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ON THE KO-ORIENTABILITY OF COMPLEX PROJECTIVE VARIETIES

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The essence of the Riemann-Roch theorem as generalized by P. Baum, W. Fulton, and R. MacPherson is the construction of a natural transformation

$$\alpha_0: K_0^{\mathrm{alg}} X \to K_0^{\mathrm{top}} X$$

from the Grothendieck group $K_0^{\mathrm{alg}} X$ of coherent algebraic sheaves on a complex quasi-projective variety X to the topological homology group $K_0^{\mathrm{top}} X$ complementary to the obvious natural transformation

$$\alpha^0: K^0_{\mathrm{alg}} X \to K^0_{\mathrm{top}} X$$

from the Grothendieck group $K^0_{\mathrm{alg}} X$ of algebraic vector bundles on X to the Atiyah-Hirzebruch group $K^0_{\mathrm{top}} X$ of topological vector bundles. Under this natural transformation, the class of the structure sheaf $\mathcal{O}_X$ corresponds to a homology class $\{X\}$,

$$\alpha_0[\mathcal{O}_X] = \{X\},$$

the K-orientation of X. Thus all varieties, singular or non-singular, are K-oriented, in contrast to the well-known fact that a smooth manifold M is K-orientable if and only if the Stiefel-Whitney class $w_3 M = 0 \in H^3(M, \mathbf{Z})$.

In this paper we begin the study of the problem of constructing KO-orientations for singular spaces by asking for which varieties X of complex dimension k the class $\{X\}$ lies in the image of the homomorphism

$$\varepsilon_{2k}: KO_{2k} X \to K_0 X,$$

where

$$\varepsilon_{.}: KO_{.} X \to K_{.} X$$

is the natural transformation dual to the complexification homomorphism

$$\varepsilon^{\cdot}: KO^{\cdot} X \to K^{\cdot} X$$

from the group of real vector bundles to the group of complex vector bundles. If X is non-singular, then it is necessary and sufficient that the Chern class $c_1 X = 0$.

Our principal tool in studying this question is an exact sequence

$$\cdots \to KO_n X \xrightarrow{\varepsilon_n} K_n X \xrightarrow{\gamma_{n-2}} KO_{n-2} X \xrightarrow{\sigma_{n-1}} KO_{n-1} X \to \cdots$$

dual to an exact sequence introduced by R. Bott [**Bo**] and presented in detail by M. Karoubi [**K**]. Here n denotes an integer mod 8, which must be replaced by its mod 2 residue in the expression $K_n X$.

A technical problem confronting the mathematician working in this area has been the lack of a definition of the homology theories $K.X$ and $KO.X$ as natural and elegant as Grothendieck's definition of the algebraic theory $K_0^{\mathrm{alg}} X$. Recently, P. Baum [**BD**] has introduced a geometric definition of $K.X$ which seeks to remedy this problem. Indeed, the results presented here were originally formulated and proven in the context of P. Baum's definition [**S**].

We adopt here a more primitive approach, in the hope of being briefer and more readily accessible. The notation of [**BFM$_2$**] is adopted and extended, and Alexander duality is adopted as the definition of $K.X$ and $KO.X$. The exact sequence above is then a special case of the Bott exact sequence. We prove a result reinterpreting the natural transformation $\gamma.$ which is significant both conceptually and computationally, as we illustrate by application to examples.

For a complex quasi-projective variety X of complex dimension k, the natural transformation $\gamma.$ leads to a new topological invariant $\gamma_{2k-2}\{X\}$ which generalizes the first Chern class of a non-singular variety. Those varieties for which this invariant vanishes constitute a class of examples of singular spaces which are KO-orientable.

1. K-theory and KO-theory. Let X be a closed subspace of a locally compact topological space Y, such that the pair (Y^+, X^+) of one-point compactifications is a pair of compact polyhedra. In [**BFM$_2$**], the relative group $K_X Y$ is defined as follows. Consider complexes

$$0 \to E_n \to \cdots \to E_1 \to E_0 \to 0$$

of complex vector bundles on Y which are exact off X. $K_X Y$ is the quotient of the free abelian group on the isomorphism classes of such complexes modulo the following relations:

(a) if $E. = E'. \oplus E''.$, then $[E.] = [E'.] + [E''.]$;

(b) if $E.$ is exact on Y, then $[E.] = 0$;

(c) if $E.$ is a complex on $Y \times [0,1]$, and $E.(t)$ denotes the restriction of this complex to $Y \times \{t\} = Y$, then $[E.(0)] = [E.(1)]$.

If C is a closed subpolyhedron of $Y \setminus X$, such that the inclusion is a deformation retract, then $K_X Y$ is isomorphic to $\tilde{K}^0(Y^+/C)$. If $f\colon Y' \to Y$ is a continuous map, such that $f^{-1}(X) \subseteq X'$, then there is a functorial homomorphism

$$f^*\colon K_X Y \to K_{X'} Y'.$$

If U is an open neighborhood of X in Y, and $i\colon U \to Y$ is the inclusion, then

$$i^*\colon K_X Y \to K_X U$$

is an isomorphism. The tensor product of complexes induces the exterior product

$$\times\colon K_{X_1} Y_1 \otimes_{\mathbf{Z}} K_{X_2} Y_2 \to K_{X_1 \times X_2} Y_1 \times Y_2$$

and the cup product

$$\cup : K_{X_1} Y \otimes_{\mathbf{Z}} K_{X_2} Y \cap K_{X_1 \cap X_2} Y.$$

Let $\pi\colon V \to Y$ be a real vector bundle of fibre dimension $n = 2k$ which has a particular Spin^c-structure. M. F. Atiyah, R. Bott, and A. Shapiro [**ABS**] construct a Thom class $\mu_V^c \in K_Y V$ as follows. Let $P \to Y$ be a principal Spin$^c(n)$-bundle, such that $V \approx P \times_{\mathrm{Spin}^c(n)} \mathbf{R}^n$. Let M_c be an irreducible $\mathbf{Z}/2$-graded module over the Clifford algebra $C_n \otimes_{\mathbf{R}} \mathbf{C}$ of the quadratic form $Q(x_1, \ldots, x_n) = -\sum x_i^2$ on $\mathbf{R}^n$, such that the element $e_1 \cdots e_n$ acts on M_c^0 as the complex scalar i^k. Let $E^i = P \times_{\mathrm{Spin}^c(n)} M^i$ for $i = 0, 1$. Clifford multiplication is a bilinear map

$$V \otimes_{\mathbf{R}} E^0 \to E^1.$$

The canonical section of $\pi^* V \to V$ thus determines a complex

$$0 \to \pi^* E^0 \to \pi^* E^1 \to 0$$

on V which is exact off the zero-section Y. The element of $K_Y V$ corresponding to this complex is $-\mu_V^c$. (The negative sign must be introduced to correct for the discrepancy between this complex, which has ascending indices, and the complexes in the definition of $K_Y V$, which have descending indices. In the definition, the rightmost non-zero bundle in a complex is regarded as being in the zeroth position.)

If $\pi\colon V \to Y$ is a complex vector bundle of complex fibre dimension k, then μ_V^c is also represented by the complex

$$0 \to \pi^* \Lambda^0 V \to \pi^* \Lambda^1 V \to \cdots \to \pi^* \Lambda^k V \to 0$$

determined by exterior multiplication with the canonical section of $\pi^* V \to V$. Dual to this complex is the complex

$$0 \to \pi^*\Lambda^k V^* \to \cdots \to \pi^*\Lambda^1 V^* \to \pi^*\Lambda^0 V^* \to 0$$

which represents the Koszul-Thom class $\lambda_V \in K_Y V$. Thus for a complex vector bundle,

$$\lambda_V = (-1)^k \bar{\mu}_V^c,$$

where the bar denotes the automorphism of $K_Y V$ induced by complex conjugation. For a real vector bundle of fibre dimension $n = 2k$, given a Spin^c-structure, this equation may be taken as the definition of λ_V. The Thom isomorphism

$$\phi\colon K_X Y \to K_X V$$

is then defined by

$$\phi a = \pi^* a \cup \lambda_V.$$

Graded relative groups are defined by

$$K_X^{-n} Y = K_X(Y \times \mathbf{R}^n)$$

for $n \geq 0$. The Thom isomorphism corresponds to Bott periodicity

$$\beta^{-n-2}\colon K_X^{-n} Y \to K_X^{-n-2} Y,$$

Thus $K_X^{\cdot} Y$ may be regarded as a $\mathbf{Z}/2$-graded theory.

If X is embedded as a closed subpolyhedron of $\mathbf{R}^n$, the Alexander duality isomorphism

$$K_X \mathbf{R}^n = K_n X$$

may be taken as the definition of $K_n X$ for $n \geq 0$. The Thom isomorphism again corresponds to Bott periodicity

$$\beta_{n+2}\colon K_n X \to K_{n+2} X$$

which, together with the fact that any two embeddings are isotopic if n is sufficiently large, implies that $K_n X$ is independent of the particular embedding. $K_{\cdot} X$ is also regarded as a $\mathbf{Z}/2$-graded theory.

If $f\colon X \to X'$ is a closed embedding, then

$$f_*\colon K_n X' \to K_n X$$

corresponds to the homomorphism

$$i^*\colon K_X \mathbf{R}^n \to K_{X'} \mathbf{R}^n$$

induced by the identity map on $\mathbf{R}^n$. If $f\colon X \to X'$ is a proper continuous map, then f_* may be described as follows. Let $f = h \circ g$, where $g\colon X \to X' \times D^{2k}$ is a closed embedding and $h\colon X' \times D^{2k} \to X'$ is the projection. If X' is embedded in $\mathbf{R}^n$, then there is an isomorphism

$$i^*\colon K_{X'}\mathbf{R}^{n+2k} \to K_{X' \times D^{2k}}\mathbf{R}^{n+2k}.$$

Composition with the Thom isomorphism yields an isomorphism

$$i^* \circ \phi\colon K_{X'}\mathbf{R}^n \to K_{X' \times D^{2k}}\mathbf{R}^{n+2k}$$

whose inverse is h_*. Then $f_* = h_* \circ g_*$.

The definition of relative groups $KO_X Y$ from complexes of real vector bundles on Y is identical to that of $K_X Y$. For a real vector bundle $\pi\colon V \to Y$ of fibre dimension $n = 8k$, the description of the Thom class $\mu_V \in KO_Y V$ is similar to that of μ_V^c, except that one uses an irreducible $\mathbf{Z}/2$-graded module M over C_n, such that $e_1 \cdots e_n$ acts on M^0 as the identity. The definition of the Thom isomorphism and of graded groups $KO_X^{-n} Y$ and $KO_n X$ is parallel to that of $K_X^{-n} Y$ and $K_n X$, except that $\mathbf{Z}/8$-graded theories are obtained.

2. Orientations of manifolds. Let M be a Spin^c-manifold, that is, a smooth manifold whose tangent bundle $TM \to M$ is given a particular Spin^c-structure, of dimension n. Let $f\colon M \to \mathbf{R}^{n+2k}$ be a smooth embedding. Then the Spin^c-structures on TM and $\mathbf{R}^{n+2k}$ together determine a unique Spin^c-structure on N_f, the normal bundle of the embedding (see Milnor [**M**]). Let U be a tubular neighborhood of M in $\mathbf{R}^{n+2k}$, which we identify with a neighborhood of the zero-section in N_f. The class in $K_n M$ corresponding to the Thom class $\lambda_{N_f} \in K_M N_f$ under the isomorphisms

$$K_M N_f \to K_M U \leftarrow K_M \mathbf{R}^{n+2k} = K_n M$$

is denoted by $\{M\}^c$, and is called the K-orientation of the Spin^c-manifold M.

Similarly, if M is a Spin-manifold of dimension n, then, letting $f\colon M \to \mathbf{R}^{n+8k}$, one obtains the KO-orientation $\{M\} \in KO_n M$.

There is an exact sequence

$$\cdots \to KO_X^{-n} Y \to K_X^{-n} Y \to KO_X^{-n+2} Y \to KO_X^{-n+1} Y \to \cdots$$

due to R. Bott [**Bo**]. The natural transformations which appear in this sequence are described by M. Karoubi [**K**] as follows.

$$\varepsilon^{-n}\colon KO_X^{-n} Y \to K_X^{-n} Y$$

is the homomorphism induced by complexification of a real vector bundle.

$$\gamma^{-n+2}\colon K_X^{-n}Y \to KO_X^{-n+2}Y$$

is the composite of the inverse of the complex periodicity isomorphism and the homomorphism ρ induced by regarding a complex vector bundle as a real vector bundle

$$K_X^{-n}Y \overset{\beta}{\leftarrow} K_X^{-n+2}Y \overset{\rho}{\leftarrow} KO_X^{-n+2}Y.$$

Finally

$$\sigma^{-n+1}\colon KO_X^{-n+2}Y \to KO_X^{-n+1}Y$$

is the homomorphism defined by

$$\sigma a = a \times \xi$$

where $\xi \in KO_{\mathrm{pt}}^{-1}(\mathrm{pt}) = \mathbf{Z}/2$ is the generator.

If M is a smooth manifold of dimension n, embedded in $\mathbf{R}^{n+8k}$, then the exact sequence above becomes the homology exact sequence

$$\cdots \to KO_n M \xrightarrow{\varepsilon_n} K_n M \xrightarrow{\gamma_{n-2}} KO_{n-2}M \xrightarrow{\sigma_{n-1}} KO_{n-1}M \to \cdots$$

From the short exact sequence of groups [**ABS**]

$$1 \to \mathrm{Spin}(n) \to \mathrm{Spin}^c(n) \xrightarrow{d} U(1) \to 1$$

it follows that

(a) if M is a Spin-manifold, then M can also be regarded as a Spin^c-manifold,

(c) if M is a Spin^c-manifold, then M is given a complex line bundle $L \to M$, and

(c) if M is a Spin^c-manifold, then M admits a Spin-structure inducing the given Spin^c-structure if and only if the complex line bundle $L \approx M \times \mathbf{C}$.

PROPOSITION. *If M is a Spin-manifold, then $\varepsilon_n\{M\} = \{M\}^c$.*

Proof. The construction of $\mu_N^c \in K_M N$ requires an irreducible $\mathbf{Z}/2$-graded module M_c over $C_{8k} \times_{\mathbf{R}} \mathbf{C}$ such that $e_1 \cdots e_{8k}$ acts on M_c^0 as the scalar $i^{4k} = 1$. If M is the module required in the construction of μ_N, then

$M_c \approx M \times_{\mathbf{R}} \mathbf{C}$. It follows that $\varepsilon^0 \mu_N = \mu_N^c \in K_M N$. Complex conjugation leaves invariant the image of ε, thus

$$\varepsilon^0 \mu_N = \mu_N^c = (-1)^{4k} \bar{\mu}_N^c = \lambda_N.$$

Under the isomorphisms $KO_M N = KO_n M$ and $K_M N = K_n M$, this equation corresponds to $\varepsilon_n \{M\} = \{M\}^c$.

Let M be a Spin^c-manifold, and let $L \to M$ be the associated complex line bundle. Let s: $M \to L$ be a smooth section which is transverse to the zero-section of L. Let $Z = s^{-1}(0)$. Then Z is a smooth submanifold of M of dimension $n - 2$. Let f: $Z \to M$ be the inclusion.

PROPOSITION. *If M is a Spin^c-manifold, then Z is a Spin-manifold, and*

$$\gamma_{n-2}\{M\}^c = f_*\{Z\} \in KO_{n-2} M.$$

Proof. Let e: $M \to \mathbf{R}^{n+8k-2}$ be a smooth embedding. The differential ds: $TM \to TL$, together with the canonical decomposition $TL_x = TM_x \oplus L_x$ for $x \in M$, induces an isomorphism

$$\tilde{ds}: N_f \to f^* L.$$

Thus there is an isomorphism

$$N_{e \circ f} \approx f^* N_e \oplus f^* L.$$

Note that if K is the complex line bundle associated with the Spin^c-structure on N_e, then $K \otimes_{\mathbf{C}} L \approx M \times \mathbf{C}$, so that $L \approx \bar{K}$.

Using the isomorphism [**ABS**]

$$\mathrm{Spin}^c(n) = \mathrm{Spin}(n) \times_{\mathbf{Z}/2} U(1),$$

we define a homomorphism

$$h: \mathrm{Spin}^c(n) \to \mathrm{Spin}(n+2)$$

by

$$h(x, e^{it}) = x(\cos t/2 - e_{n+1} e_{n+2} \sin t/2).$$

if $\tilde{1}$: $U(1) \to \mathrm{Spin}^c(2)$ is defined as in [**ABS**] by

$$\tilde{1}(e^{it}) = \left(\cos t/2 + e_1 e_2 \sin t/2, e^{it/2}\right)$$

then the following diagram commutes

$$\begin{array}{ccc} \mathrm{Spin}^c(U) & \overset{\mathrm{id}\times\bar{d}}{\to} & \mathrm{Spin}^c(n)\times U(1) \\ & & \downarrow \mathrm{id}\times\tilde{1} \\ \downarrow h & & \mathrm{Spin}^c(n)\times \mathrm{Spin}^c(2) \\ & & \downarrow \\ \mathrm{Spin}(n+2) & \to & \mathrm{Spin}^c(n+2) \end{array}$$

It follows that the Spin^c-structure on N_e induces a particular Spin-structure on $N_e \oplus L \approx N_e \oplus \overline{K}$, and thus on $N_{e\circ f}$. Together with the standard Spin-structure on $\mathbf{R}^{n+8k-2}$, this determines a Spin-structure on Z.

Let ϕ: $V' \to V$ be the exponential diffeomorphism of a neighborhood V' of the zero-section in N_f onto a tubular neighborhood V of Z in M. There is a vector bundle map

$$\Phi\colon \pi^* N_f|V' \to L|V$$

over ϕ, extending the map

$$ds\colon N_f \to f^*L$$

over the zero-section, such that if

$$r\colon V' \to \pi^* N_f|V'$$

is the canonical section, then the following diagram commutes

$$\begin{array}{ccc} \pi^*N_f|V' & \overset{\Phi}{\to} & L|V \\ \uparrow r & & \uparrow s \\ V' & \overset{\phi}{\to} & V. \end{array}$$

Explicitly, if $\pi(v) = x$, then

$$\Phi_v\colon (N_f) \to L_{\phi(v)}$$

is defined by

$$\Phi_v(\lambda v) = \lambda s(\phi v)$$

for $\lambda \in \mathbf{C}$, $v \in V'$, $v \neq 0$. Then

$$\lim_{\lambda\to 0} \Phi_{\lambda v}(v) = \lim_{\lambda \to 0} \frac{1}{\lambda} s\circ\phi(\lambda v) = ds_x(v)$$

so that Φ extends to the required map over the zero section.

More generally, let U and U' be tubular neighborhoods of M and Z, respectively, in $\mathbf{R}^{n+8k-2}$, such that $U' \subseteq U$. Identifying U and U' with neighborhoods of the zero sections in N_e and $N_{e\circ f} \approx f^*N_e \oplus f^*L$, there is a vector bundle map

$$\pi^*N_{e\circ f} \to \pi^*N_e \oplus \pi^*L$$

over the inclusion $U' \subseteq U$ such that, if $r'\colon U' \to \pi^*N_{e\circ f}$ and $r\colon U \to \pi^*N_e$ are the canonical sections, then the following diagram commutes

$$\begin{array}{ccc} \pi^*N_{e\circ f} & \to & \pi^*N_e \oplus \pi^*L \\ \uparrow r' & & \uparrow r \oplus \pi^*S \\ U' & \to & U. \end{array}$$

Regard $U = U \times \{1\} \subseteq U \times [0,1]$, and extend the use of π to denote the projection $U \times [0,1] \to M$. Define a section

$$U \times [0,1] \to \pi' N_e \oplus \pi^*L$$

by

$$(u, t) \to r(u) \oplus t\pi^*s(u).$$

This section, together with the Spin-structure on $N_e \oplus L$, determines, as in the construction of the Thom class, a complex of real vector bundles

$$0 \to \pi^*E^0 \to \pi^*E^1 \to 0$$

on $U \times [0,1]$ which is exact off $Z \times [0,1] \cup M \times \{0\}$.

The restricted complex

$$0 \to \pi^*E^0(1) \to \pi^*E^1(1) \to 0$$

over U corresponds under the excision isomorphisms

$$KO_Z U \to KO_Z U' \leftarrow KO_Z N_{e\circ f}$$

to the class $-\mu_{N_{e\circ f}}$, and thus, under the isomorphism

$$KO_Z U \leftarrow KO_Z \mathbf{R}^{n+8k-2} = KO_{n-2} Z$$

to the class $-\{Z\}$.

The homomorphism $h\colon \operatorname{Spin}^c(n) \to \operatorname{Spin}(n+2)$ defined earlier extends to a homomorphism of Clifford algebras

$$h\colon C_n \otimes_{\mathbf{R}} \mathbf{C} \to C_{n+2}$$

determined by

$$h(e_j \otimes 1) = e_j$$
$$h(e_j \otimes i) = -e_j e_{n+1} e_{n+2}.$$

If M is an irreducible $\mathbf{Z}/2$-graded module over C_{8k}, then via this homomorphism M can be regarded as a $\mathbf{Z}/2$-graded module over $C_{8k-2} \otimes_{\mathbf{R}} \mathbf{C}$. A dimension count [**ABS**] shows that M is irreducible over $C_{8k-2} \otimes_{\mathbf{R}} \mathbf{C}$. Moreover, if $e_1 \cdots e_{8k}$ acts on M^0 as the identity, then via this homomorphism $e_1 \cdots e_{8k-2}$ acts on M^0 as multiplication by the scalar i, rather than $i^{4k-1} = -i$.

It follows that the restricted complex

$$0 \to \pi^* E^0(0) \to \pi^* E^1(0) \to 0,$$

regarded as a complex of complex vector bundles, represents the image of the class $\mu^c_{N_c}$ under the excision isomorphism

$$K_m N_e \to K_M U.$$

Disregarding the complex structure of this complex, it represents the common image of $\mu^c_{N_e}$ and $-\lambda_{N_e} = (-1)^{4k} \bar{\mu}^c_{N_e}$ under the composition

$$K_M N_e \to K_M U \xrightarrow{\rho} KO_M U.$$

Thus this complex corresponds to the image of $-\{M\}^c \in K_n M = K_{n-2} M$ under the homomorphism

$$\rho_{n-2}\colon K_{n-2} M \to KO_{n-2} M.$$

The identity map of U induces the homomorphism

$$\mathrm{id}^*\colon KO_Z U \to KO_M U$$

which corresponds to the homomorphism

$$f_*\colon KO_{n-2} Z \to KO_{n-2} M.$$

The homotopy of the complexes above shows that they represent the same class in $KO_M U$. It follows that

$$\gamma_{n-2}\{M\}^c = f_*\{Z\} \in KO_{n-2} M.$$

3. Application to complex projective varieties. Let X be a complex quasi-projective variety of complex dimension k. Denote the image of the structure sheaf $\mathcal{O}_X$ under the natural transformation

$$\alpha_0\colon K_0^{\mathrm{alg}} X \to K_0 X$$

by $\{X\}^c$. If X is non-singular, then X is a Spin^c-manifold, and it follows from [**ABS**] and [**BFM$_2$**] that this class is identical to the class $\{X\}^c$ constructed in Section 2. The non-singular variety X admits KO-orientations compatible with its K-orientation $\{K\}^c$ if and only if $c_1 X = 0 \in H^2(X; \mathbf{Z})$, which is equivalent to the condition that $\gamma_{2k-2}\{X\}^c = 0 \in KO_{2k-2}X$.

If X is singular, then the above results may be used to calculate $\gamma_{2k-2}\{X\}^c$ by finding a sum of structure sheaves of non-singular varieties to which the structure sheaf is equivalent in the Grothendieck group.

A simple example is provided by the nodal cubic curve X. To compute $\gamma_0\{X\}^c \in KO_0X = KO_0(\mathrm{pt}) = \mathbf{Z}$, we observe that if f: $\mathbf{P}_1 \to X$ is a resolution of the singularity, and i: pt $\to X$ is the inclusion of the singular point, then

$$[\mathcal{O}_X] = f_![\mathcal{O}_{\mathbf{P}_1}] - i_![\mathcal{O}_{\mathrm{pt}}]$$

and

$$\{X\}^c = f_*\{\mathbf{P}_1\}^c - i_*\{\mathrm{pt}\}^c.$$

When computing $\gamma_0\{X\}^c$, we must exercise care to find the image of each component of $\{X\}^c$ in KO_0X. Thus the above decomposition is not suitable, but can be replaced by

$$\{X\}^c = f_*\{\mathbf{P}_1\}^c - g_*\{\mathbf{P}_1\}^c$$

where g: $\mathbf{P}_1 \to X$ collapses $\mathbf{P}_1$ onto the singular point.We now apply γ_0 to find that

$$\begin{aligned}\gamma_0\{X\}^c &= f_*\gamma_0\{\mathbf{P}_1\}^c - g_*\gamma_0\{\mathbf{P}_1\}^c \\ &= 2 - 2 = 0 \in KO_0X.\end{aligned}$$

Thus the nodal cubic admits KO-orientations compatible with its K-orientation.

A more subtle example is provided by the following example [**BFM$_1$**]. Let C be a non-singular projective curve of genus $g > 2$, and let d be an integer between g and $2g$. Let $L \to C$ be a complex line bundle, such that $c_1 L = -d$. Let X be the variety obtained from the projective completion $P = P(L \oplus 1)$ by blowing the zero-section down to a singular point. Let f: $P \to X$ be the blow-down and i: pt $\to X$ the inclusion of the singular point. Then

$$\{X\}^c = f_*\{P\}^c + ni_*\{\mathrm{pt}\}^c$$

where $n = \dim_{\mathbf{C}} H^0(C; L^*)$.

An examination of the Atiyah-Hirzebruch spectral sequence shows that

$$KO_2 X = \mathbf{Z} \oplus \mathbf{Z}/2.$$

Thus $\gamma_2\{X\}^c$ consists of an integer and an integer mod 2. The integer part is equal to the integer

$$c_1 X \in H_2(X:\mathbf{Z}) = \mathbf{Z}$$

where $c_1 X$ here denotes the component of codimension 2 of the total Chern class of X defined by R. MacPherson [M]. A calculation shows that

$$c_1 X = d + 2 - 2g.$$

The summand $\mathbf{Z}/2$ of $KO_2 X$ is merely the contribution of $KO_2(\mathrm{pt})$, thus if h: $X \to \mathrm{pt}$, then the mod 2 component of $\gamma_2\{X\}^c$ is $h_*\gamma_2\{X\}^c = \gamma_2 h_*\{X\}^c$. We see that

$$h_*\{X\}^c = h_* f_*\{P\}^c + n\{\mathrm{pt}\}^c = 1 - g + n \in K_0(\mathrm{pt}) = \mathbf{Z},$$

and that γ_2: $K_0(\mathrm{pt}) \to KO_2(\mathrm{pt}) = \mathbf{Z}/2$ is reduction mod 2; thus the mod 2 component of $\gamma_2\{X\}^c$ is the mod 2 residue of $1 - g + n$.

In particular, if L is the dual of the canonical bundle K, then $d = 2g - 2$ and $n = g$, thus $c_1 X = 0$ but $\gamma_2\{X\}^c$ is equal to the non-zero element in the $\mathbf{Z}/2$ summand.

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