

DESUSPENSIONS OF STUNTED PROJECTIVE SPACES

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For multiples of the Hopf bundles over real and complex projective spaces, the number of sections is compared with the number of times that the Thom complex desuspends. Examples of equality of these numbers, and of inequality, are given.

1. Survey of Results. For F the field of real numbers R or complex numbers C , let FP_m^{m+n} denote the stunted F -projective space FP^{m+n}/FP^{m-1} . In this paper, which combines some new results with a survey of known results, we consider the question of finding the largest s such that FP_m^{m+n} desuspends s times, i.e. for which there exists a complex Y , often denoted $\Sigma^{-s}FP_m^{m+n}$, such that $\Sigma^s Y$ and FP_m^{m+n} have the same homotopy type. This question is a good test for virtually any technique in (unstable) homotopy theory; e.g. secondary cohomology operations ([4]), Maunder operations ([5]), Adams operations in loop spaces ([12]), and unstable BP -operations ([36]). It is of particular importance because of its relation to the generalized vector field problem; since FP_m^{m+n} is the Thom complex of $m\xi_{F,n}$, where $\xi_{F,n}$ denotes the Hopf bundle over FP^n , the following proposition is immediate (and very well known).

PROPOSITION 1.1. *If $m\xi_{F,n}$ has s sections, then $\Sigma^{-s}FP_m^{m+n}$ exists.*

These sectioning results are often conveniently expressed in terms of geometric dimension (gd), for which the desuspension analogue is the number $b(FP_m^{m+n})$, where $b(X)$ is defined in §2. It is the minimal dimension of bottom cells of iterated desuspensions of iterated suspensions of X . We will show in 2.4 that for $X = FP_m^{m+m}$ the iterated suspensions can be omitted. Then 1.1 becomes

PROPOSITION 1.1'. $b(FP_m^{m+n}) \leq \text{gd}(m\xi_{F,n})$.

The major theme of this paper will be to assess how strong 1.1 is, i.e. to determine which of the known nonsectioning results are implied by nondesuspensions.

The simplest class of nonsectioning results consists of those implied by Stiefel-Whitney classes. These results all come from the nondesuspensions of the following result and 1.1'.

PROPOSITION 1.2. *If $\binom{m}{n}$ is odd, then $b(RP_m^{m+n}) \geq n$ and $b(CP_m^{m+n}) \geq 2n$.*

Proof. In $H^*(RP_m^{m+n}; Z_2)$, $Sq^n(x^m) \neq 0$. Since Sq^n vanishes on classes of degree less than n , RP_m^{m+n} cannot desuspend below degree n . CP_m^{m+n} is handled similarly using Sq^{2n} . \square

The other nonsectioning results obtained from ordinary cohomology are those of $m\xi_{C,n}$ implied by the Euler class.

PROPOSITION 1.3. *If $2m \geq n$, then $gd(m\xi_{C,n}) \geq n - \varepsilon$, where $\varepsilon = 1$ if n is odd and $\varepsilon = 0$ if n is even.*

Proof. If $m\xi$ had the stable type of a bundle θ of dimension $n - \varepsilon$, then

$$\chi(\theta)^2 = p_{(n-\varepsilon)/2}(\theta) = p_{(n-\varepsilon)/2}(m\xi) \neq 0,$$

and hence θ cannot have a section. \square

Many of these nonsectioning results are not implied by nondesuspensions, i.e. the Thom space desuspends lower than the bundle sections. Indeed, the nondesuspension analogue of 1.3 would say that no stunted complex projective space (SCP) can desuspend below the metastable range, where a complex with top dimension T and bottom dimension B is said to be metastable if $T \leq 3B + 2$. But this is patently false, because if $m \equiv 0 (A_n)$, where A_n is the Atiyah-Todd number, then $J(m\xi_{C,n}) = 0$, and hence $b(CP_m^{m+n}) = 0$. For further discussion, see §2.

A metastable example of $m\xi_{C,n}$ whose Thom complex desuspends lower than the bundle sections was communicated to the author by Mahowald in 1973. If n is a 2-power, then $CP^n \subset R^{4n-1}$ but $CP^n \not\subseteq R^{4n-2}$, i.e. its immersion and embedding dimensions are equal ([22]). Thus the normal bundle to the embedding has no sections but its Thom complex desuspends ([8]). Although the normal bundle itself is probably not a multiple of the Hopf bundle, it is J -equivalent to $(A_n - n - 1)\xi_{C,n}$. Since the fibers of $BO_k \rightarrow BO$ and $BG_k \rightarrow BG$ are equivalent in the stable range, it follows that $gd((A_n - n - 1)\xi_{C,n}) = 2n - 1$ and $b(CP_{A_n - n - 1}^{A_n - 1}) \leq 2n - 2$.

The other main class of nonsectioning results for CP 's is the following result, which has been proved by various applications of K -theory ([31], [13], [33], [24]).

PROPOSITION 1.4. $\text{gd}(m\xi_{C,n}) \geq 2 \cdot \max\{j - v_2(\binom{m}{j}): j \leq n\}$, where $v_p(N)$ is the exponent of p in the factorization of N .

These results all come from nondesuspensions, using 1.1' and

THEOREM 1.5. $b(CP_m^{m+n}) \geq 2 \cdot \max\{j - v_2(\binom{m}{j}): j \leq n\}$.

This is proved by using the Adams operation ψ^2 in $K(CP_m^{m+n})$. The result required is

THEOREM 1.6. ψ^k is divisible by k^{n+1} in $K(\Sigma^{2n+1}Y)$.

If the “+1”'s are omitted from 1.6, this follows from Adams' well-known formula $\psi^k \Sigma^2 = k \Sigma^2 \psi^k$. The full strength of 1.6 is proved in §3. In 1978 Sam Gitler and Steve Wilson independently pointed out to the author that the weaker form of 1.6 implied 1.5 within 1 dimension.

Using ψ^p for odd primes p in 1.6, we obtain the following generalization of 1.5.

THEOREM 1.7.

$$b(CP_m^{m+n}) \geq 2 \cdot \max\{j - v_p(\binom{m}{j}): p \text{ prime}, (p-1)j \leq n\}.$$

In §3 we present some discussion of when the odd primes in 1.7 improve upon the case $p = 2$ (Theorem 1.5). A major difference is that the odd primes imply results only below the metastable range, and hence they imply no new nonsectioning results.

In view of the success which K -theory has traditionally enjoyed for complex projective spaces (e.g. [17], [27], [11]) it seems reasonable to conjecture that the lower bound for $b(CP_m^{m+n})$ provided by 1.7 is sharp, i.e. all nondesuspensions of SCP's are detected by ψ^p 's.

Conjecture 1.8.

$$b(CP_m^{m+n}) = 2 \cdot \max\{j - v_p(\binom{m}{j}): p \text{ prime}, (p-1)j \leq n\}.$$

Some evidence is provided by Theorem 3.6, which shows 1.8 is true when $n = 2$, and by 3.8, which shows that 1.8 is true when $b = 0$.

There are unstable Adams operations in BP -theory with properties analogous to those in K -theory. In Theorem 3.3 we show that the only nondesuspensions of SCP's which they imply are those (1.7) implied by K -theory. This furnishes additional evidence for Conjecture 1.8.

Very recently the author and Bendersky developed a method ([9]) of obtaining nondesuspension results using the coaction in BP -homology and unstable results of Ravenel-Wilson ([32]). These, too, yielded exactly 1.7 when applied to SCP's, again supporting Conjecture 1.8.

The situation when $F = R$ is, in at least one respect, quite different. Because $J(RP^n) \approx \tilde{K}O(RP^n)$, there are no quick-and-easy examples of $b(RP_m^{m+n}) < \text{gd}(m\xi_{R,n})$. Indeed, we give what appears to be the first example of this.

THEOREM 1.9. $\text{gd}((64k + 36)\xi_{R,10}) > 5$, while $b(RP_{64k+36}^{64k+46}) = 4$.

The second part of this result utilizes a delicate stable homotopy equivalence of [18]; Mahowald suggested to the author that the stable equivalences of [18] might provide a source for examples such as 1.9. The first part of 1.9 was proved by Adams in [2] using representation theory.

Two major families of nonsectioning results for RP^n are known. The first ([10], [19], [15]) uses K -theory to prove: If $\binom{m-1}{n}$ is odd, then $\text{gd}(m\xi_{R,n}) \geq n - 2\nu_2(m) + \varepsilon$ where ε depends upon mod 4 values of n and $\nu_2(m)$ as in [15; 2.1] or 1.10 below. In §5 we prove the nondesuspension analogues.

THEOREM 1.10. *If $\binom{m-1}{n}$ is odd, then $b(RP_m^{m+n}) \geq n - 2\nu_2(m) + \varepsilon$, where ε is given by the table:*

		$n \bmod 4$			
		0	1	2	3
$\nu_2(m) \bmod 4$	0	0	1	1	0
	1	1	1	2	1
	2	2	1	2	1
	3	2	1	0	-1

This improves slightly upon the results of [12]. It answers Milgram's comment ([28; p. 10]): "it would be interesting to explain [10] in terms of obstructions to desuspension of Thom complex".

The other family of nonsectioning results for RP is that proved by Astey ([7]) using BP . It has been recently proved in [9] that many of these are implied by nondesuspensions.

2. Atiyah-Todd numbers. The n th Atiyah-Todd number A_n is defined by $v_p(A_n) = \max\{j + v_p(j) : (p-1)j \leq n\}$. It was proved in [3] that $J(m\xi_{C,n}) = 0$ iff $m \equiv 0 (A_n)$. Thus by [8] $CP_m^{m+n} = T(m\xi_{C,n})$ and CP_0^n are S -equivalent when $m \equiv 0 (A_n)$, i.e. $\Sigma^l CP_m^{m+n}$ and $\Sigma^{l+2m} CP_0^n$ have the same homotopy type for some l . As we show below, this implies that $CP_m^{m+n} \simeq \Sigma^{2m} CP_0^n$, a fact used in §1.

DEFINITION 2.1. If X is a CW-complex, $b'(X)$ is the dimension of the bottom cell of $\Sigma^{-s}X$ when s is maximal.

REMARK 2.2. $b'(\Sigma X) \leq b'(X)$. An example in which these are unequal is $b'(S^n \times S^n) = n$, $b'(\Sigma(S^n \times S^n)) = 0$.

DEFINITION 2.3. $b(X) = \lim_{s \rightarrow \infty} b'(\Sigma^s X)$.

PROPOSITION 2.4. $b'(FP_m^{m+n}) = b'(\Sigma^l FP_m^{m+n})$ for all m, n , and l . Hence $b(FP_m^{m+n}) = b'(FP_m^{m+n})$.

Proof. Let $F = R$. A similar argument works if $F = C$. If $n \geq m$ then $b'(RP_m^{m+n}) = b'(\Sigma^l RP_m^{m+n}) = m$ since $\text{Sq}^m x^m \neq 0$. If $n < m$ then $\Sigma^l RP_m^{m+n}$ d suspends *uniquely* to RP_m^{m+n} for dimensional reasons (e.g. [28; 5.1]). Thus if $b = b'(\Sigma^l RP_m^{m+n})$, i.e.

$$\Sigma^l RP_m^{m+n} = \Sigma^{m+l-b} X_b^{b+n} = \Sigma^l (\Sigma^{m-b} X_b^{b+n}),$$

then $RP_m^{m+n} = \Sigma^{m-b} X_b^{b+n}$, and hence $b'(RP_m^{m+n}) = b$. \square

3. Adams Operations. Theorem 1.6 follows from $\psi^k \Sigma^2 = k \Sigma^2 \psi^k$ together with

THEOREM 3.1. ψ^k is divisible by k in $\tilde{K}(\Sigma Y) \approx K^1(Y)$.

Proof (Hubbuck). Since products vanish in $\tilde{K}(\Sigma Y)$, $0 = x^p \equiv \psi^p x \pmod{p}$ for any prime p ([1]). Write $k = p_1^{n_1} \cdots p_r^{n_r}$ with p_1, \dots, p_r distinct primes. Then $\psi^k x = (\psi^{p_1})^{n_1} \cdots (\psi^{p_r})^{n_r} x$ is divisible by k . \square

Proof of 1.7. $K(CP_m^{m+n})$ is a free abelian group on generators $\alpha^m, \dots, \alpha^{m+n}$. Let $M = \min\{i + v_p\binom{m}{i} : (p-1)(m-i) \leq n\} = m - \max$, where \max is as in 1.7. By 1.6 it suffices to show $v_p(\psi^p \alpha^m) = M$.

$$\begin{aligned} \psi^p \alpha^m &= ((1 + \alpha)^p - 1)^m = \alpha^m (p \cdot g(\alpha) + \alpha^{p-1})^m \\ &= \alpha^m \Sigma \binom{m}{i} p^i g(\alpha)^i \alpha^{(p-1)(m-i)}, \end{aligned}$$

where $g(\alpha) = 1 + \cdots + \alpha^{p-2}$ with intermediate terms having integer coefficients. All terms have $\nu_p(\) \geq M$. If i is the largest value for which this minimum is achieved, then $\nu_p(\text{coef of } \alpha^{(p-1)(m-i)}) = M$, since only smaller i can have any terms with $\nu_p(\)$ this small, but only larger i can have any $\alpha^{(p-1)(m-i)}$ -terms. \square

In order that the nondesuspension of 1.7 be in the metastable range, and hence imply a nonsectioning result not already implied by the Euler class, we would need $2(j - \nu_p(\binom{m}{j})) > n$ for some j with $(p-1)j \leq n$. This would imply $0 > \nu_p(\binom{m}{j}) + (p-3)j$, which is clearly impossible if $p \geq 3$.

To illustrate the strength of the various ψ^p , we consider the lower bounds for $b(CP_m^{m+5})$ implied by each. According to [17] there are 2880 distinct stable homotopy types of such spaces, and, as the table below illustrates, ψ^3 is needed to give the best result for 70 of them, and ψ^5 for 4 of them; for the rest ψ^2 gives best results.

Lower bound for $b(CP_m^{m+5})$	Smallest p Such that ψ^p implies result	Conditions on m	# of cases
10	2	5(8), 7(8)	720
8	2	4(8), 6(8), 9(16), 11(16)	1080
6	2	3(16), 8(16), 10(16), 17(32)	630
4	2	2(16), 16(32), 33(64)	315
4	3	32(96), 65(192)	45
2	2	1, 96, 129, 160(192)	60
2	3	64(192), 192(576), 384(576)	25
2	5	576 k (2880), $1 \leq k \leq 4$	4
0	2	0(2880)	1

Novikov ([30]) introduced Adams operations ψ^k in rational MU -theory and Araki ([6]) showed that they are also defined in rational BP -theory. Wilson ([36]) showed that $k^i \psi^k$ is integral on $MU^{2i+\varepsilon}(X)$ and $BP^{2i+\varepsilon}(X)$ for $\varepsilon = 0$ or 1, but was not optimistic about their utility. We reinforce his opinion, and our Conjecture 1.8, in the following theorem.

THEOREM 3.3. *The only nondesuspensions of SCP's implied by the BP-Adams operations are those already implied by the KU-Adams operations.*

Proof. It suffices to consider ψ^p in BPQ , localized at p . $BP^* = Z_{(p)}[v_i: i \geq 1]$, and $BP^*(CP_m^{m+n})$ is a free BP^* -module with generators

$\alpha^m, \dots, \alpha^{m+n}$. In $BP^*(CP^\infty)$

$$\begin{aligned}\psi^p(\alpha) &= \frac{1}{p}[p](\alpha) = \frac{1}{p}(p\alpha - (p^{p-1} - 1)v_1\alpha^p + \dots) \\ &= \frac{1}{p}\alpha \sum_{i=0}^{\infty} c_i \alpha^{(p-1)i}\end{aligned}$$

where $c_0 = p$, $c_1 = (1 - p^{p-1})v_1$, and c_i is an integer combination of v_i 's, since the p -series $[p](\alpha)$ is induced by a map $CP^\infty \rightarrow CP^\infty$.

Then

$$\psi^p(\alpha^m) = \alpha^m \cdot \sum_{j \leq [n/(p-1)]} \alpha^{(p-1)j} C_j,$$

where

$$C_j = \sum \frac{1}{p^{i_1 + \dots + i_r}} \frac{m!}{i_1! \dots i_r! (m - i_1 - \dots - i_r)!} c_1^{i_1} \dots c_r^{i_r},$$

this sum taken over tuples (i_1, \dots, i_r) of nonnegative integers with $i_1 + 2i_2 + \dots + ri_r = j$ and $i_1 + \dots + i_r \leq m$. C_j can be written as a polynomial in the v_i 's, and $\nu_p(C_j)$ means ν_p (this polynomial in the v_i 's). We will show below that

$$(3.4) \quad \min \left\{ \nu_p(C_j) : j \leq \left\lfloor \frac{n}{p-1} \right\rfloor \right\} = \min \left\{ \nu_p \binom{m}{j} - j : j \leq \left\lfloor \frac{n}{p-1} \right\rfloor \right\},$$

and this together with Wilson's integrality result mentioned above implies exactly the nondesuspensions of 1.7 by a similar argument.

To see (3.4), we note that

$$(3.5) \quad \nu_p(\text{coef of } c_1^{i_1} \dots c_r^{i_r}) \geq \nu_p(\text{coef of } c_1^{i_1 + \dots + i_r}) = \nu_p \binom{m}{j} - j,$$

which immediately implies \geq in (3.4). Choose the smallest j realizing the minimum value, M , of $\nu_p \binom{m}{j} - j$. The coefficient of v_1^j contains a term

$$\binom{m}{j} \left(\frac{p^{p-1} - 1}{p} \right)^j$$

from c_1^j with $\nu_p(\) = M$. Other terms $c_1^{k_1} \dots c_r^{k_r}$ contributing v_1^j -terms have $\nu_p(\text{coef of } v_1^j) > M$, by (3.5), since $k_1 + \dots + rk_r = j$ and so $k_1 + \dots + k_r < j$. Therefore $\nu_p(\text{coef of } v_1^j) = M$. \square

The next result verifies Conjecture 1.8 when $n = 2$ by showing that the Adams operations detect *all* nondesuspensions of SCP's with three cells.

THEOREM 3.6.

$$b(CP_m^{m+2}) = 2 \cdot \max\{1 - \nu_2(m), 2 - \nu_2\binom{m}{2}, 1 - \nu_3(m), 0\}.$$

Proof. By [17] there are 14 stable homotopy types of SCP's, which depend upon the mod 24 value of m . The required values of b are listed below (where “= ” is from [17], e.g. $CP_{24k+1}^{24k+3} \simeq_s CP_{24l+17}^{24l+19}$):

b	$m \bmod 24$
0	0
2	1 = 17, 4 = 20, 5 = 13, 8 = 16, 9, 12, 21
4	2 = 22, 3 = 15, 6 = 18, 7 = 11, 10 = 14, 19 = 23

The \geq -part of this result is 1.7; the work required here is to show that these SCP's desuspend as far as claimed. The case $m \equiv 0$ was handled in §2. The cases in which $b = 4$ are implied for dimensional reasons, since $\Sigma^{2m-4}\Omega^{2m-4}CP_m^{m+2} \rightarrow CP_m^{m+2}$ is a $(2m + 4)$ -equivalence.

The four cases m odd, $b = 2$ are handled by noting that

$$\Sigma^{2m-2}CP^2 \simeq CP_m^{m+1} \quad \text{and} \quad \pi_5(CP^2) \xrightarrow{\Sigma^{2m-2}} \pi_{2m+3}(CP_m^{m+1})$$

is a surjection $Z \rightarrow Z_{12}$; thus the attaching map for the top cell of CP_m^{m+2} is the image of $\alpha \in \pi_5 CP^2$, and so $CP_m^{m+2} \simeq \Sigma^{2m-2}(CP^2 \cup_\alpha e^6)$.

The cases $m \equiv 4$ and 12 can be proved using James' cell decomposition of $\Omega S^3 = S^2 \cup e^4 \cup e^6 \cup \dots$. $\pi_5(\Omega S^3) \approx Z_{12}$ and the generator α stabilizes to $2g + \eta \in \pi_{n+3}S^n \oplus \pi_{n+3}S^{n+2}$. (η is present by consideration of the Hopf invariant $H: \pi_5\Omega S^3 \rightarrow \pi_5\Omega S^5$.)

$$\Sigma^{2m-2}(\Omega S^3)^{(4)} \simeq S^{2m} \vee S^{2m+2} \simeq CP_m^{m+1},$$

and

$$\pi_5(\Omega S^3)^{(4)} \xrightarrow{\Sigma^{2m-2}} \pi_{2m+3}CP_m^{m+1} = Z_{24} \oplus Z_2$$

has image all multiples of $2g + \eta$. The attaching map for the top cell of CP_m^{m+2} is of this form: η is present since Sq^2 is nonzero, and the attaching map to the bottom cell is respectively $2g$ or $6g$ as can be seen by considering $CP_m^{m+2} \rightarrow QP_{m/2}^{m/2+1} = T(mH/2)$.

Finally we prove it for $m \equiv 8$ by showing

$$\pi_5(S^2 \cup_{4h} e^4) \xrightarrow{\Sigma^{2m-2}} \pi_{2m+3}(S^{2m} \vee S^{2m+2}) \approx Z_{24} \oplus Z_2$$

has in its image the class $4g_1 + \eta$, which is the attaching map for the top cell of CP_m^{m+2} . It suffices to consider the single suspension and show

$$\pi_5(S^2 \cup_{4h} e^4) \xrightarrow{\Sigma} \pi_6(S^3 \vee S^5) \approx Z_{12} + Z_2$$

has in its image the class $2g + \eta$. If we let $J(\) = \Omega\Sigma(\)$ and use the homomorphism of [35, p. 547], the result follows from

LEMMA 3.7. *If $X = S^2 \cup_{4h} e^4$, then in*

$$\begin{array}{ccc} \pi_6(S^3 \vee S^5) = \pi_5(JX) \xrightarrow{i_*} \pi_5(JX, X) \xrightarrow{\bar{h}_*} \pi_5(J(X \wedge X)) & & \\ \Downarrow & & \Downarrow \\ Z_{12} \oplus Z_2 & & Z_2 \end{array}$$

- (i) $\bar{h}_* i_*(\eta) = 0$ (and hence $\bar{h}_* i_*(2g + \eta) = 0$) and
(ii) $\ker(\bar{h}_*)$ is torsion free.

Proof. (i) The 5-skeleton $(JX)^{(5)}$ is $S^2 \cup_{4h} e^4 \cup_{2h} e^4$. There is an equivalence $(S^2 \cup_{2g} e^4) \vee S^4 \rightarrow (JX)^{(5)}$ which sends the split S^4 to the first 4-cell by degree 1 and to the second 4-cell by degree -2 . The class η under consideration is the image of the nonzero element of $\pi_5 S^4$ in $\pi_5(JX)$ (since $Z_{12} \oplus Z = \pi_5(S^2 \cup_{2g} e^4) \rightarrow \pi_5 JX = Z_{12}$ sends the Z to 0). When X is collapsed, we have $\pi_5 S^4 \xrightarrow{-2} \pi_5 S^4$, which is 0.

(ii) \bar{h}_* can be factored into $\pi_5(JX, X) \rightarrow \pi_5(JX/X) \rightarrow \pi_5(J(X \wedge X))$. The second homomorphism is injective since through dimension 6 it is a compression $J(X \wedge X) \cup e^6 \rightarrow J(X \wedge X)$. The first has torsion-free kernel by standard stability arguments since X and JX have only even-dimensional cells and JX/X is 3-connected. [Let $F = \text{fibre}(X \rightarrow JX)$. The Serre spectral sequence shows $H^5(F)$ torsion-free. (Everything below it is torsion-free, so a torsion class could not support a nonzero differential, but $H^5 X = 0$.) Therefore $H_5(\Sigma F)$ is torsion-free, hence so is H_6 (cofibre $(\Sigma F \rightarrow JX/X)$), and thus also π_5 (fibre $(\Sigma F \rightarrow JX/X)$).] \square

The following result provides some evidence for Conjecture 1.8 by showing that in all cases in which the Adams operations merely show $b \geq 0$, we do indeed have $b = 0$.

PROPOSITION 3.8. $J(m\xi_{C,n}) = 0$ iff

$$\max\{j - v_p\binom{m}{j} : p \text{ prime}, (p-1)j \leq n\} = 0.$$

Proof. By [3] we are reduced to showing, if p , m , and J are fixed, then $\nu_p(m) \geq j + \nu_p(j)$ for all $j \leq J$ iff $\nu_p(m) \geq j$ for all $j \leq J$. This follows easily from (i)

$$\binom{m}{j} = \frac{m}{j} \binom{m-1}{j-1}$$

and (ii) $j \leq p^r | m$ implies $\binom{m-1}{j-1} \not\equiv 0 \pmod{p}$. \square

4. Example of $b(RP_m^{m+n}) < \text{gd}(m\xi_{R,n})$. In this section we prove Theorem 1.9.

Proof that $b(RP_{64k+36}^{64k+46}) = 4$. Sq^4 will not allow it to desuspend below 4. The desuspension to 4 follows from 2.1 and the fact ([18]) that P_{64k+36}^{64k+46} and P_4^{14} have the same stable homotopy type. The result there is stated only for SRP's with at least 13 cells; moreover, this case would use their Lemma 3.6, of which only the barest hint of proof is given. So we prove it.

Since $J(RP^{10}) \approx Z_{64}$, it suffices to prove it in the single case P_{100}^{110} . Since $32\xi_9$ is trivial, the classifying map for $100\xi_{10}$ can be factored as:

$$P_{10} \xrightarrow{\Delta} P^{10} \times P^{10} \xrightarrow{1 \times p} P^{10} \times S^{10} \rightarrow BO_{68} \times BO_{32} \rightarrow BO_{100},$$

$$\searrow \text{68}\xi \times \text{32}\xi$$

and the induced map of Thom spaces is

$$P_{100}^{110} \rightarrow P_{68}^{78} \wedge (S^{32} \cup_{\eta^2\sigma} e^{42}),$$

which compresses into the 110-skeleton $\Sigma^{32}P_{68}^{78} \cup_{\eta^2\sigma} e^{110}$. By [25, p. 57] the image of $\eta^2\sigma$ in $\pi_{77}(S^{68}) \xrightarrow{i_*} \pi_{77}(P_{68}^{78})$ has filtration ≥ 4 , and multiplication by η^2 is injective on classes of $\pi_{77}(P_{68}^{78})$ of filtration ≥ 4 . Since $\eta^4 = 0$, this implies $i_*(\eta^2\sigma) = 0$. Therefore $P_{68}^{78} \cup_{\eta^2\sigma} e^{78} \simeq P_{68}^{78} \vee S^{78}$, and the following composite is an equivalence:

$$P_{100}^{110} \rightarrow \Sigma^{32}P_{68}^{78} \cup_{\eta^2\sigma} e^{110} \simeq \Sigma^{32}P_{68}^{78} \vee S^{110} \xrightarrow{P} \Sigma^{32}P_{68}^{78}$$

$$\simeq \Sigma^{96}P_4^{14}. \quad \square$$

5. Nondesuspensions of SRP's implied by J . In this section we prove Theorem 1.10. We use the method of [28; 5.1]. In this section $\nu(m) = \nu_2(m)$. Let $d = m - n + 2\nu(m) - \varepsilon + 1$. If P_m^{m+n} desuspends d

times, the dotted line in the diagram

$$\begin{array}{ccc}
 \Sigma^{-d}\Sigma^{-1}P_{m+1}^{m+n} & \dashrightarrow & S^{m-d} \\
 \downarrow & & \downarrow \\
 \Omega^d\Sigma^{-1}P_{m+1}^{m+n} & \xrightarrow{\Omega^d\alpha} & \Omega^dS^m \\
 & & \downarrow s \\
 & & Q(\Sigma^{m-d}P_{m-d})
 \end{array}$$

can be filled in, where α is the attaching map, $Q = \Omega^\infty\Sigma^\infty$, and s is the Snaith map ([34]). Thus it suffices to prove nontriviality of the composite

$$(5.1) \quad P_{m+1}^{m+n} \rightarrow \Sigma^{d+1}\Omega^dS^m \rightarrow \Sigma^{d+1}Q(\Sigma^{m-d}P_{m-d}) \rightarrow \Sigma^{m+1}P_{m-d} \wedge J,$$

where J is the spectrum called bJ in [15; §2] and J in [26].

The attaching map α factors through $P_{m+\rho}^{m+n}$, where $\rho = \rho(m) = 8a + 2^b$ if $\nu = \nu(m) = 4a + b$ with $0 \leq b \leq 3$. The class of (5.1) lies in the middle group of the exact sequence

$$\begin{aligned}
 \psi^{3-1} \rightarrow [P_{m+\rho}^{m+n}, \Sigma^{m+1}P_{m-d} \wedge \Sigma^3bsp] &\xrightarrow{\delta} [P_{m+\rho}^{m+n}, \Sigma^{m+1}P_{m-d} \wedge J] \\
 &\rightarrow [P_{m+\rho}^{m+n}, \Sigma^{m+1}P_{m-d} \wedge bo] \xrightarrow{\psi^{3-1}}
 \end{aligned}$$

which is calculated as in [15; p. 42]. We will prove

PROPOSITION 5.2. *The class of (5.1) is the image under δ of a class of filtration $\nu(m) - 3$.*

This is nontrivial exactly as in [15; §2], establishing 1.10.

To do this, we consider the following prototype diagram, whose right vertical composite is (5.1).

(5.3)

$$\begin{array}{ccccccc}
 P_{m+\rho}^{m+\rho} & \rightarrow & P_{m+\rho}^{m+2\rho} & \rightarrow & \cdots & \rightarrow & P_{m+\rho}^{m+t\rho} & \rightarrow & P_{m+\rho}^{m+n} \\
 r_1 \downarrow & \tilde{r}_1 \swarrow & r_2 \downarrow & \swarrow \tilde{r}_2 & & \swarrow & \downarrow r_t & \tilde{r}_t \swarrow & \downarrow r_{t+1} \\
 \Sigma^m\Omega^{m-1}S^m & \rightarrow & \Sigma^{m-\rho}\Omega^{m-\rho-1}S^m & \rightarrow & \cdots & \rightarrow & \Sigma^{m-(t-1)\rho}\Omega^{m-(t-1)\rho-1}S^m & \rightarrow & \Sigma^{d+1}\Omega^dS^m \\
 s_1 \downarrow & & s_2 \downarrow & & & & \downarrow s_t & & \downarrow s_{t+1} \\
 \Sigma^{m+1}P_1 \wedge J & \rightarrow & \Sigma^{m+1}P_{\rho+1} \wedge J & \rightarrow & \cdots & \rightarrow & \Sigma^{m+1}P_{(t-1)\rho+1} \wedge J & \rightarrow & \Sigma^{m+1}P_{m-d} \wedge J
 \end{array}$$

Here $t = [n/\rho]$ and \tilde{r}_j is the composite

$$\Sigma^k\Sigma^{-k}P_{m+\rho}^{m+(j+1)\rho} \rightarrow \Sigma^k\Omega^{k+1}\Sigma^{-1}P_{m+\rho}^{m+(j+1)\rho} \xrightarrow{\Sigma^k\Omega^{k+1}\alpha} \Sigma^k\Omega^{k+1}S^m$$

where $k = m - (j - 1)\rho - 1$, the desuspension existing for dimensional reasons.

We make the following minor modifications in (5.3):

If $\rho \equiv 1$ (8), replace $j\rho$ by $j(\rho - 1)$ in bottom two rows and by $j(\rho - 1) + 2$ in top row when $j \geq 2$.

If $\rho \equiv 2$ (8), add 1 to top dimensional $m + j\rho$ in the top row when j is even.

If $\rho \equiv 0$ (4), replace $j\rho$ by $j(\rho - 2) + 2$ throughout the diagram, and replace bottom dimension of spaces in top row by $m + \rho - 1$.

By [29; 5.4.1], r_1 is a generator of the image of J , and so [26; 8.3] implies that the left vertical composite is the image under δ of a class of filtration $\nu(m) - 3$, where δ is as in 5.2. The induction step uses the prototype diagram

$$(5.4) \quad \begin{array}{c} [P_{m+\rho}^{m+(j+1)\rho}, \Sigma^{m+1}P_{(j-1)\rho+1} \wedge J] \xrightarrow{c^*} [P_{m+\rho}^{m+(j+1)\rho}, \Sigma^{m+1}P_{j\rho+1} \wedge J] \\ \downarrow i^* \\ [P_{m+\rho}^{m+j\rho}, \Sigma^{m+1}P_{(j-1)\rho+1} \wedge J] \end{array}$$

modified as above. The homomorphisms are surjections of groups in $\text{im}(\delta)$:

$$(5.5) \quad \begin{array}{c} \mathbf{Z}/2^{2\nu-2} \xrightarrow{c^*} \mathbf{Z}/2^{\nu-2} \\ \downarrow i_* \\ \mathbf{Z}/2^{\nu-2} \end{array}$$

We ignore here low filtration \mathbf{Z}_2 's, which can be shown by naturality to play no significant role. The order of the $\mathbf{Z}/2^{2\nu-2}$ is only approximate; the important part is that $2^{\nu-3}$ is nonzero in all groups.

(5.3) implies that in (5.4)

$$(5.6) \quad \begin{array}{c} [s_j \tilde{r}_j] \mapsto [s_{j+1} r_{j+1}] \\ \downarrow \\ [s_j r_j] \end{array}$$

The induction hypothesis is that $[s_j r_j]$ is nonzero (in fact, is $2^{\nu-3}$ times a generator), and (5.5) and (5.6) then imply the same of $[s_{j+1} r_{j+1}]$, proving 5.2 and hence 1.10.

In [15] (see also [16]), sectioning results were obtained which showed that for certain n and m ($n/4 + 4 \log_2(n/2 - \nu(m) + 4) + 11 \leq \nu(m) \leq n/2 - 75$), the nonsectioning results implied by 1.10 are best possible. In these cases we have $\text{gd}(m\xi_n) = b(P_m^{m+n}) = n - 2\nu(m) + \varepsilon$; in particular, the gd and b here depend only upon n and $\nu(m)$. This is not always true when n and m do not satisfy these conditions.

One case of particular interest concerns $2^i\xi_{2^i-1}$ and $(2^L - 2^i)\xi_{2^i-1}$, which are stably the tangent bundle and normal bundle of RP^{2^i-1} . Since the number of vector fields on RP^n equals the number on S^n , we have $\text{gd}(2^i\xi_{2^i-1}) = 2^i - 2i - \Delta$ where

$$\Delta = \begin{cases} 1 \\ 0 \\ 2 \end{cases} \quad \text{if } i \equiv \begin{cases} 0 \\ 1, 2(4) \\ 3 \end{cases} \quad ([1]).$$

James ([23]) showed $\text{gd}((2^L - 2^i)\xi_{2^i-1}) \geq 2^i - 2i - \varepsilon$, and Gitler and Mahowald ([20], [21]) presented an argument which purported to show $\text{gd}((2^L - 2^i)\xi_{2^i-1}) = \text{gd}(2^i\xi_{2^i-1})$ by showing that the relevant map of classifying spaces is an H -map. In 1979 Crabb and Steer noted a mistake in the argument of [21], so that $\text{gd}((2^L - 2^i)\xi_{2^i-1})$ is still an open question. Of course this is equivalent to determining the immersion dimension of RP^{2^i-1} . The situation of $2^i\xi_{2^i-1}$ is rather unusual in that the stable geometric dimension of [15] is already realized, i.e. $\text{gd}(2^i\xi_{2^i-1}) = \text{gd}(2^{i+4k}\xi_{2^{i+4k}-1})$ for all k . For large k the bundle is defined on a stable complex, and so multiplication of the bundle by an odd number doesn't change its gd . If this were true for odd multiples of $2^i\xi_{2^i-1}$, then the desired immersion of RP^{2^i-1} would follow immediately, but here it is not clear.

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