PACIFIC JOURNAL OF MATHEMATICS
Vol 111, No 1, 1984

SOME CONDITIONS ON THE HOMOLOGY GROUPS OF THE KOSZUL COMPLEX

CARLA MASSAZA AND ALFIO RAGUSA

In this paper we introduce the concept of a (d, i)-sequence $(d, i \in N)$ in a commutative ring A, noetherian and with identity (cf. Def. 1.1). Let $K(\underline{z}, A)$ be the Koszul complex on A, with respect to the sequence $\underline{z} = z_1, \ldots, z_n$: the concept of a (d, i)-sequence is expressed in terms of the structure of $H_i(K(\underline{z}, A))$; in particular, it turns out that $\underline{z}$ is an (n, i)-sequence iff $H_i(K(\underline{z}, A)) = 0$, and such a condition implies $\underline{z}$ is a (d, i)-sequence for any $d \leq n$. If $\bar{z}_1, \ldots, \bar{z}_h$ is a (d, i)-sequence in ${}_h\bar{A} = A/(z_{h+1}, \ldots, z_n)$, $d \leq h \leq n$, then $\underline{z}$ is seen to be a (d, i)-sequence in A; so, in particular, if $H_i(K(\bar{\underline{z}}; {}_d\bar{A})) = 0$ in ${}_d\bar{A}$, then $\underline{z}$ is a (d, i)-sequence. Moreover, for $i = 1$, the two conditions are equivalent, so that *$\underline{z}$ is a $(d, 1)$-sequence* means precisely that $\bar{z}_1, \ldots, \bar{z}_d$ is regular in ${}_d\bar{A}$. For $i > 1$, examples show that *$\underline{z}$ is a (d, i)-sequence* is a condition strictly weaker than *$\bar{z}_1, \ldots, \bar{z}_h$ is a (d, i)-sequence in ${}_h\bar{A}$*, and we investigate the relationship between those two properties. In fact, their equivalence allows us to read the depth of a quotient ring $A/(z_{h+1}, \ldots, z_n)$ in terms of the Koszul complex $K(\underline{z}; A)$ and implies, for (d, i)-sequences, properties which are a natural generalization of *good properties* satisfied by regular sequences, such as the depth-sensitivity of the Koszul complex. A characteristic condition for their equivalence is a kind of weak surjectivity of a natural map acting between $\operatorname{syz}^{i+1}(K(\underline{z}; A))$ and $\operatorname{syz}^{i+1}(K(\bar{\underline{z}}; {}_h\bar{A}))$.

From an algebraic form of that weak surjectivity we get some sufficient conditions, in terms of weak regularity of the sequence $z_{h+1}, \ldots, z_n$. For instance, if $z_{h+1}, \ldots, z_n$ is a d-sequence, or a relative regular sequence, or less, if $z_{h+1}, \ldots, z_n$ is a relative regular A-sequence with respect to a convenient set of ideals, then *$\underline{z}$ is a (d, i)-sequence in A* implies *$\bar{z}_1, \ldots, \bar{z}_h$ is a (d, i)-sequence in ${}_h\bar{A}$*.

Moreover, if $\underline{z}$ is a (d, i)-sequence and $z_{d+1}, \ldots, z_n$ is a regular sequence, then $H_i(K(\underline{z}; A)) = 0$, while this vanishing implies that it is possible to find $x_i, \ldots, x_n$ in $I = (z_1, \ldots, z_n)$ such that $z_1, \ldots, z_{i-1}, x_i, \ldots, x_n$ is a (d, i)-sequence and $x_{d+1}, \ldots, x_n$ is a regular sequence.

In the last section we give an interpretation of our results in terms of the behaviour of some systems of linear equations.

N. 1. Let A be a noetherian ring (with 1) and $\underline{z} = z_1, \ldots, z_n$ a sequence of elements of A such that $(z_1, \ldots, z_n)A \neq A$. We denote by $K(\underline{z}; A)$ the Koszul complex with respect to $\underline{z}$, i.e. the differential graded algebra (DGA for short) (cf. **[G-L]** cap. I for a definition)

$$0 \to \wedge^n A^n \xrightarrow{d_n} \wedge^{n-1} A^n \to \cdots \to \wedge^2 A^n \xrightarrow{d_2} A^n \xrightarrow{d_1} A \xrightarrow{d_0} s A/(\underline{z})A \to 0$$

generated by e_i, $i = 1,\ldots,n$, with differential

$$d_j\left(e_{j_1} \wedge \cdots \wedge e_{i_j}\right) = \sum_{t=1,\ldots,j} (-1)^{t+1} z_{i_t} \cdot e_{i_1} \wedge \cdots \wedge \check{e}_{i_t} \wedge \cdots \wedge e_{i_j}.$$

Also we write

$$\operatorname{syz}^i(K(\underline{z}; A)) = \ker(d_{i-1}) \subseteq \wedge^{i-1} A^n$$

for $i = 1,\ldots,n+1$. As in **[M-R]**, for every $1 \leq i \leq d \leq n$, $T_i^{(n,d)}$ will mean the free A-module generated by $e_{j_1\cdots j_i} = e_{j_1} \wedge \cdots \wedge e_{j_i}$, with $1 \leq j_1 < \cdots < j_i \leq n$ and $j_i > d$, which is a complementary module of $\wedge^i(Ae_1 \oplus \cdots \oplus Ae_d)$, briefly $\wedge^i A_{1\cdots d}$, in $\wedge^i A^n$, so $\wedge^i A^n = \wedge^i A_{1\cdots d} \oplus T_i^{(n,d)}$.

Then

$$\pi_i : \wedge^i A^n \to T_i^{(n,d)},$$
$$\chi_i : \wedge^i A^n \to \wedge^i A_{1\cdots d}$$

will be the usual projections, i.e.

$$\pi_i\left(\sum_{1\leq j_1<\cdots<j_i\leq n} a_{j_1\cdots j_i} e_{j_1\cdots j_i}\right) = \sum_{1\leq j_1<\cdots<j_i\leq n} a_{j_1\cdots j_i} e_{j_1\cdots j_i},$$

$$\chi_i\left(\sum_{1\leq j_1<\cdots<j_i\leq n} a_{j_1\cdots j_i} e_{j_1\cdots j_i}\right) = \sum_{1\leq j_1<\cdots<j_i\leq d} a_{j_1\cdots j_i} e_{j_1\cdots j_i}$$

and, more generally,

$$\chi_i^h : \wedge^i A^n \to \wedge^i A_{1\cdots h} \qquad (h \leq n)$$

will be like χ_i when we set $d = h$.

When $z_1,\ldots,z_n$ are fixed elements of A, we write

$${}_t\bar{A} = A/(z_{t+1},\ldots,z_n)A$$

for $t = 0,\ldots,n$ (${}_n\bar{A} = A$), and $\bar{z}_i \in {}_t\bar{A}$ means the image of z_i by the natural map ρ_t: $A \to {}_t\bar{A}$.

For every two integers $s \geq r$ we can define a map of DGAs,

$$\Psi^{(s,r)}: K\left(\bar{z}_1,\ldots,\bar{z}_s; {}_s\bar{A}\right) \to K\left(\bar{z}_1,\ldots,\bar{z}_r; {}_r\bar{A}\right), \tag{1}$$

by

$$\Psi_0^{(s,r)} = \text{the natural map } {}_s\bar{A} \to {}_r\bar{A}$$

and

$$\Psi_1^{(s,r)}(e_i) = \begin{cases} f_i & \text{for } 1 \leq i \leq r, \\ 0 & \text{for } r < i \leq s, \end{cases}$$

where $e_1,\ldots,e_s$ and $f_1,\ldots,f_r$ are, respectively, the free generators of $K(\bar{z}_1,\ldots,\bar{z}_s;\ {}_s\bar{A})$ and $K(\bar{z}_1,\ldots,\bar{z}_r;\ {}_r\bar{A})$. $\Psi^{(n;h)}$ will be denoted simply by Ψ^h. Finally, we make the usual convention of setting

$$c_{j_1\cdots j_t} = (-1)^\delta c_{\tau(j_1)\cdots\tau(j_t)},$$

where τ is a permutation on $\{j_1,\ldots,j_t\}$ and δ is 0 or 1 according to whether τ is an even or an odd permutation.

DEFINITION 1.1. Let $\underline{z} = z_1,\ldots,z_n$ be a system of non invertible elements of a ring A and i, d integers such that $1 \le i \le d \le n$. We say that $\underline{z}$ is a (d, i)*-sequence if $H_i(K(\underline{z};\ A))$ can be generated by the image of $T_i^{(n,d)}$. We say that $\underline{z}$ is a (d, i)-sequence if it is a (d, i)*-sequence and $(z_1,\ldots,z_n)A \subseteq \operatorname{rad} A$.

REMARK 1.2. (i) Obviously $\{z_1,\ldots,z_d, z_{d+1},\ldots,z_n\}$ is a (d, i)-sequence if and only if $\{z_{\tau(1)},\ldots,z_{\tau(d)}, z_{\sigma(d+1)},\ldots,z_{\sigma(n)}\}$ is a (d, i)-sequence, where τ and σ are permutations on $\{1,\ldots,d\}$ and $\{d+1,\ldots,n\}$, respectively.

(ii) Any (d, i)-sequence is a (d', i)-sequence, for $d' \le d$.

(iii) (d, i)*-sequences go up and down by faithful flatness. In fact, if $f\colon A \to B$ is a morphism, then

$$K(\underline{z};\ A) \otimes_A B = K(f(\underline{z});\ B) \quad \text{and} \quad d_i\left(\wedge^i A^n\right) \otimes_A B = d_i\left(\wedge^i B^n\right).$$

Now, the (d, i)* condition says in A or, respectively, in B

1. $\operatorname{syz}_A^{i+1}(A/\underline{z}A) = d_{i+1}\left(\wedge^{i+1}A^n\right) + \left[T_i^{(n,d)}(A) \cap \operatorname{syz}_A^{i+1}(A/\underline{z}A)\right]$

2. $\operatorname{syz}_B^{i+1}(B/f(\underline{z})B)$
$$= d_{i+1}\left(\wedge^{i+1}B^n\right) + \left[T_i^{(n,d)}(B) \cap \operatorname{syz}_B^{i+1}(B/f(\underline{z})B)\right].$$

If f is faithfully flat, then

$$\operatorname{syz}^{i+1}(A/\underline{z}A) \otimes_A B \simeq \operatorname{syz}^{i+1}(B/f(\underline{z})B),$$

and for every two A-modules M and N,

$$(M \otimes_A B) \cap (N \otimes_A B) \simeq (M \cap N) \otimes_A B$$

so that 2. comes from 1. by tensoring with B; again by faithful flatness the conclusion follows.

(iv) For $n = d$, $\underline{z}$ is an (n, i)-sequence iff $H_i(K(\underline{z};\ A)) = 0$; so, in particular, if $\operatorname{depth}(z_1,\ldots,z_n) \ge n - i + 1$, then $\underline{z}$ is a (d, j)-sequence, for every d, j such that $i \leq j \leq d \leq n$ (because of (ii) and the depth-sensitivity of the Koszul complex [**A-B**]).

(v) If d_i is the largest integer such that $\underline{z}$ is a (d_i, i)-sequence, $n - d_i$ gives a *measure* of the obstruction to $\underline{z}$ having depth bigger than or equal to $n - i + 1$; in particular, for $i = 1$, $n - d_1$ says how far $z_1, \dots, z_n$ is from regularity.

THEOREM 1.3. *Let $\underline{z} = z_1, \dots, z_n$ be a system of elements of a ring A, with $(z_1, \dots, z_n)A \subseteq \operatorname{rad} A$, and i, d, h integers such that $1 \leq i \leq d \leq h \leq n$. If $\bar{z}_1, \dots, \bar{z}_h$ is a (d, i)-sequence in ${}_h\bar{A}$, then $z_1, \dots, z_n$ is a (d, i)-sequence in A.*

Proof. Let

$$\alpha = \sum_{1 \leq j_1 < \cdots < j_i \leq n} a_{j_1 \cdots j_i} e_{j_1 \cdots j_i}$$

be an element in $\operatorname{syz}^{i+1}(K(\underline{z}; A))$. Since Ψ^h is a map of complexes,

$$\Psi_i^h(\alpha) = \sum_{1 \leq j_1 < \cdots < j_i \leq h} \bar{a}_{j_1 \cdots j_i} f_{j_1 \cdots j_i}$$

will be in $\operatorname{syz}^{i+1}(K(\bar{\underline{z}}; \bar{A}))$. Then the hypothesis on ${}_h\bar{A}$ says

$$\sum_{1 \leq j_1 < \cdots < j_i \leq d} \bar{a}_{j_1 \cdots j_i} f_{j_1 \cdots j_i} = d_{i+1}\bar{\beta} + \sum_{\substack{1 \leq j_1 < \cdots < j_i \leq h \\ j_i > d}} \bar{b}_{j_1 \cdots j_i} f_{j_1 \cdots j_i}$$

for some $\bar{\beta} \in \wedge^{i+1}{}_h\bar{A}^h$ and $\bar{b}_{j_1 \cdots j_i} \in {}_h\bar{A}$.

From this we get

$$\begin{aligned}\sum_{1 \leq j_1 < \cdots < j_i \leq d} a_{j_1 \cdots j_i} e_{j_1 \cdots j_i} = d_{i+1}\beta &+ \sum_{\substack{1 \leq j_1 < \cdots < j_i \leq h \\ j_i > d}} b_{j_1 \cdots j_i} e_{j_1 \cdots j_i} \\ &+ \sum_{\substack{1 \leq j_1 < \cdots < j_i \leq n \\ j_i > h}} c_{j_1 \cdots j_i} e_{j_1 \cdots j_i} \\ &+ \sum_{1 \leq j_1 < \cdots < j_i \leq h} \left(\sum_{t=h+1}^{n} c_{j_1 \cdots j_i} z_t e_{j_1 \cdots j_i} \right)\end{aligned}$$

for some $\beta \in \wedge^{i+1}A^n$, $c_{j_1 \cdots j_i} \in A$, for $1 \leq j_1 < \cdots < j_i \leq n$ and for any lifting $b_{j_1 \cdots j_i}$ of $\bar{b}_{j_1 \cdots j_i}$.

Now we can conclude the proof just by remarking that

$$z_k e_{j_1 \cdots j_i} = d_{i+1}(e_{j_1 \cdots j_i}) + \alpha \wedge e_k$$

for every $k \neq j_1, \dots, j_i$, for some $\alpha \in \wedge^{i-1}A^n$.

COROLLARY 1.4. *If* $\operatorname{depth}(\bar{z}_1,\ldots,\bar{z}_d) \geq d-i+1$ *in* ${}_d\bar{A}$, *then* $z_1,\ldots,z_n$ *is a* (d, i)*-sequence in* A.

Proof. Use Remark 1.2 and then apply Theorem 1.3 for $h = d$.

Theorem 1.3 allows us to lift (d, i)-sequences from ${}_h\bar{A}$ to A, but, conversely, a (d, i)-sequence in A does not necessarily remain a (d, i)-sequence in ${}_h\bar{A}$, as we can see from the following

EXAMPLE 1.5. Let $A = k[[X, Y, Z]]/I$, $I = (X^2 - Z^2, XY, XZ)$, k a field, and denote by x, y, z the images of X, Y, Z in A. We show that x, y, z is a $(2,2)$-sequence in A, but, in $\bar{A} = A/(z)$, $\bar{x}, \bar{y}$ is not, i.e. $\operatorname{depth}(\bar{x}, \bar{y}) = 0$ in $\bar{A}$. The second fact is trivial since $A/(z) = k[[X, Y]]/X(X, Y)$ so $\operatorname{depth}(A/(z)) = 0$.

Now let $\beta = a_{12}e_1 \wedge e_2 + a_{13}e_1 \wedge e_3 + a_{23}e_2 \wedge e_3$ be an element of $\operatorname{syz}^3(K(x, y, z; A))$; this says

$$a_{12}y + a_{13}z = 0, \quad -a_{12}x + a_{23}z = 0, \quad a_{13}x + a_{23}y = 0, \tag{2}$$

and we have to show $a_{12} \in (z)A$. Since $\bar{y}$ is a regular element in $A/(x)$, from the third equation in (2) we get $a_{23} = \lambda x$, for some $\lambda \in A$. On the other hand, in $A/(z)A$ we have $\operatorname{ann}(\bar{x}, \bar{y}) = (\bar{x})\bar{A}$, so from the first and second equations of (2) we have $a_{12} = \mu x + \nu z$ for some $\mu, \nu \in A$. Now the second equation in (2) becomes $\mu x^2 = 0$, so $\mu x \in \operatorname{ann}(x)$, but $\mu x \in \operatorname{ann}(y)$ and in A we have $\operatorname{ann}(x) \cap \operatorname{ann}(y) = (y, z)A \cap (x)A = (z^2)A$. Thus $a_{12} \in (z)A$.

Corollary 1.4 suggests investigating the condition we need in order to have the relationship

$$\begin{aligned} &z_1,\ldots,z_n \text{ is a } (d, i)\text{-sequence in } A \\ &\Rightarrow \operatorname{depth}(\bar{z}_1,\ldots,\bar{z}_d) \geq d-i+1 \quad \text{in } {}_d\bar{A}. \end{aligned} \tag{3}$$

We know this is always verified, for $i = 1$, for a local ring $(A, \mathfrak{m})$ and $(z_1,\ldots,z_n)A = \mathfrak{m}$ (see **[M-R]** Theorem 2.5); here we prove it without any assumption on A and $z_1,\ldots,z_n$ (cf. Corollary 2.4).

For $i \geq 2$ we certainly need some conditions (Example 1.5 says (3) is not generally true). We will see that (3) holds whenever $z_{d+1},\ldots,z_n$ is a regular sequence; nevertheless there are weaker conditions which imply our relation. We will dedicate the next two sections to investigating such a question and some related ones. For this, most of our technique depends on the solutions of some systems of linear equations and their subsystems, and we will devote the last part of the paper to explaining this idea.

N. 2. The equality $\Psi_i^h(\operatorname{syz}^{i+1}(K(\underline{z}; A))) = \operatorname{syz}^{i+1}(K(\underline{\bar{z}}; {}_h\bar{A}))$ (cf. (1) N. 1) clearly implies that if $z_1, \ldots, z_h$ is a (d, i)-sequence in A, then $\bar{z}_1, \ldots, \bar{z}_h$ is a (d, i)-sequence of ${}_h\bar{A}$. However, such a condition is not a necessary one, because the (d, i)-condition gives a link only between those components of the elements of $\operatorname{syz}^{i+1}(K(\underline{z}; A))$ which lie in $\wedge^i A_{1\ldots d}$. It is easy to check that (d, i)-sequences descend from A to ${}_h\bar{A}$ if the following condition is verified:

Weak Lifting Condition $(\text{W.L.C.})_{h,i}$. Let

$$\bar{\alpha} = \bar{a} + \bar{b} \in \operatorname{syz}^{i+1}\big(K\big(\underline{\bar{z}}; {}_h\bar{A}\big)\big),$$

where $\bar{a} \in \wedge^i {}_h\bar{A}_{1\ldots d}$, $\bar{b} \in T_i^{(h,d)}$ (as ${}_h\bar{A}$-module). Then there is $\alpha = a + b \in \operatorname{syz}^{i+1}(K(\underline{z}; A))$, where $a \in \wedge^i A_{1\ldots d}$, $b \in T_i^{(n,d)}$ (as A-module), such that $\Psi_i^h(a) = \bar{a}$. We will call α a weak lifting of $\bar{\alpha}$.

REMARK 2.1. $(\text{W.L.C.})_{d,i}$ is equivalent to $\Psi_i^d(\operatorname{syz}^{i+1}(K(\underline{z}; A))) = \operatorname{syz}^{i+1}(K(\underline{\bar{z}}; {}_d\bar{A}))$, because in such a situation we have $\bar{b} = 0$; in other words, $(\text{W.L.C.})_{d,i}$ is exactly the surjectivity of the restriction of Ψ_i^d to the syzygies.

Now, let us prove that $(\text{W.L.C.})_{h,i}$ is also necessary to pass (d, i)-sequences from A to ${}_h\bar{A}$.

PROPOSITION 2.2. *Let $z_1, \ldots, z_n$ be a (d, i)-sequence in A. Then $\bar{z}_1, \ldots, \bar{z}_h$ is a (d, i)-sequence in ${}_h\bar{A}$ if and only if* $(\text{W.L.C.})_{h,i}$ *holds.*

Proof. We already observed that $(\text{W.L.C.})_{h,i}$ is sufficient. Let us prove its necessity. With the same notation as in $(\text{W.L.C.})_{h,i}$, let $\bar{\alpha} = \bar{a} + \bar{b} \in \operatorname{syz}^{i+1}(K(\underline{\bar{z}}; {}_h\bar{A}))$, where

$$\bar{a} = \sum_{1 \le j_1 < \cdots < j_i \le d} \bar{a}_{j_1\cdots j_i} f_{j_1\cdots j_i}.$$

The (d, i)-condition on the sequence says that

$$\bar{a}_{j_1\cdots j_i} = \sum_{t \in \{1,\ldots,h\} - \{j_1,\ldots,j_i\}} \bar{c}_{j_1\cdots t\cdots j_i} \cdot \bar{z}_t, \qquad 1 \le j_1 < \cdots < j_i \le d.$$

We lift these relations to A, defining

$$a_{j_1\cdots j_i} = \sum_{t \in \{1,\ldots,h\} - \{j_1,\ldots,j_i\}} c_{j_1\cdots t\cdots j_i} \cdot z_t, \qquad 1 \le j_1 < \cdots < j_i \le d,$$

with $\rho_h(c_{j_1\cdots t\cdots j_i}) = \bar{c}_{j_1\cdots t\cdots j_i}$, and finally

$$a = \sum_{1\le j_1<\cdots<j_i\le d} a_{j_1\cdots j_i} e_{j_1\cdots j_i}.$$

Moreover, we define $b \in T^{(n,h)}$ as follows:

$$b = \sum_{\substack{1\le j_1<\cdots<j_{i-1}\le d\\ d<j_i\le h}} b_{j_1\cdots j_i} e_{j_1\cdots j_i},$$

where

$$b_{j_1\cdots j_i} = \sum_{t\in\{1,\ldots,d\}-\{j_1,\ldots,j_i\}} c_{j_1\cdots t\cdots j_i}\cdot z_t,$$
$$1\le j_1<\cdots<j_{i-1}\le d,\quad d<j_i\le h.$$

Since trivially $\Psi_i^h(a) = \bar{a}$, it is enough to prove that

$$a + b \in \operatorname{syz}^{i+1}(K(\underline{z}; A)).$$

The coefficient of $e_{j_1\cdots j_{i-1}}$, $1\le j_1<\cdots<j_{i-1}\le d$, in $d_i(a+b)$, is

$$\begin{aligned}
&\sum_{u\in\{1,\ldots,d\}-\{j_1,\ldots,j_{i-1}\}} a_{j_1\cdots u\cdots j_{i-1}}\cdot z_u + \sum_{r=d+1}^{h} b_{j_1\cdots j_{i-1}r}\cdot z_r\\
&\quad= \sum_{u\in\{1,\ldots,d\}-\{j_1,\ldots,j_{i-1}\}}\Bigg(\sum_{t\in\{1,\ldots,d\}-\{j_1,\ldots,j_{i-1},u\}}\\
&\qquad\qquad\times c_{j_1\cdots u\cdots t\cdots j_{i-1}}\cdot z_t + \sum_{t=d+1}^{h} c_{j_1\cdots u\cdots j_{i-1}t}\cdot z_t\Bigg) z_u\\
&\qquad+ \sum_{r=d+1}^{h}\Bigg(\sum_{t\in\{1,\ldots,d\}-\{j_1,\ldots,j_{i-1}\}} c_{j_1\cdots t\cdots j_{i-1}r}\cdot z_t\Bigg) z_r\\
&\quad= \sum_{\substack{u,t\in\{1,\ldots,d\}-\{j_1,\ldots,j_{i-1}\}\\ u\ne t}} c_{j_1\cdots u\cdots t\cdots j_{i-1}}\cdot z_t z_u\\
&\qquad+ \sum_{u\in\{1,\ldots,d\}-\{j_1,\ldots,j_{i-1}\}}\ \sum_{t=d+1}^{h} c_{j_1\cdots u\cdots j_{i-1}t}\cdot z_t z_u\\
&\qquad+ \sum_{r=d+1}^{h}\ \sum_{t\in\{1,\ldots,d\}-\{j_1,\ldots,j_{i-1}\}} c_{j_1\cdots t\cdots j_{i-1}r}\cdot z_t z_r,
\end{aligned}$$

which is zero, because for $p \ne m$, p, $m = 1,\ldots,d$, the coefficient of $z_p z_m$ is

$$(\dagger)\qquad c_{j_1\cdots p\cdots m\cdots j_{i-1}} + c_{j_1\cdots m\cdots p\cdots j_{i-1}} = 0,$$

coming from the first Σ, and for $1 \leq p \leq_i$ and $d < m \leq h$, the coefficient of $z_p z_m$ is again (†), coming from the second and the third Σ. Similarly, the coefficient of $e_{j_1 \cdots j_{i-1}}$, $1 \leq j_1 < \cdots < j_{i-2} \leq d$, $d < j_{i-1} \leq h$, in $d_i(a+b)$ is

$$\sum_{u \in \{1,\ldots,d\} - \{j_1,\ldots,j_{i-2}\}} b_{j_1 \cdots u \cdots j_{i-1}} \cdot z_u =$$
$$= \sum_{u \in \{1,\ldots,d\} - \{j_1,\ldots,j_{i-2}\}} \; \sum_{t \in \{1,\ldots,d\} - \{j_1,\ldots,j_{i-2},u\}} c_{j_1 \cdots u \cdots t \cdots j_{i-2}} \cdot z_u z_t = 0$$

with the same computation.

Theorem 1.3 and Proposition 2.2 can be restated as follows.

THEOREM 2.3. *The following conditions are equivalent*
(1) $z_1, \ldots, z_n$ *is a* (d, i)*-sequence in* A *and* (W.L.C.)$_{h,i}$ *holds.*
(2) $\bar{z}_1, \ldots, \bar{z}_h$ *is a* (d, i)*-sequence in* ${}_h\bar{A}$.

In particular, for $h = d$, Theorem 2.3 becomes

COROLLARY 2.4. *The following conditions are equivalent*:
(1) $z_1, \ldots, z_n$ *is a* (d, i)*-sequence in* A *and* (W.L.C.)$_{d,i}$ *holds.*
(2) $\operatorname{depth}(\bar{z}_1, \ldots, \bar{z}_d) \geq d - i + 1$ *in* ${}_d\bar{A}$.

The case $i = 1$ looks much simpler because of

PROPOSITION 2.5. *Condition* (W.L.C.)$_{h,1}$ *is always verified, as* $\Psi_1^h(\operatorname{syz}^2(K(\underline{z}; A))) = \operatorname{syz}^2(K(\underline{\bar{z}}; {}_h\bar{A}))$.

Proof. It is clearly enough to show the equality for $h = n - 1$ because then we use induction. If

$$\sum_{i=1}^{n-1} \bar{a}_i f_i \in \operatorname{syz}^2\big(K\big(\underline{\bar{z}}; {}_{n-1}\bar{A}\big)\big),$$

we have $\sum_{i=1}^{n-1} \bar{a}_i \bar{z}_i = 0$; so there exist $a_1, \ldots, a_n$ such that $\sum_{i=1}^{n} a_i z_i = 0$, which implies $\sum_{i=1}^{n} a_i e_i \in \operatorname{syz}^2(K(\underline{z}; A))$ and $\Psi_1^{(n-1)}(\sum_{i=1}^{n} a_i e_i) = \sum_{i=1}^{n-1} \bar{a}_i f_i$.

As a consequence of Proposition 2.5, we get

PROPOSITION 2.6. *The following are equivalent*:
(1) $z_1, \ldots, z_n$ *is a* $(d, 1)$*-sequence in* A.
(2) $\bar{z}_1, \ldots, \bar{z}_d$ *is a* $(d, 1)$*-sequence in* ${}_d\bar{A}$.

(3) $\bar{z}_1,\ldots,\bar{z}_d$ *is a regular sequence of* ${}_d\bar{A}$.

(4) $(z_1,\ldots,z_{i-1}, z_{d+1},\ldots,z_n)\colon z_i = (z_1,\ldots,z_{i-1}, z_{d+1},\ldots,z_n)$, $i = 1,\ldots,d$, $(z_0 = 0)$.

Proof. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from Proposition 2.5, Corollary 2.4 and Remark 1.2. The equivalence between (3) and (4) is just an easy computation, translating the definition of regularity for the sequence $\bar{z}_1,\ldots,\bar{z}_d$.

From condition (3) of Proposition 2.6 and Remark 1.2(i), we immediately get

COROLLARY 2.7. *If* $z_1,\ldots,z_d, z_{d+1},\ldots,z_n$ *is a* $(d,1)$*-sequence, then* $x_1,\ldots,x_{d-s}, z_{d+1},\ldots,z_n$ *is a* $(d-s,1)$*-sequence, where* $x_1,\ldots,x_{d-s}$ *is any non empty subset of* $z_1,\ldots,z_d$.

REMARK 2.8. If $(\text{W.L.C.})_{d,i}$ holds, Corollary 2.7 can easily be generalized to the i case; so, in this case, if $z_1,\ldots,z_n$ is a (d,i)-sequence, then $x_1,\ldots,x_{d-s}, z_{d+1},\ldots,z_n$ is a $(d-s,i)$-sequence for $x_1,\ldots,x_{d-s}$ any subset of $z_1,\ldots,z_d$, $0 \le s \le d-i$.

REMARK 2.9. Let $(A,\mathfrak{m})$ be a local ring and $z_1,\ldots,z_n$ a set of generators of $\mathfrak{m}$; then $\bar{z}_1,\ldots,\bar{z}_d$ is a set of generators of $\bar{\mathfrak{m}}$ in ${}_d\bar{A}$, so condition (3) of Proposition 2.6 says that ${}_d\bar{A}$ is a regular ring. Moreover, the $(d,1)$-condition on $\underline{z}$ is the same as condition $\mathbf{R}_2^d$ defined for a projective resolution of $A/\mathfrak{m}$ in [**M-R**]; so, Proposition 2.6 implies Theorems 2.5 and 2.7 of loc. cit.

Now we can generalize Corollary 2.12 of [**M-R**].

PROPOSITION 2.10. *If* $z_1,\ldots,z_n$ *is a* $(d,1)$*-sequence of* A *and* $I = (z_{d+1},\ldots,z_n)$ *has co-height* $\le d$ *(i.e.* $\dim(A/I) \le d$*), then* A/I *is Cohen-Macaulay. In particular, if* $(A,\mathfrak{m})$ *is local and* $(z_1,\ldots,z_n) = \mathfrak{m}$, *then* A/I *is regular.*

Proof. Applying Remark 1.2(iii) and Proposition 2.6 we get $\operatorname{depth}(A/I)_{\mathfrak{p}} \ge d$ for every $\mathfrak{p} \in \operatorname{Max}(A/I)$; since, by hypothesis, $\dim(A/I) \le d$, the conclusion follows.

By Remark 1.2(ii) if $z_1,\ldots,z_n$ is a (d,i)-sequence, it is also a (d',i)-sequence for every $d' \le d$ $(i \le d')$. It seems meaningful to ask whether every (d,i)-sequence is also a (d,j)-sequence for $j \ge i$ $(j \le d)$. For $d = n$

this becomes the well-known rigidity of the Koszul complex (see for instance **[G-L]**).

Another partial answer to this question is given by

PROPOSITION 2.11. *If $z_1,\dots,z_n$ is a $(d,1)$-sequence, then it is a (d,i)-sequence for every $i \geq 1$.*

Proof. By Proposition 2.6 and the (d, 1)-condition on $\underline{z}$, we have $\bar{z}_1,\dots,\bar{z}_d$ is a $(d,1)$-sequence in ${}_d\bar{A}$, which means $H_1(K(\underline{\bar{z}}; {}_d\bar{A})) = 0$. Now, the mentioned rigidity of the Koszul complex implies $H_i(K(\underline{\bar{z}}; {}_d\bar{A})) = 0$ for every $i > 1$. Then Theorem 1.3 says $z_1,\dots,z_n$ is a (d,i)-sequence in A.

The well-known depth-sensitivity of the Koszul complex says, in particular, that if $H_i(K(\underline{z}; A)) = 0$ then there exist $n-i+1$ elements $x_1,\dots,x_{n-i+1}$ in $(z_1,\dots,z_n)$ which form a regular sequence, i.e. $H_1(K(x_1,\dots,x_{n-i+1}; A)) = 0$. Now we prove a sort of $(d,1)$-*sensitivity* of the Koszul complex. Namely, we have

THEOREM 2.12. *If $z_1,\dots,z_n$ is a (d,i)-sequence and* $(\mathrm{W.L.C.})_{\mathrm{d,i}}$ *holds, then for every s, $0 \leq s < i$, there exist $x_1,\dots,x_{d-s} \in (z_1,\dots,z_d)$ such that $x_1,\dots,x_{d-s}, z_{d+1},\dots,z_n$ is a $(d, i-s)$-sequence. In particular, we can find $d-i+1$ elements in $(z_1,\dots,z_d)$ such that $x_1,\dots,x_{d-i+1}, z_{d+1},\dots,z_n$ is a $(d,1)$-sequence.*

Proof. By Corollary 2.4 $H_i(K(\underline{\bar{z}}; {}_d\bar{A})) = 0$, so, for $0 \leq s < i$, we can find $\bar{x}_1,\dots,\bar{x}_{d-s} \in (\bar{z}_1,\dots,\bar{z}_d)$ such that $H_{i-s}(K(\bar{x}_1,\dots,x_{d-s}; {}_d\bar{A})) = 0$. Now from Corollary 1.4 we get the desired result.

The $(\mathrm{W.L.C.})_{d,i}$ condition in the previous theorem seems to be necessary (in some sense) to the above $(d,1)$-*sensitivity* of the Koszul complex. In fact, if $(\mathrm{W.L.C.})_{d,i}$ does not hold in ${}_d\bar{A} = A/(z_{d+1},\dots,z_n)$, then Proposition 2.2 says we can find a (d,i)-sequence, $z_1,\dots,z_d, z_{d+1},\dots,z_n$, such that $\bar{z}_1,\dots,\bar{z}_d$ is not a (d,i)-sequence in ${}_d\bar{A}$. Now, let $1 \leq j < i$ be such that $(\mathrm{W.L.C.})_{d,j}$ holds in ${}_d\bar{A}$; then we cannot find $d-i+j$ elements, say $x_1,\dots,x_{d-i+j} \in (z_1,\dots,z_d)$, such that $x_1,\dots,x_{d-i+j}, z_{d+1},\dots,z_n$ is a $(d-i+j, j)$-sequence. Namely, otherwise it should be $H_j(K(\bar{x}_1,\dots,\bar{x}_{d-i+j}; {}_d\bar{A})) = 0$, which implies $H_i(K(\bar{x}_1,\dots,\bar{x}_{d-i+j}; {}_d\bar{A})) = 0$, and also $H_i(K(\bar{z}_1,\dots,\bar{z}_d; {}_d\bar{A})) = 0$, which means, by Corollary 2.4, that $\bar{z}_1,\dots,\bar{z}_d$ should be a (d,i)-sequence.

Therefore, for $j = 1$, using Proposition 2.5, we get

PROPOSITION 2.13. *If for any (d,i)-sequence $z_1,\dots,z_d, z_{d+1},\dots,z_n$, with fixed tail $z_{d+1},\dots,z_n$, we can find a $(d-i+1, 1)$-sequence*

$x_1, \ldots, x_{d-i+1}$, $z_{d+1}, \ldots, z_n$ *with* $x_1, \ldots, x_{d-i+1} \in (z_1, \ldots, z_d)$, *then* (W.L.C.)$_{d,i}$ *must hold for* ${}_d\bar{A} = A/(z_{d+1}, \ldots, z_n)$.

In order to investigate the behavior of (d, i)-sequences when we pass to a quotient with respect to elements of its *head* (the first d elements), let us denote by

$$\phi\colon K(z_1, \ldots, z_n; A) \to K\big(\tilde{z}_2, \ldots, \tilde{z}_n; \tilde{A} = A/(z_1)\big)$$

the usual map of DGA, defined by

$$\phi_0\colon A \overset{\text{nat}}{\to} A/(z_1),$$

$$\phi_1(e_i) = \begin{cases} f_i & \text{for } i \geq 2, \\ 0 & \text{for } i = 1, \end{cases}$$

where $\{e_i\}_{i=1,\ldots,n}$ and $\{f_i\}_{i=2,\ldots,n}$ are free generators of $K_1(\underline{z}; A)$ and $K_1(\underline{\tilde{z}}; \tilde{A})$, respectively, and denote by

$$\tilde{\phi}_i\colon H_i \to \tilde{H}_i \quad \text{and} \quad \phi_i^*\colon H_i/T_i^{(n,d)}H_i \to \tilde{H}_i/\tilde{T}_i^{(n-1,d-1)}\tilde{H}_i$$

the induced maps, where $H_i = H_i(K(\underline{z}; A))$ and $\tilde{H}_i = H_i(K(\underline{\tilde{z}}; \tilde{A}))$.

The crucial fact for our purpose is

LEMMA 2.14. *With the above notation,*
(i) *if* z_1 *is regular in* A, *then* ϕ_i^* *is surjective,*
(ii) *if* $\bar{z}_1$ *is regular in* ${}_d\bar{A}$, *then* ϕ_i^* *is injective.*

Proof. (i) is trivial since the regularity of z_1 in A implies the surjectivity on the induced map

$$\tilde{\phi}_i\colon \operatorname{syz}^{i+1}(K(\underline{z}; A)) \to \operatorname{syz}^{i+1}\big(K(\underline{\tilde{z}}; \tilde{A})\big).$$

(ii) Let $[\alpha]$ be an element of $H_i(K(\underline{z}; A))$ and suppose $\tilde{\phi}_i([\alpha]) \in \tilde{T}_i^{(n-1,d-1)}\tilde{H}_i$; then

$$[\alpha] = \left[\sum_{\substack{2 \leq j_1 < \cdots < j_i \leq n \\ j_i > d}} a_{j_1 \cdots j_i} e_{j_1 \cdots j_i} + \sum_{2 \leq j_2 < \cdots < j_i \leq n} a_{1 j_2 \cdots j_i} e_{1 j_2 \cdots j_i}\right].$$

Since α is a cycle, for every $2 \leq j_2 < \cdots < j_i \leq d$, we have

$$a_{1 j_2 \cdots j_i} \cdot z_1 + \sum_{k=d+1}^{n} a_{j_2 \cdots j_i k} \cdot z_k = 0.$$

So the regularity of $\bar{z}_1$ in ${}_d\bar{A}$ implies

$$a_{1j_2\cdots j_i} = \sum_{k=d+1}^{n} c_{j_2\cdots j_i k} z_k$$

for every $2 \leq j_2 < \cdots < j_i \leq d$. Then

$$[\alpha] = \left[\sum_{\substack{2\leq j_1 < \cdots < j_i \leq n \\ j_i > d}} b_{j_1\cdots j_i} e_{j_1\cdots j_i} + \sum_{\substack{2\leq j_2 < \cdots < j_i \leq n \\ j_i > d}} a_{1j_2\cdots j_i} e_{1j_2\cdots j_i} \right]$$

for some $b_{j_1\cdots j_i}$. Thus $[\alpha] \in T_i^{(n,d)} H_i$.

PROPOSITION 2.15. *If $z_1,\ldots,z_n$ is a (d, i)-sequence and $z_1,\ldots,z_s$, $1 \leq s \leq d - i$, is a regular A-sequence, then $\tilde{z}_{s+1},\ldots,\tilde{z}_n$ is a $(d - s, i)$-sequence in $\tilde{A} = A/(z_1,\ldots,z_s)$.*

Proof. By induction reduce to the case $s = 1$, then apply Lemma 2.14(i) to conclude $\tilde{H}_i/\tilde{T}_i^{(n-1,d-1)}\tilde{H}_i = 0$, so $\tilde{z}_2,\ldots,\tilde{z}_n$ is a $(d - 1, i)$-sequence in $\tilde{A} = A/(z_1)$.

Conversely, we have

PROPOSITION 2.16. *If $z_1,\ldots,z_n$ is a sequence (in* rad A*) such that $\tilde{z}_{s+1},\ldots,\tilde{z}_n$ is a $(d - s, i)$-sequence in $\tilde{A} = A/(z_1,\ldots,z_s)$ and $\bar{z}_1,\ldots,\bar{z}_s$ is a regular sequence in ${}_d\bar{A}$, then it is a (d, i)-sequence.*

Proof. Again by induction reduce to $s = 1$, then apply Lemma 2.14 (ii), so $H_i/T_i^{(n,d)}H_i = 0$, i.e. $z_1,\ldots,z_n$ is a (d, i)-sequence.

N. 3. Now, our aim is to translate $(\text{W.L.C.})_{h,i}$ into an algebraic form. The next proposition will be helpful; it says, roughly, that, if $(\text{W.L.C.})_{h,i}$ holds, we can build a weak lifting of $\bar{\alpha} = \bar{a} + \bar{b}$ starting from any lifting a of $\bar{a}$.

PROPOSITION 3.1. *Let*

$$\bar{a} + \bar{b} \in \operatorname{syz}^{i+1}\left(K\left(\underline{\bar{z}};\, {}_h\bar{A}\right)\right), \quad a + b \in \operatorname{syz}^{i+1}(K(\underline{z};\, A)),$$

where $a \in \bigwedge^i A_{1\ldots d}$, $b \in T_i^{(n,d)}$, $\Psi_i^h(a) = \bar{a}$. Then, for any $a' \in (\Psi_i^h)^{-1}(\bar{a})$, there exists $b' \in T_i^{(n,d)}$ such that $a' + b' \in \operatorname{syz}^{i+1}(K(\underline{z};\, A))$.

Proof. If

$$a = \sum_{1\leq j_1 < \cdots < j_i \leq d} a_{j_1\cdots j_i} e_{j_1\cdots j_i}$$

and

$$a' = \sum_{1 \le j_1 < \cdots < j_i \le d} a'_{j_1 \cdots j_i} e_{j_1 \cdots j_i},$$

then $\Psi_i^h(a) = \Psi_i^h(a')$ is equivalent to

$$a_{j_1 \cdots j_i} = a'_{j_1 \cdots j_i} + \sum_{t=h+1}^{n} \alpha_{j_1 \cdots j_i t} \cdot z_t, \qquad 1 \le j_1 < \cdots < j_i \le d.$$

The element we are looking for is

$$\begin{aligned} b' = & \sum_{\substack{d \le j_1 < \cdots < j_i \le h \\ j_{i-1} > d}} a_{j_1 \cdots j_i} e_{j_1 \cdots j_i} + \\ & + \sum_{\substack{1 \le j_1 < \cdots < j_{i-1} \le d \\ j_i > h}} \left(a_{j_1 \cdots j_i} + \sum_{k=1}^{d} \alpha_{j_1 \cdots k \cdots j_i} \cdot z_k \right) e_{j_1 \cdots j_i} \end{aligned}$$

because it is a matter of computation to see that $d_i(a' + b') = 0$.

We recall the notation (cf. N. 1): $\Psi_i^h \colon \bigwedge^i A^n \to \bigwedge^i A_{1 \cdots h}$ defined by

$$\Psi_i^h \left(\sum_{1 \le j_1 < \cdots < j_i \le n} a_{j_1 \cdots j_i} e_{j_1 \cdots j_i} \right) = \sum_{1 \le j_1 < \cdots < j_i \le h} a_{j_1 \cdots j_i} f_{j_1 \cdots j_i}.$$

LEMMA 3.2. *Let* $\bar{a} + \bar{b} \in \operatorname{syz}^{i+1}(K(\underline{z};\ {}_h\bar{A}))$, *where* $\bar{a} \in \bigwedge^i {}_h\bar{A}_{1 \cdots d}$, $\bar{b} \in T_i^{(h,d)}$, *and let* $a \in (\Psi_i^h)^{-1}(\bar{a})$, $b \in (\Psi_i^h)^{-1}(\bar{b})$. *Then there exists* $c \in T_i^{(n,h)}$, *whose components with at least two indices bigger than h are zero, such that* $\chi_{i-1}^h d_i(a + b + c) = 0$.

Proof. The hypothesis $d_i(\bar{a} + \bar{b}) = 0$ implies $d_i(a + b) \in (z_{h+1}, \ldots, z_n) \bigwedge^{i-1} A_{1 \cdots h}$, that is

$$d_i(a + b) = \sum_{1 \le j_1 < \cdots < j_{i-1} \le h} \sum_{t=h+1}^{n} c_{j_1 \cdots j_{i-1} t} \cdot z_t e_{j_1 \cdots j_{i-1}}.$$

It is easy to verify that we can choose

$$c = \sum_{\substack{1 \le j_1 < \cdots < j_{i-1} \le h \\ t = h+1, \ldots, n}} -c_{j_1 \cdots j_{i-1} t} e_{j_1 \cdots j_{i-1} t}.$$

According to Lemma 3.2, we can give the following

DEFINITION 3.3. Let $\bar{a} + \bar{b} \in \operatorname{syz}^{i+1}(K(\underline{z};\, {}_h\bar{A}))$, where $\bar{a} \in \wedge^i {}_h\bar{A}_{1\cdots d}$, $\bar{b} \in T_i^{(h,d)}$, and let $a \in (\Psi_i^h)^{-1}(\bar{a})$. We set $\phi(a) = a + b'$, where $b' \in T_i^{(n,d)}$ is chosen such that:

(α) $\chi_{i-1}^h d_i(a + b') = 0$;

(β) $\Psi_i^h(b') = \bar{b}$;

(γ) *the components of $\phi(a)$ with at least two indices bigger than h are zero.*

PROPOSITION 3.4. *The following conditions are equivalent*:

(i) $(\text{W.L.C.})_{h,i}$.

(ii) *Let* $\bar{a} + \bar{b} \in \operatorname{syz}^{i+1}(K(\underline{z};\, {}_h\bar{A}))$, $\bar{a} \in \wedge^i {}_h\bar{A}_{1\cdots d}$, $\bar{b} \in T_i^{(n,d)}$, $a \in (\Psi_i^h)^{-1}(\bar{a})$. *Then there exists* $\lambda \in T_i^{(n,d)}$ *such that*

(1) $\pi_{i-1} d_i(\phi(a)) = \pi_{i-1} d_i(\lambda)$;

(2) $\chi_{i-1}^h d_i(\lambda) = 0$.

Proof. (i) $\Rightarrow$ (ii). Let $a + b$ be a weak lifting of $\bar{a} + \bar{b}$ (cf. Proposition 3.1) and $\phi(a) = a + b'$; then $\lambda = b' - b$ is the required element. In fact $\lambda \in T_i^{(n,d)}$, as b and b' belong to that module; moreover from $d_i(a + b) = 0$ we get

$$\begin{aligned}\pi_{i-1} d_i(\phi(a)) &= \pi_{i-1} d_i(a) + \pi_{i-1} d_i(b') \\ &= -\pi_{i-1} d_i(b) + \pi_{i-1} d_i(b') = \pi_{i-1} d_i(b - b');\end{aligned}$$

so condition (1) is verified. Now, since we have $\chi_{i-1}^h d_i(a + b') = 0$ and $\chi_{i-1}^h d_i(a + b) = 0$, then $\chi_{i-1}^h d_i(b' - b) = 0$ and (2) follows.

(ii) $\Rightarrow$ (i). From $\phi(a) = a + b'$, we obtain a weak lifting $a + b$ of $\bar{a} + \bar{b}$ by choosing $b = b' - \lambda$. In fact $b' - \lambda \in T_i^{(n,d)}$ and, moreover,

$$\pi_{i-1} d_i(a + b' - \lambda) = \pi_{i-1} d_i(\phi(a)) - \pi_{i-1} d_i(\lambda) = 0,$$

$$\chi_{i-1}^d d_i(a + b' - \lambda) = \chi_{i-1}^d d_i(\phi(a)) - \chi_{i-1}^d d_i(\lambda) = 0.$$

REMARK 3.5. Condition (1) of (ii) in Proposition 3.4 can be replaced by $\pi_{i-1}^h d_i(\phi(a)) = \pi_{i-1}^h d_i(\lambda)$, because of condition (2).

Now, let us look for conditions stronger than those in Proposition 3.4 which are easier to formulate and verify. First, we point out that $d_i(\phi(a))$ is a boundary with some zero components, so that it is of the form

$$\beta = \sum_{1 \le k_1 < \cdots < k_{i-2} \le h} \beta_{k_1 \cdots k_{i-1}} e_{k_1 \cdots k_{i-1}},$$

where

I. $$\beta_{k_1\cdots k_{i-1}} \in I^h_{k_1\cdots k_{i-2}} = \left(z_1,\ldots,\check{z}_{k_1},\ldots,\check{z}_{k_{i-2}},\ldots,z_h\right)A,$$

II. $$\sum_{t\in\{1,\ldots,n\}-\{j_1,\ldots,j_{i-2}\}} \beta_{j_1\cdots t\cdots j_{i-2}}\cdot z_t = 0$$

(the β's with more than one index bigger than h are zero). From this, taking into account only relations II, corresponding to $1 \le j_1 < \cdots < j_{i-2} \le h$, we easily get

COROLLARY 3.6. $(\text{W.L.C.})_{h,i}$ *is verified if* $\sum_{t=h+1}^{n}\beta_{k_1\cdots k_{i-2}t}\cdot z_t = 0$, *with* $\beta_{k_1\cdots k_{i-2}t} \in I^h_{k_1\cdots k_{i-2}}$, *implies*

$$\beta_{k_1\cdots k_{i-2}t} = \sum_{u\in\{1,\ldots,h\}-\{k_1,\ldots,k_{i-2}\}} \lambda_{k_1\cdots u\cdots k_{i-2}t}\cdot z_u$$

for $1 \le k_1 < \cdots < k_{i-2} \le h$, *where, for every* $1 \le k_1 < \cdots < k_{i-1} \le h$,

(4) $$\sum_{v=h+1,\ldots,n} \lambda_{k_1\cdots k_{i-1}v} z_v = 0.$$

Proof. As we just observed,

$$d_i(\phi(a)) = \sum_{\substack{1\le k_1<\cdots<k_{i-2}\le h\\ k_{i-1}>h}} \beta_{k_1\cdots k_{i-1}} e_{k_1\cdots k_{i-1}},$$

where

$$\beta_{k_1\cdots k_{i-1}} \in I^h_{k_1\cdots k_{i-2}} \quad\text{and}\quad \sum_{t=h+1,\ldots,n} \beta_{k_1\cdots k_{i-2}t}\cdot z_t = 0,$$

so

$$\beta_{k_1\cdots k_{i-1}} = \sum_{u\in\{1,\ldots,h\}-\{k_1,\ldots,k_{i-2}\}} \lambda_{k_1\cdots u\cdots k_{i-1}}\cdot z_u$$

and this shows condition (1) of (ii) in Proposition 3.4 holds. Condition (2) is implied by (4) if we choose $\lambda_{k_1\cdots k_{i-1}r} = 0$ for $r \le h$.

Two weaker versions of Corollary 3.6 are the following:

COROLLARY 3.7. $(\text{W.L.C.})_{h,i}$ *is verified if*

$$\sum_{t=h+1}^{n} \beta_{k_1\cdots k_{i-2}t}\cdot z_t = 0,$$

$$\beta_{k_1\cdots k_{i-2}t} \in I^h_{k_1\cdots k_{i-2}} \left(1 \le k_1 < \cdots < k_{i-2} \le h\right)$$

implies

$$\beta_{k_1 \cdots k_{t-2} t} = \sum_{u=h+1}^{n} \beta_{k_1 \cdots k_{t-2} u t} \cdot z_u + \varepsilon_{k_1 \cdots k_{t-2} t}$$

for some $\varepsilon_{k_1 \cdots k_{t-2} t} \in I^h_{k_1 \cdots k_{t-2}} \cdot (0 : z_t)$.

Proof. Just check that (4) holds.

According to the following definition (Fiorentini [**F**]), *if* $N \subseteq M$ *are* A*-modules*, $x_1, \ldots, x_n$ *is said to be a relative regular* M*-sequence with respect to* N *if*

$$((x_1, \ldots, x_i)N : x_{i+1}) \cap N = (x_1, \ldots, x_i)M, \qquad i = 0, \ldots, n-1,$$

we have

COROLLARY 3.8. $(\text{W.L.C.})_{h,i}$ *is verified if* $z_{h+1}, \ldots, z_n$ *is a relative* A*-sequence with respect to* $I^h_{k_1 \cdots k_{i-2}}$ *for every* $1 \leq k_1 < \cdots < k_{i-2} \leq h$. *In particular*, $(\text{W.L.C.})_{h,i}$ *is verified if* $z_{h+1}, \ldots, z_n$ *is a relative regular sequence* (cf. [**F**]) *or a d-sequence* (*see* [**H**]).

Proof. Just apply Corollary 3.7, since from the hypothesis we get the required implication for $\varepsilon = 0$.

Finally we have the result quoted in N. 1.

COROLLARY 3.9. *If* $z_{d+1}, \ldots, z_n$ *is a regular sequence, the following are equivalent*:

(i) $z_1, \ldots, z_n$ *is a* (d, i)*-sequence*;

(ii) $\operatorname{depth}(\bar{z}_1, \ldots, \bar{z}_d) \geq d - i + 1$ *in* ${}_d\bar{A}$.

REMARK 3.10. The interesting particular case $h = n - 1$ gives rises to an easy way of expressing the conditions of both Corollaries 3.6 and 3.7 (which in such a case coincide); precisely, with the above notation,

$$(0 : z_n) \cap I^{n-1}_{k_1 \cdots k_{i-2}} = (0 : z_n) \cdot I^{n-1}_{k_1 \cdots k_{i-2}}.$$

Moreover the condition of Corollary 3.8 simply becomes

$$(0 : z_n) \cap I^{n-1}_{k_1 \cdots k_{i-2}} = 0.$$

As an easy application of the previous remark, we have

PROPOSITION 3.11. *Let* a, b, c *be a* $(2,2)$*-sequence of a local ring* $(A, \mathfrak{m})$, *and suppose* $\operatorname{Tor}_1(A/(a, b), A/\operatorname{ann}(c)) = 0$. *Then* $\operatorname{depth}(\bar{a}, \bar{b}) \geq 1$ *in* $\bar{A} = A/(c)$.

Proof. The Tor condition says that $(a, b) \cap \operatorname{ann}(c) = (a, b) \cdot \operatorname{ann}(c)$. From the previous remark this implies $(\bar{a}, \bar{b})$ is a $(2,2)$-sequence, and this just means $\operatorname{depth}(\bar{a}, \bar{b}) \geq 1$.

Now we are able to give an application which is a sort of generalization of what we proved in Propositionn 2.10.

PROPOSITION 3.12. *Let $I = (z_1,\dots,z_m)$ be an ideal of A generated by a relative regular sequence (in particular a d-sequence or a regular sequence), with $\dim(A/I) \leq d$. If, for some $i > 0$, there exist $d + i - 1$ elements $x_1,\dots,x_{d+i-1}$ such that $x_1,\dots,x_{d+i-1}, z_1,\dots,z_m$ is a $(d + i - 1, i)$-sequence, then A/I is Cohen-Macaulay. Moreover, if $(A, \mathfrak{m})$ is local and $x_1,\dots,x_{d+i-1}, z_1,\dots,z_m$ is a system of generators of $\mathfrak{m}$, then A/I is regular.*

Proof. By Corollary 3.8 (or Corollary 3.9) and Proposition 2.2, we have, in A/I, $\operatorname{depth}(\bar{x}_1,\dots,\bar{x}_{d+i-1}) \geq d + i - 1 - i + 1 = d$, so, for every $\mathfrak{p} \in \operatorname{Max}(A/I)$, $\operatorname{depth}(A/I)_{\mathfrak{p}} \geq d$ and the conclusion follows.

We can realize how much the condition *$z_{d+1},\dots,z_n$ is a regular sequence* is stronger than $(\text{W.L.C.})_{d,i}$ by looking at the next proposition, which points out a strict relation between (d, i)-sequences, the regularity of the tails of sequences and the vanishing of the Koszul homology.

PROPOSITION 3.13. *For a sequence $z_1,\dots,z_n$ (in* rad A*) we have:*

(1) *If $z_1,\dots,z_n$ is a (d, i)-sequence and $z_{d+1},\dots,z_n$ is a regular sequence, then $H_i(K(\underline{z}; A)) = 0$.*

(2) *if $H_i(K(\underline{z}; A)) = 0$, then there exist $x_1,\dots,x_{n-i+1} \in (z_1,\dots,z_n)$ such that $z_1,\dots,z_{i-1}, x_1,\dots,x_{n-i+1}$ is a (d, i)-sequence and $x_{d-i+2},\dots,x_{n-i+1}$ is a regular sequence.*

Proof. (1) From Corollary 3.9, the hypothesis implies $\operatorname{depth}(\bar{z}_1,\dots,\bar{z}_d) \geq d - i + 1$; as $z_{d+1},\dots,z_n$ is a regular sequence, we get $\operatorname{depth}(z_1,\dots,z_n) \geq n - i + 1$, which implies $H_i(K(\underline{z}; A)) = 0$.

(2) The hypothesis is equivalent to $\operatorname{depth}(z_1,\dots,z_n) \geq n - i + 1$, so we can take a regular sequence $x_1,\dots,x_{n-i+1}$ in $(z_1,\dots,z_n)$. So we have

$$\operatorname{depth}(z_1,\dots,z_{i-1}, x_1,\dots,x_{n-i+1}) \geq n - i + 1,$$

with $x_{d-i+2},\dots,x_{n-i+1}$ a regular sequence. This implies

$$\operatorname{depth}(\bar{z}_1,\dots,\bar{z}_{i-1}, \bar{x}_1,\dots,\bar{x}_{d-i+1}) \geq d - i + 1,$$

so, by Corollary 3.9 and the regularity of $x_{d-i+2},\dots,x_{n-i+1}$, we have $z_1,\dots,z_{i-1}, x_1,\dots,x_{n-i+1}$ is a (d, i)-sequence.

Let us remark that, for $i=1$, part (1) of the previous proposition becomes the well-known result

$$H_1(K(\bar{\underline{z}};\, {}_d\bar{A})) = 0 \textit{ and } z_{d+1},\dots,z_n \textit{ a regular sequence}$$
$$\Rightarrow H_1(K(\underline{z};A)) = 0.$$

If $(\text{W.L.C.})_{d,i}$ holds for every i, the smallest i for which $z_1,\dots,\underline{z_n}$ is a (d,i)-sequence says exactly that $\operatorname{depth}(\bar{z}_1,\dots,\bar{z}_d) = d-i+1$ in ${}_d\bar{A}$ (cf. Corollary 2.4); if, moreover, $z_{d+1},\dots,z_n$ is a regular sequence, such an i says that $\operatorname{depth}(z_1,\dots,z_n) = n-i+1$ (cf. Proposition 3.13). If we do not assume $(\text{W.L.C.})_{d,i}$ holds, we can only say that $\operatorname{depth}(\bar{z}_1,\dots,\bar{z}_d) = s$ implies $z_1,\dots,z_n$ is a (d,i)-sequence for $i \geq d+1-s$; however it may be a (d,i)-sequence for smaller i's, as we saw in Example 1.5, where $\operatorname{depth}(\bar{x},\bar{y}) = 0$, and x, y, z is a $(2,2)$-sequence.

Let us give more examples of (d,i)-sequences $z_1,\dots,z_n$ which do not pass to the quotient ${}_d\bar{A}$, that is where $(\text{W.L.C.})_{d,i}$ does not hold.

Example 3.14. We consider again the local ring of Example 1.5:

$$A = k[[X,Y,Z]]/(X^2-Z^2, XY, XZ) = k[[x,y,z]].$$

The element z is not a d-sequence (according to [H]): in fact $(0:z) = (x)$ and $(0:z^2) = (x,z)$. This remark agrees with what we proved in Example 1.5, that is, (x,y,z) is a $(2,2)$-sequence and $(\bar{x},\bar{y})$ is not a $(2,2)$-sequence in $A/(z)$ (cf. Corollary 3.8).

As in $A/(z)$ every ideal can be generated by two elements. Let us examine all the sequences (α,β,z) in A. They cannot be $(2,1)$-sequences, because, in that case, they should pass to $A/(z)$ (cf. Proposition 2.5), but $\operatorname{depth} A/(z) = 0$. So, the only meaningful question is whether or not they are $(2,2)$-sequences. It is a matter of computation to show that they are essentially of the following three types:

$$\begin{aligned}
s_1 &= (x + y^m\cdot u,\, y^n,\, z), && 1 \leq m < n,\ u \text{ invertible in } A;\\
s_2 &= (y^m,\, y^n,\, z), && m,n \geq 1;\\
s_3 &= (x,\, y^m,\, z), && m \geq 1;
\end{aligned}$$

now s_1 and s_2 are not $(2,2)$-sequences (for instance the cycle

$$xe_{12} - u^{-1}y^{n-m}ze_{13} + ze_{23} \notin \wedge^3 A^3 + T_2^{(3,2)}$$

in $K(x+y^m u,\, y^n,\, z;\, A)$ and, respectively, the cycle

$$x(e_{12}+e_{13}+e_{23}) \notin \wedge^3 A^3 + T_2^{(3,2)}$$

in $K(y^m, y^n, z; A)$), and s_3 is a $(2,2)$-sequence which does not pass to the quotient $A/(z)$.

By using Proposition 1.3, the previous example gives rise to the following one: in the ring $B = k[[X, Y, Z]]$, the sequences

$$(X, Y^h, Z, X^2 - Z^2, XY, XZ), \qquad h \geq 1,$$

are $(2,2)$-sequences which $\operatorname{mod}(X^2 - Z^2, XY, XZ)$ remain $(2,2)$-sequences (note that $X^2 - Z^2$, XY, XZ is a d-sequence); however they do not give rise to $(2,2)$-sequences in $B/(Z, X^2 - Y^2, XY, XZ) = A/(z)$, so, in particular, $(Z, X^2 - Z^2, XY, XZ)$ is not a d-sequence in B.

EXAMPLE 3.15. Let $B = k[[X, Y]]$, $\mathfrak{n} = (X, Y)$, $A = B/\mathfrak{n}^3 = k[[x, y]]$. Then:

(a) $(x, y; x^2, xy, y^2)$ is a $(2,2)$-sequence in A.

(b) $(\bar{x}, \bar{y}; \bar{x}^2, \overline{xy})$ is a $(2,2)$-sequence in $A/(y^2)$, though y^2 is not a d-sequence. This fact shows that *the condition to be generated by a d-sequence is strictly stronger than* $(\text{W.L.C.})_{d,t}$ (cf. Corollary 3.8).

(c) $(\bar{x}, \bar{y})$ is not a $(2,2)$-sequence in $A/(x^2, xy, y^2)$.

Let us prove (a). It is equivalent to show that

$$(5) \qquad \begin{cases} a_{12}y + a_{13}x^2 + a_{14}xy + a_{15}y^2 = 0, \\ a_{12}x - a_{23}x^2 - a_{24}xy - a_{25}y^2 = 0, \\ a_{13}x + a_{23}y - a_{34}xy - a_{35}y^2 = 0, \\ a_{14}x + a_{24}y + a_{34}x^2 - a_{45}y^2 = 0, \\ a_{15}x + a_{25}y + a_{35}x^2 + a_{45}xy = 0 \end{cases}$$

implies $a_{12} \in (x^2, xy, y^2)$. Now, working $\operatorname{mod}(x^2, xy, y^2)$ we can see that a_{1i}, a_{2i}, $i > 2$, are not invertible, so (5) becomes $a_{12}y = a_{12}x = 0$. From this the conclusion follows easily.

The proof of (b) is similar; (c) is trivial, as $\dim A/(x^2, xy, y^2) = 0$.

N. 4. In this last section we just want to give a new version of the results we obtained in the previous ones. Here we use essentially the idea of looking at the syzygies of the Koszul complex as particular systems of linear equations, so that conditions on syzygies can be seen as conditions on the solutions of these systems. For a better understanding, we introduce some general notation and definitions.

Let (F) be a system of linear equations with coefficients in a ring A and with indeterminates $\underline{X} = \{X_1, \ldots, X_n\}$; let $g\colon A \to B$ be any ring homomorphism and denote by $(g(F))$ the system we get from (F) when we apply g to the coefficients of (F). So $(g(F))$ is a system of linear equations with coefficients in B and with indeterminates $\{X_{i_1},\dots,X_{i_r}\}$ where $\{X_{i_1},\dots,X_{i_n}\} \subseteq \{X_1,\dots,X_n\}$ (we delete all indeterminates with coefficient zero).

DEFINITION 4.1. (F) is said to be admissible with respect to $g: A \to B$ (or $(g(F))$) if for every solution $\beta = \{\beta_{i_1},\dots,\beta_{i_r}\}$ of $(g(F))$, $\beta_{i_j} \in B$, $j = 1,\dots,r$, there exists a solution $\alpha = \{\alpha_1,\dots,\alpha_n\}$, $\alpha_k \in A$, $k = 1,\dots,n$, of (F) such that $g(\alpha_{i_j}) = \beta_{i_j}$, $j = 1,\dots,r$.

An easy consequence of the previous definition is

PROPOSITION 4.2. *Let g: $A \to B$ a surjective morphism and* $\mathrm{Ker}(g) =$ $(u_1,\dots,u_r)$; *consider a system of the form*

$$(F) \qquad \sum_{j=1}^{n_i} a_j^{(i)} X_j^{(i)} + \sum_{t=1}^{r} u_t Y_t^{(i)} = 0, \qquad i = 1,\dots,h.$$

Then (F) is admissible with respect to g.

Proof. It is almost trivial since every solution β of $(g(F))$,

$$\sum_{j=1}^{n_i} g\left(a_j^{(i)}\right) X_j^{(i)} = 0, \qquad i = 1,\dots,h,$$

can be lifted to $\alpha = \{\alpha_j^{(i)}\}_{j=1,\dots,n_i; i=1,\dots,h}$ in A, so $\sum_{j=1}^{n_i} a_j^{(i)}\alpha_j^{(i)} \in \mathrm{Ker}(g)$. Then we can find elements in A, $\gamma = \{\gamma_t^{(i)}\}_{t=1,\dots,r; i=1,\dots,h}$, with

$$\sum_{j=1}^{n_i} a_j^{(i)}\alpha_j^{(i)} + \sum_{t=1}^{r} \gamma_t^{(i)} u_t = 0.$$

Now (α, γ) is a solution of (F) and $g(\alpha) = \beta$.

We point out that Proposition 4.2 can be easily generalized by letting g be surjective only on the solutions of $(g(F))$. We now introduce a similar terminology to deal with a subsystem of a system of linear equations.

DEFINITION 4.3. Let (F) be a system of linear equations with coefficients in A and indeterminates $\underline{X}$, $\underline{Y}$, where $\underline{X} = \{X_1,\dots,X_n\}$ and $\underline{Y} = \{Y_1,\dots,Y_m\}$, and let (F') be a subsystem of (F) with indeterminates $\underline{X}$. We say that (F) is admissible with respect to (F') if, for every solution $\underline{X} = \alpha$ of (F'), there exists a solution of (F) of the form $\underline{X} = \alpha$, $\underline{Y} = \beta$.

EXAMPLE. In $\mathbf{Z}$ it is easy to check that

$$(F) \qquad \begin{cases} x + y + z = 0, \\ 3x - y + z = 0, \\ 3x + 4y + z + 5t = 0 \end{cases}$$

is admissible with respect to

$$(F') \qquad \begin{cases} x + y + z = 0, \\ 3x - y + z = 0. \end{cases}$$

Now let us return to our subject and let $z = \{z_1, \ldots, z_n\}$ be elements of A with $(z_1, \ldots, z_n) \subseteq \operatorname{rad}(A)$.

For a fixed i consider the system

$$(S) \qquad \sum_{\substack{t=1 \\ t \neq j_1, \ldots, j_{i-1}}}^{n} (-1)^{s_t} z_t X_{j_1 \cdots t \cdots j_{i-1}} = 0, \qquad 1 \leq j_1 < \cdots < j_{i-1} \leq n,$$

with $\binom{n}{i-1}$ linear equations and $\binom{n}{i}$ indeterminates, and s_t = number of j's preceding t. There is a natural bijection between the set of solutions of (S) and $\operatorname{syz}^{i+1}(K(\underline{z}; A))$, so the definition of (d, i)-sequence can be restated as follows:

Every solution of (S) must have the form

$$\langle d,i \rangle \quad X_{j_1 \cdots j_i} = \sum_{\substack{s=1 \\ s \neq j_1 \cdots j_i}}^{n} \alpha_{j_1 \cdots j_i s} z_s \quad \text{for every } 1 \leq j_1 < \cdots < j_i \leq d,$$

with $\alpha_{j_1 \cdots j_i s} \in A$ and the usual convention on the α's.

We remark that the condition $\langle d, i \rangle$ concerns only some components of every solution of (S).

Now let us fix an integer h, $d \leq h \leq n$, and denote by

$$(S_h) \qquad \sum_{\substack{t=1 \\ t \neq j_1 \cdots j_{i-1}}}^{n} (-1)^{s_t} z_t X_{j_1 \cdots t \cdots j_{i-1}} = 0, \qquad 1 \leq j_i < \cdots < j_{i-1} \leq h,$$

the subsystem of (S) corresponding to the indices $1, 2, \ldots, h$.

Proposition 4.2 implies (S_h) is admissible with respect to the natural map $\phi_h \colon A \to_h \bar{A}$, i.e. with respect to the system

$$(\bar{S}_h) \qquad \sum_{\substack{t=1 \\ t \neq j_1 \cdots j_{i-1}}}^{h} (-1)^{s_t} \bar{z}_t X_{j_1 \cdots t \cdots j_{i-1}} = 0, \qquad 1 \leq j_1 < \cdots < j_{i-1} \leq h,$$

as we already knew by Proposition 3.2.

Clearly every solution of (S) gives a solution of (S_h) and then a solution of $(\bar{S}_h)$; so, if there is an integer h, $d \le h \le n$, such that the solution of $(\bar{S}_h)$ has the form $\langle d, i \rangle$ in ${}_h\bar{A}$, i.e.

$$X_{j_1 \cdots j_i} = \sum_{\substack{s=1 \\ s \neq j_1 \cdots j_i}}^{h} \bar{\alpha}_{j_1 \cdots j_i s} \bar{z}_s, \qquad 1 \le j_1 < \cdots < j_i \le d,$$

then every solution of (S) in A will be in the form $\langle d, i \rangle$, i.e.

$$X_{j_1 \cdots j_i} = \sum_{\substack{s=1 \\ s \neq j_1 \cdots j_i}}^{n} \alpha_{j_1 \cdots j_i s} z_s, \qquad 1 \le j_1 < \cdots < j_i \le d.$$

This simply says that if there exists h such that $\bar{z}_1, \ldots, \bar{z}_h$ is a (d, i)-sequence in ${}_h\bar{A}$, then $z_1, \ldots, z_n$ is a (d, i)-sequence in A, and that is Theorem 1.3.

When $n = d$, condition $\langle d, i \rangle$ concerns the whole solution, so in this case to say that every solution of (S) has the form $\langle n, i \rangle$ is equivalent to $\operatorname{depth}(z_1, \ldots, z_n) \ge n - i + 1$. The Corollary 1.4 becomes: if the solutions of $(\bar{S}_d)$ have the form $\langle d, i \rangle$, then the same is true for the solutions of (S).

Now we want to study how a property $\langle \mathcal{P} \rangle$, in particular $\langle d, i \rangle$, passes from the solutions of a system (F) to the solutions of a subsystem (F'). We have this first easy result.

Lemma 4.4. *If (F) is a system of linear equations with two sets of indeterminates $\underline{X}$, $\underline{Y}$, and if the solutions of (F) satisfy a property $\langle \mathcal{P} \rangle$ related to the part concerning the $\underline{X}$ indeterminates, then every admissible subsystem (F'), with indeterminates $\underline{X}$, has all the solutions satisfying $\langle \mathcal{P} \rangle$.*

Proof. It is trivial; just take a solution α for (F') and (α, β) the solution of (F) arising from the admissibility; then since (α, β) has $\langle \mathcal{P} \rangle$, which is related to the $\underline{X}$'s, α has $\langle \mathcal{P} \rangle$.

Corollary 4.5. *If $z_1, \ldots, z_n$ is a (d, i)-sequence and our system (S) is admissible with respect to (S_d), then* $\operatorname{depth}(\bar{z}_1, \ldots, \bar{z}_d) \ge d - 1 + i$.

Remark 4.6. For $i = 1$, $(S) \equiv (S_h)$ for every $h \le n$, so Corollary 4.5 gives again: $z = \{z_1, \ldots, z_n\}$ is a $(d, 1)$-sequence implies $\bar{z}_1, \ldots, \bar{z}_d$ is a regular sequence in ${}_d\bar{A}$ (cf. Proposition 2.6).

Corollary 4.5 can be easily generalized to

COROLLARY 4.7. *If the composition of system maps*

$$(S) \xrightarrow{\mu} (S_h) \xrightarrow{\nu} (\bar{S}_h),$$

where μ *means to pass to a subsystem and* ν *is the induced map of the natural one* $A \to_h \bar{A}$, *is admissible* (*i.e. every solution of* (S_h) *in* ${}_h\bar{A}$ *can be lifted to a solution of* (S) *in* A), *then the condition* $\langle d, i \rangle$ *descends from* A *to* ${}_h\bar{A}$.

We observe that the hypothesis of Corollary 4.7 is really weaker than in Corollary 4.5 (also if we use there h instead of d), since compositions of admissible maps (of systems) are admissible, but conversely if the composition is admissible (and the second map is too) the first map is not necessarily admissible. In fact, the admissibility of (S) with respect to $(\bar{S}_h)$ simply means the surjectivity of ψ_i^h: $\operatorname{syz}^{i+1}(K(\underline{z}; A)) \to \operatorname{syz}^{i+1}(K(\underline{\bar{z}}; {}_h\bar{A}))$, while the admissibility of (S) with respect to (S_h) means the strongest relation:

$$(\psi_i^h)^{-1}\left(\operatorname{syz}^{i+1}\left(K\left(\underline{\bar{z}}, {}_h\bar{A}\right)\right)\right) = \operatorname{syz}^{i+1}(K(z; A)).$$

Nevertheless the hypothesis of Corollary 4.7 is still not necessary to pass the $\langle d, i \rangle$-condition from A to ${}_h\bar{A}$.

From now on, in our system (S) we denote by $\underline{X}$ the set of indeterminates $\{X_{j_1 \cdots j_i}\}_{1 \le j_1 < \cdots < j_i \le d}$ and by $\underline{Y}$ all the remaining indeterminates, i.e. $\{X_{j_1 \cdots j_i}\}_{1 \le j_1 < \cdots < j_i \le n;\, j_i > d}$; h is always an integer such that $d \le h \le n$.

We need a weak version of admissibility.

DEFINITION 4.8. Let (F) be a system of linear equations with coefficients in A and with two sets of indeterminates $\underline{X}$, $\underline{Y}$; let g: $A \to B$ be a ring homomorphism and (F') the system induced from (F) by g with indeterminates $\underline{X}$ and $\underline{Y}'$, where $\underline{Y}' \subseteq \underline{Y}$. We say that (F) and (F') are admissible with respect to $\underline{X}$ (or weakly admissible when there is no chance of confusion) if for every solution $(\bar{a}, \bar{b})$ of $(F'$ in B, with $\underline{X} = \bar{a}$, $\underline{Y}' = \bar{b}$, there is a solution (a, c) of (F) in A, with $\underline{X} = a$, $\underline{Y} = c$, such that $g(a) = \bar{a}$ (more precisely, putting $a = \{a_1, \ldots, a_t\}$ and $\bar{a} = \{\bar{a}_1, \ldots, \bar{a}_t\}$, $g(a_i) = \bar{a}_i$).

EXAMPLE. Take $\mathbf{Z} \underset{\text{nat}}{\to} \mathbf{Z}/6\mathbf{Z}$ and

$$(F) \qquad \begin{cases} x + y + 2z + 6t = 0, \\ 3x - y + 2z + 6t = 0, \\ 6x + 6y + 6z + 24t = 0, \end{cases}$$

so

$$(F') \qquad \begin{cases} x+y+2z=0, \\ 3x-y+2z=0 \end{cases} \quad \text{in } \mathbf{Z}/6\mathbf{Z}.$$

As is easy to see, they are not admissible (for instance, we cannot lift the solution of (F') $x=y=\bar{0}$, $z=\bar{3}$), but with respect to the set of indeterminates (x, y) they are; namely, the only solutions of (F') in $\mathbf{Z}/6\mathbf{Z}$ have the form $(\bar{\lambda}, \bar{\lambda}, -\bar{\lambda})$ or $(\bar{\lambda}, \bar{\lambda}, -\bar{\lambda}+\bar{3})$, with $\bar{\lambda} \in \mathbf{Z}/6\mathbf{Z}$; so they can be lifted to a solution of (F) in $\mathbf{Z}$, for instance $(\lambda, \lambda, 2\lambda, -\lambda)$, for some $\lambda \in \mathbf{Z}$ whose image in $\mathbf{Z}/6\mathbf{Z}$ is $\bar{\lambda}$.

Of course, when $\underline{Y}' = \varnothing$, admissibility coincides with weak admissibility; in particular, this happens for (S) and $(\bar{S}_d)$.

Let us go back to our system (S); now Proposition 3.1 can be restated as follows.

LEMMA 4.9. *If* $(S) \xrightarrow{\mu} (\bar{S}_h)$ *(the usual composition* $(S) \to (S_h) \to (\bar{S}_h)$*) is weakly admissible and* $(\bar{a}, \bar{b})$ *is a solution of* $(\bar{S}_h)$ *in* ${}_h\bar{A}$, *for every* $a' \in \mu^{-1}(\bar{a})$, *we can find* c', *set of elements in* A, *such that* (a', c') *is a solution of* (S).

Finally, the new version of Proposition 2.2 is

THEOREM 4.10. *For* $z_1, \ldots, z_n$ *in* A, *with* $(z_1, \ldots, z_n) \subseteq \operatorname{rad}(A)$, *the following are equivalent*:

(i) $z_1, \ldots, z_n$ *is a* (d, i)*-sequence in* A *and* $(S) \to (\bar{\bar{S}}_h)$ *is weakly admissible.*

(ii) $\bar{z}_1, \ldots, \bar{z}_h$ *is a* (d, i)*-sequence in* ${}_h\bar{A}$.

REMARK 4.11. The conditions in Corollaries 3.6–3.9 are all sufficient in order to have $(S) \to (\bar{S}_d)$ admissible.

Just to show how one can deal with these problems in terms of linear systems, let us rewrite the proof of Proposition 3.11.

The admissibility of the systems

$$(S_2) \quad \begin{cases} bx + xy = 0, \\ -ax + cz = 0, \end{cases} \qquad\qquad (\bar{S}_2) \quad \begin{cases} \bar{b}x = 0, \\ \bar{a}x = 0, \end{cases}$$

says that, for some lifting α of a solution of $\bar{\alpha}$ of $(\bar{S}_2)$, there exists a solution (α, β, γ) of (S_2); since $\beta a + \gamma b \in (a, b) \cap (0:c)$, for the Tor-condition, $\beta a + \gamma b \in (a, b) \cdot (0:c)$, so we have elements $\beta', \gamma' \in (0:c)$

such that $\beta a + \gamma b = \beta' a + \gamma' b$. Now $(\alpha, \beta - \beta', \gamma - \gamma')$ is a solution of

$$(S) \qquad \begin{cases} bx + cy = 0, \\ -ax + cz = 0, \\ ay + bz = 0, \end{cases}$$

that is, (S) and $(\bar{S}_2)$ are admissible. The $(2,2)$-condition implies $\alpha = \lambda c$, for some $\lambda \in A$, i.e. $\bar{\alpha} = 0$.

REFERENCES

[A-B] M. Auslander and D. Buchsbaum, *Codimension and multiplicity*, Ann. of Math., **94** (1969), 23–58.

[F] M. Fiorentini, *On relative regular sequences*, J. Algebra, **18** (1971), 384–389.

[G-L] T. Gulliksen and G. Levine, *Homology of local rings*, Queen's Papers in Pure and Applied Math. N., **20** (1969), Queen's Univ., Kingston.

[H] C. Hunecke, *The theory of d-sequences and powers of ideals*, Adv. in Math., (to appear).

[M-R] C. Massaza and A. Ragusa, *On some conditions of regularity, in terms of existence of a good basis of syzygies*, J. Algebra, **70** N. 2, (June 1981), 493–516.

Received October 30, 1981 and in revised form July 10, 1982. This paper was written while both authors were members of C.N.R., sez. 3 of G.N.S.A.G.A.

UNIVERSITÀ DI SIENA
SIENA, ITALY

AND

UNIVERSITÀ DI CATANIA
CATANIA, ITALY