j \gamma_j(t)\right]$$
$$\times\left[1 + \sum_{j=1}^{l} \left(y_j \gamma_j(t)\right)^2\right]^z \frac{dt_1}{t_1} \cdot \ldots \cdot \frac{dt_k}{t_k}$$

for $x \in R^k$, $y \in R^l$, and $z \in \mathbf{C}$. Of course, $m_{\varepsilon,N,0} = m_{\varepsilon,N}$, so we taken z in a certain vertical strip $\{z \in \mathbf{C}: -a \leq \mathrm{Re}(z) \leq b\}$, where $a > 0$ and $b > 0$ are to be determined by the exponents $\alpha_{i,j}$ and the dimensions k and l. Our application of Stein's theorem is akin to its use in proving L^p inequalities for Hilbert transforms along *curves* in [**12**, Theorem 11, pg. 1271], [**3**, Theorem 1, pg. 397], and especially [**5**, Theorem 3.1, pg. 243]; precedents are also found in the study of related maximal functions, as in [**4**], [**9**], and [**12**, Theorem 12, pg. 1276].

In §III we study $m_{\varepsilon,N,z}$ for $\mathrm{Re}(z) \geq 0$. In spite of the growth of the factor $[1 + \Sigma_{j=1}^{l}(y_j\gamma_j(t))^2]^z$, we shall see that these "worsened multipliers" will be bounded on R^n uniformly in ε and N, so long as $\mathrm{Re}(z)$ is not too large. In estimating certain trigonometric integrals, we shall apply in a crucial way results of Nagel and Wainger from their L^2 study [**6**].

In Section IV we consider the $m_{\varepsilon,N,z}$ for $\mathrm{Re}(z) < 0$. If $\mathrm{Re}(z)$ is sufficiently large negative, then the decay of $[1 + \Sigma_{j=1}^{l}(y_j\gamma_j(t))^2]^z$ will enable us to show that these "improved multipliers" satisfy the hypotheses of the Marcinkiewicz multiplier theorem [**10**, pg. 109], with estimates uniform in ε and N. The technique employed here is elementary but cumbersome, requiring many integrations by parts.

At this point we can define operators $T_{\varepsilon,N,z}$ by

$$(T_{\varepsilon,N,z}f)\hat{} = m_{\varepsilon,N,z}\hat{f} \quad \text{for } f \in L^2(R^n).$$

Section III shows that $T_{\varepsilon,N,z}$ is bounded on $L^2(R^n)$ if $\mathrm{Re}(z)$ is not too large. Section IV shows that $T_{\varepsilon,N,z}$ is bounded on $L^p(R^n)$ for $1 < p < \infty$ if $\mathrm{Re}(z)$ is sufficiently large negative. Stein's analytic interpolation theorem thus shows that $T_{\varepsilon,N} = T_{\varepsilon,N,0}$ is bounded on $L^p(R^n)$ for certain p near 2. Section V contains the details of this argument. We also comment on the limitations of our method and cite some related questions about maximal functions associated with the surface σ.

III. The worsened multipliers: $m_{\varepsilon,N,z}$ **for** $\mathrm{Re}(z) \geq 0$. In this section, we shall prove that if $\mathrm{Re}(z)$ is not too large, then $m_{\varepsilon,N,z}$ will be bounded on R^n.

1. *Preliminaries*. Our inequalities depend on the following three results, proven by Nagel and Wainger in their development of the L^2 theory for T:

THEOREM A [**6**, *Theorem* 3.1, *pg*. 768]. *Let* $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, *and suppose that* $\alpha_j > 0$ *for some fixed index* j. *Then if* $x_j = 1$, *and* $x_1, \ldots, x_n$ *are otherwise arbitrary real numbers, we have*

$$\left|\int_0^b \exp i[x_1 t^{\alpha_1} + \cdots + x_n t^{\alpha_n}]\, dt\right| \le c[1 + b^{(1-\alpha_j/n)}]$$

where $c = c(\alpha_1, \ldots, \alpha_n) > 0$ *is independent of the* $x_1, \ldots, x_n$, *and* b.

COROLLARY B [**6**, *Cor*. 3.6, *pg*. 772]. *Let* $\alpha_1, \ldots, \alpha_n \in R$. *Suppose* $\alpha_j \neq 1, j = 1, 2, \ldots, N$ *and* $a \ge 1$. *Then*

$$\left|\int_1^B \exp i\left[as + \sum_{j=1}^{N} y_j s^{\alpha_j}\right]\frac{ds}{s}\right| \le ca^{-1/(N+1)}$$

where $c = c(\alpha_1, \ldots, \alpha_N) > 0$ *is independent of the real variables* $y_1, \ldots, y_N$, $B \ge 1$, *and* a.

LEMMA C [**6**, *Lemma* 3.7, *pg*. 773].

$$\int_1^\infty \cdots \int_1^\infty [\max(t_1, \ldots, t_n)]^{-\varepsilon} \frac{dt_n}{t_n} \cdots \frac{dt_1}{t_1} < \infty \quad \textit{for every } \varepsilon > 0.$$

Nagel and Wainger employed a Van der Corput Lemma technique to prove A and B. C is elementary.

2. *An estimate of a trigonometric integral*. Our central tool in estimating the worsened multipliers will be the inequality given in the following

MAIN LEMMA. (i) *Suppose* $0 < b_j \neq 1$ *for* $1 \le j \le l$; $b_j \neq b_i$ *if* $j \neq i$; *and*

$$0 < U < \frac{\min(1, b_1, \ldots, b_l)}{2(l+1)\max(1, b_1, \ldots, b_l)}.$$

Then there are constants $C < \infty$ *and* $\kappa > 0$, *depending only on the above parameters, such that if* $\operatorname{Re}(z) \le U, y \in R^l$, $a \ge 1$, *and* $N \ge 1$ *then*

$$\left|\int_1^N \exp i\left[as + \sum_{j=1}^{l} y_j s^{b_j}\right]\left[1 + \sum_{j=1}^{l} \left(y_j s^{b_j}\right)^2\right]^z \frac{ds}{s}\right| \le \frac{C(1 + |z|)}{a^\kappa}.$$

(ii) *The same estimate holds for the integral*

$$\int_1^N \exp i\left[as + \sum_{j=1}^{l} y_j s^{b_j}\right]\left[1 + (as)^2 + \sum_{j=1}^{l} \left(y_j s^{b_j}\right)^2\right]^z \frac{ds}{s}.$$

Proof. We give the proof of (i) only, that for (ii) being virtually identical.

Let l, b_j, z, y, a, and N be as above. Let $\delta = \mathrm{Re}(z)$. $\mathcal{I}$ will denote the integral to be estimated; b will be $\max(b_1,\ldots,b_l)$; b' will be $\min(b_1,\ldots,b_l)$.

Case A. Suppose $|y_j| \le a$ for all j. Let ϕ be defined by

$$\phi(s) = -\int_s^N \exp i\left[at + \sum_{j=1}^{l} y_j t^{b_j}\right]\frac{dt}{t}.$$

Integration by parts then shows that $\mathcal{I} = BT - IT$, where

$$BT = -\phi(1)\cdot\left[1 + \sum_{j=1}^{l} y_j^2\right]^z$$

and

$$IT = \int_1^N \phi(s)\cdot z\cdot\left[1 + \sum_{j=1}^{l}\left(y_j s^{b_j}\right)^2\right]^{z-1}\cdot\sum_{j=1}^{l} 2b_j\left(y_j s^{b_j}\right)^2 s^{-1}\,ds.$$

A change of variables in the integral defining ϕ gives

$$\phi(s) = -\int_1^{N/s} \exp i\left[(as)t + \sum_{j=1}^{l}\left(y_j s^{b_j}\right)t^{b_j}\right]\frac{dt}{t}$$

and we see by Corollary B that there is a constant c_0, depending only on l and the exponents b_j, such that

$$|\phi(s)| \le c_0\cdot (as)^{-1/(l+1)} \quad \text{if } 1 \le s \le N.$$

Thus we have

$$|BT| \le c_0\cdot a^{-1/(l+1)}\cdot\left(1 + \sum_{j=1}^{l} y_j^2\right)^{\delta} \le c_0\cdot a^{-1/(l+1)}\cdot(1 + la^2)^{\delta}$$
$$\le c_0(l+1)\cdot a^{2\delta - 1/(l+1)} \le c_0(l+1)\cdot a^{2U - 1(l+1)}.$$

Notice that the exponent upon a is negative, due to our restriction on U.

Likewise, we estimate IT by

$$
\begin{aligned}
|IT| &\leq \int_1^N c_0(as)^{-1/(l+1)}|z|\left[1+\sum_{j=1}^{l}\left(y_js^{b_j}\right)^2\right]^{U-1} 2b\sum_{j=1}^{l}\left(y_js^{b_j}\right)^2\frac{ds}{s}\\
&\leq 2bc_0|z|a^{-1/(l+1)}\int_1^N\left[1+\sum_{j=1}^{l}\left(y_js^{b_j}\right)^2\right]^{U} s^{-1-1/(l+1)}\,ds\\
&\leq 2bc_0|z|a^{-1/(l+1)}\int_1^N\left[(as^b)^2+\sum_{j=1}^{l}(as^b)^2\right]^{U} s^{-1-1/(l+1)}\,ds\\
&= 2bc_0|z|a^{-1/(l+1)}a^{2U}(l+1)^U\int_1^N s^{2bU-1-1/(l+1)}\,ds\\
&\leq C\cdot|z|\cdot a^{2U-1/(l+1)}.
\end{aligned}
$$

C is given by

$$C = 2bc_0(l+1)\int_1^\infty s^{2bU-1-1/(l+1)}\,ds$$

and is finite since, by our choice of U, $2bU-1-1/(l+1) < -1$. Note also that the exponent upon a is negative. This gives the required estimate, and Case A is completed.

Case B. Suppose that

$$|y_r| \equiv \max(|y_1|,\ldots,|y_l|) > a. \tag{3}$$

Let α be chosen so that $(l+1)/b' < \alpha < 1/(2Ub)$; say, let α be the average of these two numbers. This is possible by our choice of U. Replacing s by s^α in $\mathcal{I}$ gives us

$$\mathcal{I} = \alpha\int_1^{N^{1/\alpha}}\exp i\left[as^\alpha+\sum_{j=1}^{l}y_js^{\alpha b_j}\right]\left[1+\sum_{j=1}^{l}\left(y_js^{\alpha b_j}\right)^2\right]^{z}\frac{ds}{s}. \tag{4}$$

Now let ϕ be defined by

$$\phi(s) = -\int_s^{N^{1/\alpha}}\exp i\left[at^\alpha+\sum_{j=1}^{l}y_jt^{\alpha b_j}\right]dt.$$

The change of variables $t = |y_r|^{-1/(\alpha b_r)}\cdot\tau$ gives us

$$\phi(s) = -|y_r|^{-1/(\alpha b_r)}\int_p^q \exp i\left[a|y_r|^{-1/b_r}\cdot\tau^\alpha+\sum_{j=1}^{l}y_j|y_r|^{-b_j/b_r}\cdot\tau^{\alpha b_j}\right]d\tau$$

where $p = |y_r|^{1/(\alpha b_r)}s$ and $q = |y_r|^{1/(\alpha b_r)}N^{1/\alpha}$. In the last integral notice that

(a) the exponents on τ in the exponential are positive and distinct,

(b) the coefficient corresponding to $j = r$ is ± 1, and

(c) the exponent on τ corresponding to $j = r$, αb_r, satisfies $1 - \alpha b_r/(l+1) < 0$, by choice of α.

Now (a) and (b) show that the hypotheses of Theorem A are satisfied, and (c) shows that the conclusion of this theorem implies that

$$|\phi(s)| \le C \cdot |y_r|^{-1/(\alpha b_r)} \tag{5}$$

where C depends only on the dimension l and the exponents α and αb_j, hence only on l, U, and the b_j.

Now in (4) we integrate by parts in the way indicated by our definition of ϕ.

By (3) and (5), the boundary term BT satisfies

$$\begin{aligned}|BT| &= \left| -\alpha \cdot \phi(1) \cdot \left[1 + \sum_{j=1}^{l} y_j^2\right]^z \right| \le \alpha \cdot C \cdot |y_r|^{-1/(\alpha b_r)} \cdot \left[1 + l y_r^2\right]^U \\ &\le \alpha \cdot C \cdot (l+1) \cdot |y_r|^{2U - 1/(\alpha b_r)} \le \alpha \cdot C \cdot (l+1) \cdot |y_r|^{2U - 1/(\alpha b)} \\ &\le \alpha \cdot C \cdot (l+1) \cdot a^{2U - 1/(\alpha b)},\end{aligned}$$

where we note that the exponent upon a is negative, by choice of α.

There are two integrated terms, $IT1$ and $IT2$. We have, again using (3) and (5), that

$$\begin{aligned}|IT1| &= \left| \alpha \int_1^{N^{1/\alpha}} \phi(s) \cdot z \cdot \left[1 + \sum_{j=1}^{l} \left(y_j s^{\alpha b_j}\right)^2\right]^{z-1} \cdot 2\alpha \sum_{j=1}^{l} b_j \left(y_j s^{\alpha b_j}\right)^2 s^{-2}\, ds \right| \\ &\le 2\alpha^2 Cb\,|z| \cdot |y_r|^{-1/(\alpha b_r)} \int_1^{\infty} \left[1 + \sum_{j=1}^{l} \left(y_j s^{\alpha b_j}\right)^2\right]^{U-1} \\ &\quad \cdot \sum_{j=1}^{l} \left(y_j s^{\alpha b_j}\right)^2 s^{-2}\, ds \\ &\le 2\alpha^2 Cb\,|z| \cdot |y_r|^{-1/(\alpha b)} \int_1^{\infty} \left[1 + \sum_{j=1}^{l} \left(y_j s^{\alpha b_j}\right)^2\right]^{U} s^{-2}\, ds \\ &\le 2\alpha^2 Cb\,|z| \cdot |y_r|^{-1/(\alpha b)} (l+1)\, |y_r|^{2U} \int_1^{\infty} s^{2\alpha b U} s^{-2}\, ds \\ &\le \left[2\alpha^2 Cb(l+1) \int_1^{\infty} s^{2\alpha b U - 2} ds\right] \cdot |z| \cdot a^{2U - 1/(\alpha b)}.\end{aligned}$$

Notice that the constant preceding $|z|$ is finite since, by choice of α, $2\alpha bU - 2 < -1$, and likewise that the power on a is negative.

The second integrated term $IT2$ satisfies

$$|IT2| = \left| \alpha \int_1^{N^{1/\alpha}} \phi(s) \cdot \left[1 + \sum_{j=1}^{l} \left(y_j s^{\alpha b_j} \right)^2 \right]^z \cdot s^{-2}\, ds \right|$$

$$\leq \alpha C |y_r|^{-1/(\alpha b_r)} \int_1^{\infty} \left[1 + \sum_{j=1}^{l} \left(y_j s^{\alpha b_j} \right)^2 \right]^U \cdot s^{-2}\, ds$$

$$\leq \alpha C |y_r|^{-1/(\alpha b)} (l+1) |y_r|^{2U} \int_1^{\infty} s^{2\alpha b U} s^{-2}\, ds$$

$$\leq \alpha C (l+1) \int_1^{\infty} s^{2\alpha b U - 2}\, ds \cdot a^{2U - 1/(\alpha b)},$$

and we thus see that $IT2$ is bounded in the desired way. This completes Case B, and the lemma is proven, with

$$\kappa = \min(2U - 1/(l+1), 2U - 1/(\alpha b)).$$

3. *Boundedness result for the worsened multipliers.* We now can state and prove our boundedness result for the worsened multipliers.

The needed assumptions on the exponents $\alpha_{i,j}$ are the following:

$$(6) \qquad \begin{array}{ll} 1 \neq \alpha_{i,j} > 0 & \text{for } 1 \leq i \leq k,\ 1 \leq j \leq l \text{ and} \\ \alpha_{i,j} \neq \alpha_{i,\mathfrak{j}} & \text{for } 1 \leq i \leq k,\ 1 \leq j \leq l,\ 1 \leq \mathfrak{j} \leq l, j \neq \mathfrak{j}. \end{array}$$

$\operatorname{Re}(z)$ must not exceed an upper bound U_0, where U_0 is a positive number satisfying

$$(7) \qquad U_0 < \min_{1 \leq i \leq k} \frac{\min(1, \alpha_{i,1}, \ldots, \alpha_{i,l})}{2(l+1)\max(1, \alpha_{i,1}, \ldots, \alpha_{i,l})} \equiv U_0^*.$$

PROPOSITION 1. *Suppose the exponents $\alpha_{i,j}$ satisfy (6) and U_0 satisfies (7). Then there is a finite constant C_0, depending only on the dimensions k and l, the exponents $\alpha_{i,j}$, and U_0, so that*

if $\operatorname{Re}(z) \leq U_0$, $x \in R^k$, $y \in R^l$, *and* $0 < \varepsilon \leq N < \infty$
then $|m_{\varepsilon,N,z}(x, y)| \leq C_0(1 + |z|)$.

Proof. We follow the proof of [**6**, Theorem 4.1, pg. 774]. The change of variables $t_i = x_i^{-1}s_i$ in (2) gives us

$$(8)\qquad m_{\varepsilon,N,z}(x, y) = \pm \underset{\substack{\varepsilon_i \le |s_i| \le N_i \\ (1\le i\le k)}}{\int\cdots\int} \exp i\left[\sum_{i=1}^{k} s_i + \sum_{j=1}^{l} y_j'\gamma_j(s)\right] \times\left[1 + \sum_{j=1}^{l} \left(y_j'\gamma_j(s)\right)^2\right]^z \frac{ds_1}{s_1}\cdot\cdots\cdot\frac{ds_k}{s_k}$$

where $\varepsilon_i = \varepsilon\,|\,x_i\,|$, $N_i = N\,|\,x_i\,|$, and $y_j' = y_j/\gamma_j(x_1,\dots,x_k)$.

We now split the region of integration in (8) into three parts:

(i) the region where $|\,s_i\,| \ge 1$ for all i,

(ii) the region where $|\,s_i\,| \le 1$ for some but not all i, and

(iii) the region where $|\,s_i\,| \le 1$ for all i.

Region (i). The set S over which we integrate is given by $S = \{s \in R^k\colon \max(1, \varepsilon_i) \le |\,s_i\,| \le N_i \text{ for } 1 \le i \le k\}$. We may assume that for each i, $\max(1, \varepsilon_i) = 1$. (For if $\varepsilon_i > 1$, write $[\varepsilon_i, N_i] = [1, N_i] - [1, \varepsilon_i)$; then the integral over S may be written as a sum and difference of at most 2^k integrals over sets of the form $\{s \in R^k\colon 1 \le |\,s_i\,| \le B_i \text{ for } 1 \le i \le k\}$.) We further split S into the k subregions where $|\,s_1\,|,\dots,|\,s_k\,|$ respectively is the maximum of $\{|\,s_1\,|,\dots,|\,s_k\,|\}$. By symmetry we may consider only the last of these subregions: $|\,s_k\,| = \max(|\,s_1\,|,\dots,|\,s_k\,|)$. The integral I to be estimated is therefore given by

$$(9)\quad I = \underset{\substack{1 \le |s_i| \le N_i \\ (1\le i\le k) \\ |s_k| = \max(|s_1|,\dots,|s_k|)}}{\int\cdots\int} \exp i\left[\sum_{i=1}^{k} s_i + \sum_{j=1}^{l} y_j'\gamma_j(s)\right] \times\left[1 + \sum_{j=1}^{l} \left(y_j'\gamma_j(s)\right)^2\right]^z \frac{ds_1}{s_1}\cdots\frac{ds_k}{s_k},$$

$$= \int_{1\le|s_1|\le N_1} \frac{\exp i s_1}{s_1}\cdot\cdots\cdot \int_{1\le|s_{k-1}|\le N_{k-1}} \frac{\exp i s_{k-1}}{s_{k-1}}\,[\mathcal{I}]\, ds_{k-1}\cdot\cdots\cdot ds_1$$

where the inner integral $\mathcal{I}$ is

$$(10)\quad \mathcal{I} = \int_{M\le|s_k|\le N_k} \exp i\left[s_k + \sum_{j=1}^{l} y_j'\gamma_j(s)\right]\left[1 + \sum_{j=1}^{l} \left(y_j'\gamma_j(s)\right)^2\right]^z \frac{ds_k}{s_k}$$

$$= \int_{1\le|s_k|\le N_k/M} \exp i\left[M s_k + \sum_{j=1}^{l} \tilde{y}_j\gamma_j(s)\right]\left[1 + \sum_{j=1}^{l} \left(\tilde{y}_j\gamma_j(s)\right)^2\right]^z \frac{ds_k}{s_k}$$

with $M = \max(|\,s_1\,|,\dots,|\,s_{k-1}\,|)$ and $\tilde{y}_j = M^{\alpha_{k,j}}y_j'$.

Now, viewing $\tilde{y}_j\gamma_j(s)$ as $(\tilde{y}_j\Pi_{i=1}^{k-1}|s_i|^{\alpha_{i,j}})\cdot|s_k|^{\alpha_{k,j}}$, and observing that in view of (9) we have $M \geq 1$, we apply the Main Lemma, (i), to (10) and conclude that $|\mathcal{G}| \leq C(1+|z|)/M^\kappa$, where C and κ depend only on U_0, the dimension, and the exponents $\alpha_{k,j}$; in particular they do not depend on $s_1,\ldots,s_{k-1}$, x, or y. (Note: when applying the lemma we consider $\bar{\mathcal{G}}$ for $-N_k/M \leq s_k \leq -1$.)

Thus, since $\mathcal{G}$ is an even function of $s_1,\ldots,s_{k-1}$, we have

$$\begin{aligned}|I| &\leq 2^{k-1}\int_1^\infty \frac{1}{s_1}\cdots\int_1^\infty \frac{1}{s_{k-1}}\cdot\frac{C(1+|z|)}{M^\kappa}ds_{k-1}\cdot\cdots\cdot ds_1\\ &= C_1(1+|z|),\end{aligned}$$

where the finite constant C_1 is given by

$$\begin{aligned}C_1 = 2^{k-1}\cdot C\cdot\int_1^\infty\cdots\int_1^\infty &\frac{1}{s_1\cdot\cdots\cdot s_{k-1}}\\ &\cdot\frac{1}{[\max(s_1,\ldots,s_{k-1})]^\kappa}ds_{k-1}\cdot\cdots\cdot ds_1.\end{aligned}$$

(See Lemma C.) This completes (i).

Region (ii). The set S over which we integrate is $\{s \in R^k\colon \varepsilon_i \leq |s_i| \leq \min(1, N_i)$ for $i \in \Omega_1$ and $\max(1,\varepsilon_i) \leq |s_i| \leq N_i$ for $i \in \Omega_2\}$, where Ω_1 is a proper nonempty subset of the indices $\{1,2,\ldots,k\}$ and Ω_2 is the complementary set of indices. By symmetry we can take Ω_1 to be $\{1,2,\ldots,\tilde{i}\}$ where $1 \leq \tilde{i} < k$. We need to estimate I, where

$$I = \underset{\substack{\varepsilon_i\leq|s_i|\leq\min(1,N_i)\\(1\leq i\leq\tilde{i})}}{\int\cdots\int} \mathcal{G}\cdot\prod_{i=1}^{\tilde{i}}\frac{\exp \mathrm{i}s_i}{s_i}\cdot ds_1\cdot\cdots\cdot ds_{\tilde{i}} \quad\text{and} \tag{11}$$

$$\begin{aligned}\mathcal{G} = \underset{\substack{\max(1,\varepsilon_i)\leq|s_i|\leq N_i\\(\tilde{i}<i\leq k)}}{\int\cdots\int} &\exp \mathrm{i}\left[\sum_{i=\tilde{i}+1}^{k} s_i + \sum_{j=1}^{l} y_j'\gamma_j(s)\right]\\ &\times\left[1+\sum_{j=1}^{l}\left(y_j'\gamma_j(s)\right)^2\right]^z\frac{ds_{\tilde{i}+1}}{s_{\tilde{i}+1}}\cdot\cdots\cdot\frac{ds_k}{s_k}.\end{aligned} \tag{12}$$

Since $\mathcal{G}$ is an even function of $s_1,\ldots,s_{\tilde{i}}$, we may replace $\exp \mathrm{i}s_i/s_i$ by the bounded function $\mathrm{i}\sin(s_i)/s_i$ in (11), for $1 \leq i \leq \tilde{i}$. Also in (12) if we view $y_j'\gamma_j(s)$ as $(y_j'\Pi_{i=1}^{\tilde{i}}|s_i|^{\alpha_{i,j}})\cdot\gamma_j'(s')$ where we write

$$\gamma_j'(s') = \gamma_j'(s_{\tilde{i}+1},\ldots,s_k) = \prod_{i=\tilde{i}+1}^{k}|s_i|^{\alpha_{i,j}},$$

then we see that (12) is an integral of the type considered in (i), with the dimension k replaced by $k - \tilde{i}$. Thus we have $|\mathcal{I}| \leq C_1(1 + |z|)$ and therefore

$$|I| \leq 2^{\tilde{i}} \cdot C_1 \cdot (1 + |z|) < 2^k \cdot C_1 \cdot (1 + |z|).$$

This completes (ii).

Region (iii). The set S over which we integrate is $\{s \in R^k \colon \varepsilon_i \leq |s_i| \leq \min(1, N_i) \text{ for } 1 \leq i \leq k\}$, and the integral I to be estimated is

$$(13)\; I = \int \cdots \int \prod_{i=1}^{k-1} \frac{\exp \mathrm{i} s_i}{s_i} \left(\int \exp \mathrm{i}\left[s_k + \sum_{j=1}^{l} y_j' \gamma_j(s) \right] \times \left[1 + \sum_{j=1}^{l} \left(y_j' \gamma_j(s) \right)^2 \right]^z \frac{ds_k}{s_k} \right) ds_{k-1} \cdot \cdots \cdot ds_1$$

with limits as indicated in S. As in (ii), we may replace $\exp \mathrm{i} s_i / s_i$ by $\mathrm{i} \sin(s_i)/s_i$ in the integral (13), for $1 \leq i \leq k$, and we thus need only obtain a favorable estimate for the inner integral $\mathcal{I}$ in (13).

Now, letting $A = \min(1, N_k)$, we see that

$$(14) \qquad \mathcal{I} = 2\mathrm{i} \int_{\varepsilon_k}^{A} \frac{\sin(s)}{s} \exp \mathrm{i}\left[\sum_{j=1}^{l} \tilde{y}_j s^{\beta_j} \right] \left[1 + \sum_{j=1}^{l} \left(\tilde{y}_j s^{\beta_j} \right)^2 \right]^z ds$$

where $\tilde{y}_j = y_j' \prod_{i=1}^{k-1} |s_i|^{\alpha_{i,j}}$ and $\beta_j = \alpha_{k,j}$. Let s_0 be the unique positive number such that $\sum_{j=1}^{l} (\tilde{y}_j s_0^{\beta_j})^2 = l$. (Such a number is unique since the left side is an increasing function of s_0.) The integrand in (14) is bounded by $(1 + l)^{U_0}$ if $0 \leq s \leq s_0$, so, since $A \leq 1$, we may assume that $s_0 \leq \varepsilon_k$. Now define ϕ by

$$(15) \qquad \phi(s) = 2\mathrm{i} \int_{\varepsilon_k}^{s} \exp \mathrm{i}\left[\sum_{j=1}^{l} \tilde{y}_j t^{\beta_j} \right] \left[1 + \sum_{j=1}^{l} \left(\tilde{y}_j t^{\beta_j} \right)^2 \right]^z \frac{dt}{t}.$$

Then $\mathcal{I} = \phi(s)\sin(s)\big|_{\varepsilon_k}^{A} - \int_{\varepsilon_k}^{A} \phi(s)\cos(s)\, ds$, so it suffices (again since $A \leq 1$) to estimate $\phi(s)$ for $\varepsilon_k \leq s \leq A$.

In (15) replace t by $(t/|\tilde{y}_{\mathrm{j}}|)^{1/\beta_{\mathrm{j}}}$ where the index j is chosen so that $|\tilde{y}_{\mathrm{j}}| s_0^{\beta_{\mathrm{j}}} \geq 1$. There is such an index by our choice of s_0. We obtain

$$\phi(s) = \frac{2i}{\beta_{\mathrm{j}}} \int_{|\tilde{y}_{\mathrm{j}}|\varepsilon_k^{\beta_{\mathrm{j}}}}^{|\tilde{y}_{\mathrm{j}}| s^{\beta_{\mathrm{j}}}} \exp \mathrm{i}\left[\sum_{j=1}^{l} \tilde{y}_j |\tilde{y}_{\mathrm{j}}|^{-\beta_j/\beta_{\mathrm{j}}} t^{\beta_j/\beta_{\mathrm{j}}} \right] \times \left[1 + \sum_{j=1}^{l} \left(\tilde{y}_j |\tilde{y}_{\mathrm{j}}|^{-\beta_j/\beta_{\mathrm{j}}} t^{\beta_j/\beta_{\mathrm{j}}} \right)^2 \right]^z \frac{dt}{t}.$$

In the term corresponding to $j = \mathrm{j}$, the exponent upon t is 1 and the coefficient is $\mathrm{sgn}(\tilde{y}_{\mathrm{j}})$. Also, since $s_0 \leq \varepsilon_k$, we have by choice of j that the lower limit of integration is at least 1. The Main Lemma, (ii), (applied to $\bar{\phi}$ if $\mathrm{sgn}(\tilde{y}_{\mathrm{j}}) = -1$) thus shows that ϕ is bounded by $C(1 + |z|)/1^{\kappa} = C(1 + |z|)$, where C depends only on U_0, the dimension, and the exponents $\beta_j/\beta_{\mathrm{j}}$, hence the exponents $\alpha_{i,j}$. This completes (iii), and Proposition 1 is proven.

IV. The improved multipliers: $m_{\varepsilon,N,z}$ for $\mathrm{Re}(z) < 0$. In this section we shall prove that if $\mathrm{Re}(z)$ is sufficiently large negative, then $m_{\varepsilon,N,z}$ satisfies the hypotheses of the Marcinkiewicz multiplier theorem [**10**, pg. 109]. It is clear that $m_{\varepsilon,N,z} \in C^{\infty}(R^n)$, so our task is to show that $x^{\lambda}y^{\eta} \cdot \partial_x^{\lambda}\partial_y^{\eta}(m_{\varepsilon,N,z})(x, y)$ is bounded on R^n for all k-dimensional multi-indices λ and l-dimensional multi-indices η each of whose entries is either 0 or 1. For such λ and η, a computation shows that for all $x \in R^k$ and $y \in R^l$, we have

$$(16) \qquad x^{\lambda}y^{\eta} \cdot \partial_x^{\lambda}\partial_y^{\eta}(m_{\varepsilon,N,z})(x, y) = \sum_{\eta' + \eta'' = \eta} \mathrm{i}^{|\lambda| + |\eta'|} \cdot 2^{|\eta''|} \cdot C_{|\eta''|}(z) \cdot I$$

where the sum runs over l-dimensional multi-indices η' and η'', $C_p(z) = z(z-1) \cdot \cdots \cdot (z - p + 1)$, and the integral I is defined by

$$(17) \quad I = \underset{\substack{\varepsilon < |t_i| < N \\ (1 \leq i \leq k)}}{\int \cdots \int} \exp \mathrm{i}\left[\sum_{i=1}^{k} x_i t_i + \sum_{j=1}^{l} y_j \gamma_j(t)\right]\left[1 + \sum_{j=1}^{l} \left(y_j \gamma_j(t)\right)^2\right]^{z - |\eta''|}$$

$$\times \prod_{i \in \Lambda} (x_i t_i) \prod_{j \in \Omega_1} \left(y_j \gamma_j(t)\right) \prod_{j \in \Omega_2} \left(y_j \gamma_j(t)\right)^2 \frac{dt_1}{t_1} \cdot \cdots \cdot \frac{dt_k}{t_k},$$

$$\Lambda = \{i \colon \lambda_i = 1\}, \quad \Omega_1 = \{j \colon \eta'_j = 1\}, \quad \Omega_2 = \{j \colon \eta''_j = 1\}.$$

(Note: for a multi-index $\lambda = (\lambda_1, \ldots, \lambda_k)$, we write $|\lambda|$ for $\lambda_1 + \cdots + \lambda_k$, and if also $x \in R^k$, we write x^{λ} for $x_1^{\lambda_1} \cdot \cdots \cdot x_k^{\lambda_k}$.)

Thus, we need to estimate integrals of the kind in (17). For the case $l = k = 1$, Nagel and Wainger have already obtained suitable inequalities in [**5**, pg. 244], and extension to $l > 1$ presents no problem. In the general case we argue by induction on k. The details of this proof are somewhat technical, so we relegate the proof to the appendix and present here instead a rough outline of the argument.

We view a $k + 1$-fold integral of the form (17) as

$$(18) \qquad \int \exp \mathrm{i} x_{k+1} t_{k+1}\, \mathcal{G}\left(x_{k+1} t_{k+1}\right)^{E} \frac{dt_{k+1}}{t_{k+1}}$$

where E can be 0 or 1 according to the index set Λ and $\mathcal{G}$ is a k-fold integral similar to (17). A change of variables reduces us to the case $x_{k+1} = 1$, and a computation shows that $\partial \mathcal{G}/\partial t_{k+1}$ is a sum of k-fold integrals each of which is again similar to (17), but including an extra factor of t_{k+1}^{-1}. Integration by parts (once or twice, according to whether $E = 0$ or 1) on t_{k+1} in the $k + 1$-fold integral (18) thus leads to integrals of the form $\int \exp it_{k+1} \mathcal{G} t_{k+1}^{-2} dt_{k+1}$ with $\mathcal{G}$ as above. An inductive assumption that k-fold integrals, such as $\mathcal{G}$, are bounded then leads to the same conclusion in the $k + 1$-fold situation. The interested reader is referred to the appendix for details. Lemma A2, presented there, shows that if the exponents $\alpha_{i,j}$ are all positive, then the integral I of (17) satisfies

$$(19) \qquad |I| \leq C \cdot (1 + |z|)^{2k} \quad \text{if } \mathrm{Re}(z) \leq L_0^* \equiv -(l + 2k - 1/2)$$

where C is a finite constant independent of x, y, z, ε, and N. Thus, referring to (16), we deduce immediately

PROPOSITION 2. *If λ and η are k- and l-dimensional multi-indices respectively all of whose entries are 0 or 1, and the exponents $\alpha_{i,j}$ are all positive, then*

$$|x^\lambda y^\eta \cdot \partial_x^\lambda \partial_y^\eta (m_{\varepsilon,N,z})(x, y)| \leq C_1 \cdot (1 + |z|)^{2k+l} \quad \textit{if } \mathrm{Re}(z) \leq L_0^*$$

where C_1 is a finite constant independent of $x \in R^k$, $y \in R^l$, z, ε, and N.

V. Conclusion.

1. *L^p-boundedness of the multiple Hilbert transform.* We know by Proposition 1 that $m_{\varepsilon,N,z}$ is bounded on R^n for $\mathrm{Re}(z) \leq U_0$ (see (7)), so for these z the equation

$$(T_{\varepsilon,N,z} f)\hat{} = m_{\varepsilon,N,z} \hat{f}$$

defines the operator $T_{\varepsilon,N,z}$ on all of $L^2(R^n)$. Since our estimates on the size of $m_{\varepsilon,N,z}$ grow at most polynomially in $|z|$, it follows that the operators $T_{\varepsilon,N,z}$ are an analytic family admissible for the Stein analytic interpolation theorem [**13**, pg. 205], defined for z in the strip $S = \{z \in \mathbf{C}: L_0^* \leq \mathrm{Re}(z) \leq U_0\}$ where U_0 satisfies (7) and L_0^* is defined by (19). Proposition 1 and the Plancherel theorem show that if $\mathrm{Re}(z) = U_0$ then

$$\|T_{\varepsilon,N,z} f\|_2 \leq C_0 \cdot (1 + |z|) \cdot \|f\|_2 \quad \text{for } f \in L^2(R^n)$$

where C_0 is a finite constant independent of z, ε, N, and f. Proposition 2 and the Marcinkiewicz multiplier theorem [**10**, pg. 109] show that if

$\text{Re}(z) = L_0^*$ and $1 < p < \infty$ then

$$\|T_{\varepsilon,N,z}f\|_p \leq C_p \cdot (1 + |z|)^{2k+l} \cdot \|f\|_p \quad \text{for } f \in L^2 \cap L^p(R^n)$$

where C_p is a finite constant independent of z, ε, N, and f.

By analytic interpolation we conclude that

$$\text{(20)} \quad \text{if } \frac{2(U_0 - L_0^*)}{2U_0 - L_0^*} < p < \frac{2(U_0 - L_0^*)}{-L_0^*} \quad \text{then } \|T_{\varepsilon,N}f\|_p \leq C_p \|f\| p$$

for all simple functions f on R^n, where $C_p < \infty$ is independent of ε, N, and f. An easy limiting argument extends (20) to all $f \in L^p(R^n)$, hence to all $f \in C_c^\infty(R^n)$, our original domain of definition for T and the truncated operators $T_{\varepsilon,N}$. We may also let $U_0 \to U_0^*$ so that (20) holds for

$$\text{(21)} \qquad \frac{2(U_0^* - L_0^*)}{2U_0^* - L_0^*} < p < \frac{2(U_0^* - L_0^*)}{-L_0^*}.$$

Finally, letting $\varepsilon \to 0$ and $N \to \infty$, Fatou's lemma gives us our

THEOREM. *If p satisfies* (21) *and the exponents $\alpha_{i,j}$ satisfy* (6), *then $\|Tf\|_p \leq C_p \|f\|_p$ for $f \in C_c^\infty(R^n)$, where C_p is a finite constant independent of f. (See* (7) *and* (19) *for definitions of U_0^* and L_0^*.)*

2. *Some comments and related questions*. It seems clear that the above range of p is not best possible; thus, the interest of the theorem is that T is bounded on L^p for *some* p other than $p = 2$. In the case of $k = n - 1$, R. Strichartz **[14]** has recently shown by methods of Mellin analysis that T is bounded on $L^p(R^n)$ for $4n/(3n - 1) < p < 4n/(n + 1)$ or (according to a condition on the exponents) $4n/(3n - 2) < p < 4n/(n + 2)$.

For general k, positive results for a broader range of p might be had by examining the kernel $K_{\varepsilon,N,z}$ corresponding to the multiplier $m_{\varepsilon,N,z}$. If one could prove that $T_{\varepsilon,N,z}$ is bounded on L^p for arbitrarily small negative $\text{Re}(z)$ and all p, $1 < p < \infty$, then interpolation would imply the same for T. This kind of argument has been successfully carried out in the study of L^p estimates for Hilbert transforms along *curves*. For example, see [**12**, Thm. 11, pg. 1273]; the "improved operators" considered there are seen to be bounded on $L^p(R^n)$, $1 < p < \infty$, by an application of an extension **[8]** of the Calderón-Zygmund theory of singular integrals. In the current situation, however, it seems that the kernels $K_{\varepsilon,N,z}$ fail even to be integrable on the unit sphere in R^n uniformly in ε and N, if $\text{Re}(z)$ is small negative, and thus the Calderón-Zygmund theory does not apply.

A related operator of some interest is the maximal operator M associated with our surface σ, namely

$$Mf(x) = \sup_{h_1,\ldots,h_k>0} \frac{1}{h_1\cdot \cdots \cdot h_k}\int_0^{h_k}\cdots\int_0^{h_1}|f(x-\sigma(t))|\,dt_1\cdot\cdots\cdot dt_k,$$
$$x \in R^n.$$

No positive L^p-boundedness result has been proven (to our knowledge) for M, even in the case $p = 2$. However, as others have previously noted, positive results are readily obtained for the smaller operator M_0 for a wide variety of k-surfaces σ in R^n, where

$$M_0 f(x) = \sup_{h>0}\frac{1}{h^k}\int\cdots\int_{|t|\le h}|f(x-\sigma(t))|\,dt_1\cdot\cdots\cdot dt_k, \qquad x\in R^n$$

with $|t| = (t_1^2+\cdots+t_k^2)^{1/2}$. In fact, when the above integral is written in polar coordinates, the most elementary estimate yields immediately the inequality

$$M_0 f(x) \le \int_{\Sigma_{k-1}} M_{\sigma,u}f(x)\,du. \tag{22}$$

In (22), Σ_{k-1} is the unit sphere $|t| = 1$ in R^k and du is the corresponding "area" measure. $M_{\sigma,u}$ is the maximal operator associated with the *curve* $\gamma_{\sigma,u}$ in R^n, given for $u \in \Sigma_{k-1}$ by

$$M_{\sigma,u}f(x) = \sup_{h>0}\frac{1}{h}\int_0^h|f(x-\gamma_{\sigma,u}(s))|\,ds \quad \text{for } x\in R^n,$$
$$\gamma_{\sigma,u} = \sigma(su)\in R^n \quad \text{for } s \in R.$$

(Several L^2- and L^p-boundedness theorems are known for these maximal operators associated with curves; see for example the extensive paper [**12**] of Stein and Wainger, or more recently the Ph.D. theses [**7**] and [**15**] of Nestlerode and Weinberg.) Thus, if the surface σ is such that L^p estimates are known uniformly for the family of maximal operators $\{M_{\sigma,u}\}_{u\in\Sigma_{k-1}}$, then (22) shows that an L^p estimate holds for M_0 as well. This is the case, as [**12**, Theorem 12A, pg. 1275] shows, for the surfaces σ considered in this paper.

Appendix

Here we give a detailed proof of the estimates required for the improved multipliers, $m_{\varepsilon,N,z}$ for $\mathrm{Re}(z) < 0$, as discussed in IV. To begin with we have the inequality needed in the case $k = 1$:

LEMMA A1. *Given an integer $V \geq 0$ and $\alpha_1,\ldots,\alpha_l > 0$, there exists $C < \infty$ so that if $(y_1,\ldots,y_l) \in R^l$, $b_v \in \{1,2\}$ and $j_v \in \{1,2,\ldots,l\}$ for $1 \leq v \leq V$, $0 < \varepsilon \leq N < \infty$, and $\mathrm{Re}(z) \leq -(V+1/2)$ then*

$$\left|\int_\varepsilon^N \sin(t)\cdot \exp i \sum_{j=1}^{l} y_j t^{\alpha_j} \cdot \left[1+\sum_{j=1}^{l}\left(y_j t^{\alpha_j}\right)^2\right]^z \cdot \prod_{v=1}^{V}\left(y_{j_v} t^{\alpha_{j_v}}\right)^{b_v}\frac{dt}{t}\right|$$
$$\leq C(1+|z|).$$

(*Note*: *we allow $j_v = j_{\mathfrak{v}}$ with $v \neq \mathfrak{v}$. Also, the empty product $\prod_{v=1}^{0}\cdots$ which occurs in the case $V=0$ means* 1.)

Proof. I will denote the integral to be estimated. If $V > 0$, we have

$$\begin{aligned}
|I| &\leq \int_0^\infty \left[1+\sum_{j=1}^{l}\left(y_j t^{\alpha_j}\right)^2\right]^{\mathrm{Re}(z)} \prod_{v=1}^{V}\left|y_{j_v} t^{\alpha_{j_v}}\right|^{b_v}\frac{dt}{t}\\
&\leq \sum_{v=1}^{V}\int_0^\infty \left[1+\left(y_{j_v} t^{\alpha_{j_v}}\right)^2\right]^{\mathrm{Re}(z)}\left|y_{j_v} t^{\alpha_{j_v}}\right|^{Vb_v}\frac{dt}{t}\\
&= \sum_{v=1}^{V}\alpha_{j_v}^{-1}\int_0^\infty (1+s^2)^{\mathrm{Re}(z)} s^{Vb_v}\frac{ds}{s}\\
&\leq \sum_{v=1}^{V}\alpha_{j_v}^{-1}\int_0^\infty (1+s^2)^{-V-1/2} s^{Vb_v-1}ds.
\end{aligned}$$

Notice that $2(-V-1/2)+Vb_v \leq -1$ and $Vb_v - 1 \geq 0$, for $1 \leq v \leq V$.
In the case $V = 0$, we have using integration by parts that

$$I = \int_\varepsilon^N \sin(t)\cdot \exp i \sum_{j=1}^{l} y_j t^{\alpha_j} \cdot \left[1+\sum_{j=1}^{l}\left(y_j t^{\alpha_j}\right)^2\right]^z \frac{dt}{t} = BT - IT$$

where

$$BT = \frac{1-\cos(t)}{t}\cdot \exp i \sum_{j=1}^{l} y_j t^{\alpha_j}\cdot\left[1+\sum_{j=1}^{l}\left(y_j t^{\alpha_j}\right)^2\right]^z\Bigg|_\varepsilon^N$$

and

$$IT = \int_{\varepsilon}^{N} \frac{1-\cos(t)}{t^2} \cdot \exp \mathrm{i} \sum_{j=1}^{l} y_j t^{\alpha_j} \cdot [\mathcal{A} + \mathcal{B} + \mathcal{C}]\, dt,$$

$$\mathcal{A} = -\left[1 + \sum_{j=1}^{l} \left(y_j t^{\alpha_j}\right)^2\right]^z, \quad \mathcal{B} = \mathrm{i} \sum_{\mathrm{i}=1}^{l} \alpha_{\mathrm{i}} \cdot \left[1 + \sum_{j=1}^{l} \left(y_j t^{\alpha_j}\right)^2\right]^z y_{\mathrm{i}} t^{\alpha_{\mathrm{i}}},$$

$$\mathcal{C} = 2z \sum_{\mathrm{i}=1}^{l} \alpha_{\mathrm{i}} \cdot \left[1 + \sum_{j=1}^{l} \left(y_j t^{\alpha_j}\right)^2\right]^{z-1} \left(y_{\mathrm{i}} t^{\alpha_{\mathrm{i}}}\right)^2.$$

Notice that $|BT| \le 4$ if only $\mathrm{Re}(z) \le 0$ and, if $\mathrm{Re}(z) \le -1/2$ then $|\mathcal{A}| \le 1$, $|\mathcal{B}| \le \Sigma_{j=1}^{l} \alpha_j$, and $|\mathcal{C}| \le 2|z| \Sigma_{j=1}^{l} \alpha_j$. Thus, if $\mathrm{Re}(z) \le -1/2$ then

$$|IT| \le \left[1 = (1 + 2|z|) \sum_{j=1}^{l} \alpha_j\right] \cdot \int_0^{\infty} \frac{1-\cos(t)}{t^2}\, dt.$$

We can prove by induction on k our

LEMMA A2. (*Estimate for the improved multiplier and its derivatives.*) *Given positive integers k and l; $\alpha_{i,j} > 0$, $1 \le i \le k$, $1 \le j \le l$ (and associated surface $\sigma(t) = (t, \gamma_1(t), \ldots, \gamma_l(t))$ for $t \in R^k$, defined as in* §I); *integer $V \ge 0$; $b_v \in \{1,2\}$ and $j_v \in \{1,2,\ldots,l\}$, $1 \le v \le V$; integer $\rho, 0 \le \rho \le k$; and integers i_r, $1 \le i_1 < i_2 < \cdots < i_\rho \le k$; there exists $C < \infty$ so that if* $\mathrm{Re}(z) \le -(V + 2k - 1/2)$, $0 < \varepsilon \le N < \infty$, $(x_1, \ldots, x_k) \in R^k$, *and* $(y_1, \ldots, y_l) \in R^l$, *then*

$$\left| \underset{\substack{\varepsilon \le |t_i| \le N \\ (1 \le i \le k)}}{\int \cdots \int} \exp \mathrm{i} \left[\sum_{i=1}^{k} x_i t_i + \sum_{j=1}^{l} y_j \gamma_j(t)\right] \cdot \left[1 + \sum_{j=1}^{l} \left(y_j \gamma_j(t)\right)^2\right]^z \right.$$

$$\left. \cdot \prod_{v=1}^{V} \left(y_{j_v} \gamma_{j_v}(t)\right)^{b_v} \cdot \prod_{r=1}^{\rho} x_{i_r} t_{i_r} \frac{dt_1}{t_1} \cdot \cdots \cdot \frac{dt_k}{t_k} \right| \le C(1 + |z|)^{2k}.$$

Proof. We proceed by induction on k.

$k = 1$, $\rho = 0$. The integral I to be estimated is given by

$$I = 2\mathrm{i} \int_{\varepsilon}^{N} \sin(xt) \cdot \exp \mathrm{i} \sum_{j=1}^{l} y_j t^{\alpha_j} \cdot \left[1 + \sum_{j=1}^{l} \left(y_j t^{\alpha_j}\right)^2\right]^z \cdot \prod_{v=1}^{V} \left(y_{j_v} t^{\alpha_{j_v}}\right)^{b_v} \frac{dt}{t}$$

the subscript i having been dropped. We may assume that $x = 1$, since the change of variables $|x|t \to s$ only replaces x by $\operatorname{sgn}(x)$, ε by $|x|\varepsilon$, N by $|x|N$, and y_j by $|x|^{-\alpha_j}y_j$. (Our estimates must be independent of these parameters.) We see then that Lemma A1 gives the required estimate if $\operatorname{Re}(z) \leq -(V + 1/2)$.

$k = 1$, $\rho = 1$. The integral I to be estimated is given by

$$I = 2\int_{\varepsilon}^{N} \cos(xt)\cdot \exp \mathrm{i} \sum_{j=1}^{l} y_j t^{\alpha_j} \cdot \left[1 + \sum_{j=1}^{l} \left(y_j t^{\alpha_j}\right)^2\right]^z \cdot \prod_{v=1}^{V} \left(y_{j_v} t^{\alpha_{j_v}}\right)^{b_v} x\, dt.$$

Again we may assume that $x = 1$, and integration by parts gives us

$$\text{(A1)} \quad \frac{1}{2} I = \sin(t)\cdot \exp \mathrm{i} \sum_{j=1}^{l} y_j t^{\alpha_j} \cdot \left[1 + \sum_{j=1}^{l} \left(y_j t^{\alpha_j}\right)^2\right]^z \cdot \prod_{v=1}^{V} \left(y_{j_v} t^{\alpha_{j_v}}\right)^{b_v} \Bigg|_{t=\varepsilon}^{t=N}$$

$$- \int_{\varepsilon}^{N} \sin(t)\cdot \exp \mathrm{i} \sum_{j=1}^{l} y_j t^{\alpha_j} \cdot [\mathcal{A} + \mathcal{B} + \mathcal{C}] \frac{dt}{t},$$

$$\text{(A2)} \qquad A = \mathrm{i} \sum_{\mathrm{j}=1}^{l} \alpha_{\mathrm{j}} \cdot \left[1 + \sum_{j=1}^{l} \left(y_j t^{\alpha_j}\right)^2\right]^z \cdot \prod_{v=1}^{V+1} \left(y_{j_{v,\mathrm{j}}} t^{\alpha_{j_{v,\mathrm{j}}}}\right)^{b_v},$$

where

$$\text{(A3)} \qquad j_{v,\mathrm{j}} = \begin{cases} j_v & \text{if } 1 \leq v \leq V \\ \mathrm{j} & \text{if } v = V + 1 \end{cases} \qquad \text{and} \qquad b_{V+1} = 1,$$

$$\text{(A4)} \qquad \mathcal{B} = 2z \sum_{\mathrm{j}=1}^{l} \alpha_{\mathrm{j}} \cdot \left[1 + \sum_{j=1}^{l} \left(y_j t^{\alpha_j}\right)^2\right]^{z-1} \cdot \prod_{v=1}^{V+1} \left(y_{j_{v,\mathrm{j}}} t^{\alpha_{j_{v,\mathrm{j}}}}\right)^{b_v},$$

where $j_{v,\mathrm{j}}$ is defined as in (A3) and $b_{V+1} = 2$, and

$$\text{(A5)} \qquad \mathcal{C} = \left(\sum_{v=1}^{V} b_v \alpha_{j_v}\right) \cdot \left[1 + \sum_{j=1}^{l} \left(y_j t^{\alpha_j}\right)^2\right]^z \cdot \prod_{v=1}^{V} \left(y_{j_v} t^{\alpha_{j_v}}\right)^{b_v}.$$

We estimate the boundary terms in (A1) by

$$\sum_{v=1}^{V} \left(\left|y_{j_v} t^{\alpha_{j_v}}\right|^{Vb_v} \cdot \left[1 + \left(y_{j_v} t^{\alpha_{j_v}}\right)^2\right]^{\operatorname{Re}(z)} \right).$$

This will be at most V, for every t, if $Vb_v + 2\operatorname{Re}(z) \leq 0$ for $1 \leq v \leq V$; i.e. if $\operatorname{Re}(z) \leq -V$. Lemma A1 is used as in the case $\rho = 0$ to estimate the integrated terms in (A1) arising from $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, and we see that $\operatorname{Re}(z) \leq -(V + 3/2)$ is required. This completes the case $k = 1$.

Now suppose that the lemma holds for k-fold integrals.

Induction step, $\rho = k + 1$. The integral I to be estimated is given by

$$I = 2\int_\varepsilon^N \cos(x_{k+1}t_{k+1})\cdot \mathcal{I} \cdot x_{k+1}\, dt_{k+1} \tag{A6}$$

where the inner k-fold integral $\mathcal{I}$ is given by

$$\begin{aligned} I = \underset{\substack{\varepsilon\le|t_i|\le N\\(1\le i\le k)}}{\int\cdots\int} \exp \mathrm{i}\left[\sum_{i=1}^{k} x_i t_i + \sum_{j=1}^{l} y_j\gamma_j(t)\right]\cdot\left[1+\sum_{j=1}^{l}\left(y_j\gamma_j(t)\right)^2\right]^z \\ \times \prod_{v=1}^{V}\left(y_{J_v}\gamma_{J_v}(t)\right)^{b_v}\cdot \prod_{r=1}^{\tilde\rho} x_{i_r}t_{i_r}\frac{dt_1}{t_1}\cdot\ \cdots\ \cdot\frac{dt_k}{t_k}, \end{aligned} \tag{A7}$$

with $\tilde\rho = \rho - 1 = k$. (Notice that in (A7), t denotes the $k+1$-vector $(t_1,\dots,t_k, t_{k+1})$.) Just as in the case $k = 1$, $\rho = 0$, we may assume in (A6) that $x_{k+1} = 1$. Integration by parts in (A6) then gives us

$$\frac{1}{2}I = \sin(t_{k+1})\cdot \mathcal{I}\Big|_{t_{k+1}=\varepsilon}^{t_{k+1}=N} - \int_\varepsilon^N \sin(t_{k+1})\cdot\frac{\partial\mathcal{I}}{\partial t_{k+1}}\,dt_{k+1}. \tag{A8}$$

The induction hypothesis shows that $\mathcal{I}$ and therefore the boundary terms are no greater that $C(1+|z|)^{2k}$ provided that z satisfies $\mathrm{Re}(z) \le -(V+2k-1/2)$. In applying the induction hypothesis we of course view $y_j\gamma_j(t)$ as $(y_j t_{k+1}^{\alpha_{k+1,j}})\cdot\gamma_j'(t')$ where we write

$$\gamma_j'(t') = \gamma_j'(t_1,\dots,t_k) = \prod_{i=1}^{k}|t_i|^{\alpha_{i,j}},$$

and thus we obtain an estimate independent of t_{k+1}.

To estimate the integrated term in (A8), we first must estimate $\partial\mathcal{I}/\partial t_{k+1}$ and then integrate by parts again. For the former task, we first observe that for $t_{k+1} > 0$,

$$\begin{aligned} \frac{\partial\mathcal{I}}{\partial t_{k+1}} = t_{k+1}^{-1}\underset{\substack{\varepsilon\le|t_i|\le N\\(1\le i\le k)}}{\int\cdots\int} \exp\mathrm{i}\left[\sum_{i=1}^{k} x_i t_i + \sum_{j=1}^{l} y_j\gamma_j(t)\right] \\ \times[\mathcal{A}+\mathcal{B}+\mathcal{C}]\prod_{r=1}^{\tilde\rho} x_{i_r}t_{i_r}\frac{dt_1}{t_1}\cdot\ \cdots\ \cdot\frac{dt_k}{t_k}. \end{aligned} \tag{A9}$$

$\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are given by

$$\mathcal{A} = \mathrm{i}\sum_{\mathrm{j}=1}^{l}\alpha_{k+1,\mathrm{j}}\cdot\left[1+\sum_{j=1}^{l}\left(y_j\gamma_j(t)\right)^2\right]^z\cdot\prod_{v=1}^{V+1}\left(y_{j_{v,\mathrm{j}}}\gamma_{j_{v,\mathrm{j}}}(t)\right)^{b_v}, \tag{A10}$$

$j_{v,\mathrm{j}}$ defined as in (A3), $b_{V+1} = 1$;

$$\text{(A11)}\quad \mathfrak{B} = 2z\sum_{\mathrm{i}=1}^{l}\alpha_{k+1,\mathrm{i}}\cdot\left[1+\sum_{j=1}^{l}\left(y_j\gamma_j(t)\right)^2\right]^{z-1}\cdot\prod_{v=1}^{V+1}\left(y_{j_{v,\mathrm{i}}}\gamma_{j_{v,\mathrm{i}}}(t)\right)^{b_v},$$

$j_{v,\mathrm{i}}$ defined as in (A3), $b_{V+1} = 2$; and

$$\text{(A12)}\quad \mathfrak{C} = \left(\sum_{v=1}^{V} b_v\alpha_{k+1,j_v}\right)\cdot\left[1+\sum_{j=1}^{l}\left(y_j\gamma_j(t)\right)^2\right]^{z}\cdot\prod_{v=1}^{V}\left(y_{j_v}\gamma_{j_v}(t)\right)^{b_v}.$$

The integrals arising in (A9) from the various terms in $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ can be estimated by using the induction hypothesis. We conclude that

$$\text{(A13)}\quad \left|\frac{\partial\mathcal{G}}{\partial t_{k+1}}\right| \le t_{k+1}^{-1}\cdot C\cdot(1+|z|)^{2k+1}\quad \text{if } \mathrm{Re}(z)\le -(V+2k+1/2)$$

where C is as in the statement of this lemma and in particular independent of t_{k+1}.

Integration by parts once again in (A8) yields

$$\text{(A14)}\quad \int_\varepsilon^N \sin(t_{k+1})\cdot\frac{\partial\mathcal{G}}{\partial t_{k+1}}dt_{k+1} = (1-\cos(t_{k+1}))\cdot\frac{\partial\mathcal{G}}{\partial t_{k+1}}\bigg|_{t_{k+1}=\varepsilon}^{t_{k+1}=N} - \int_\varepsilon^N(1-\cos(t_{k+1}))\cdot\frac{\partial^2\mathcal{G}}{\partial t_{k+1}^2}dt_{k+1}.$$

The boundary terms in (A14) are estimated by use of (A13). To estimate $\partial^2\mathcal{G}/\partial t_{k+1}^2$ and thus the integrated term in (A14), we notice that formulas (A9)–(A12) show that $t_{k+1}\partial\mathcal{G}/\partial t_{k+1}$ is a linear combination of terms like $\mathcal{G}$ itself, with z and V possibly replaced by $z-1$ and $V+1$ respectively; the number of terms depends only on l; the linear coefficients depend only on the exponents $\alpha_{i,j}$ with the exception that some (see (A11)) include a factor of z. Thus, a repetition of the argument yielding (A13) shows that

$$\left|\frac{\partial^2\mathcal{G}}{\partial t_{k+1}^2}\right| \le t_{k+1}^{-2}\cdot C\cdot(1+|z|)^{2k+2}\quad \text{if } \mathrm{Re}(z)\le -(V+2k+3/2).$$

The integrated term in (A14) and therefore I itself is now dominated by $C(1+|z|)^{2k+2}$, if $\mathrm{Re}(z)\le -(V+2k+3/2)$. This completes the induction step in the case $\rho = k+1$.

Induction step, $\rho < k+1$. In this case we may assume that $i_\rho < k+1$, i.e. that $x_{k+1}t_{k+1}$ does not occur in $\prod_{r=1}^{\rho} x_{i_r}t_{i_r}$. The integral I to be estimated is given by

$$I = 2\mathrm{i}\int_\varepsilon^N \sin(x_{k+1}t_{k+1})\cdot\mathcal{G}\cdot\frac{dt_{k+1}}{t_{k+1}}$$

where $\mathcal{G}$ is given by (A7), except that in this case we have $\tilde{\rho} = \rho \leq k$. Again we may assume that $x_{k+1} = 1$, and we integrate by parts of obtain

$$(2i)^{-1}I = \frac{(1 - \cos(t_{k+1}))}{t_{k+1}} \cdot \mathcal{G}\Big|_{t_{k+1}=\varepsilon}^{t_{k+1}=N}$$

$$- \int_\varepsilon^N (1 - \cos(t_{k+1})) \cdot \left(t_{k+1}^{-1} \frac{\partial \mathcal{G}}{\partial t_{k+1}} - t_{k+1}^{-2} \mathcal{G} \right) dt_{k+1}.$$

As we observed in the case $\rho = k + 1$, $\mathcal{G}$ and $t_{k+1}\partial\mathcal{G}/\partial t_{k+1}$ are appropriately bounded, so the required estimate for I follows immediately. This completes the induction step, and Lemma A2 follows.

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