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AN ARTIN RELATION (MOD 2) FOR FINITE GROUP ACTIONS ON SPHERES

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Recently it has been shown that whenever a finite group G (not a p-group) acts on a homotopy sphere there is no general numerical relation which holds between the various formal dimensions of the fixed sets of p-subgroups (p dividing the order of G). However, if G is dihedral of order $2q$ (q an odd prime power) there is a numerical relation which holds (mod 2). In this paper, actions of groups G which are extensions of an odd order p-group by a cyclic 2-group are considered and a numerical relation (mod 2) is found to be satisfied (for such groups acting on spheres) between the various dimensions of fixed sets of certain subgroups; this relation generalises the classical Artin relation for dihedral actions on spheres.

0. Introduction. When a p-group P acts on a mod p homology n-sphere X, the fixed point set, X^H, of any subgroup H has the mod p homology of an $n(H)$-sphere, for some integer $n(H)$. The function from subgroups of P to integers defined by $H \to n(H)$ is called the dimension function and any such function arising in this way is known to originate in a real representation of P (see [**2**]). If P is elementary abelian, the Borel identity holds (see [**1**, pg. 175]):

$$n - n(P) = \sum (n(H) - n(P))$$

(sum over all $H \leq P$ such that $P/H = \mathbf{Z}_p$). The motivation for this identity comes from consideration of representations of P.

Now suppose G is the dihedral group D_p (p odd prime) (a semidirect product of $\mathbf{Z}_p$ and $\mathbf{Z}_2$ via the automorphism of $\mathbf{Z}_p$, $g \to g^{-1}$). If V is a real representation of G, one can by considering the real irreducible representations of G, write down the following Artin relation,

$$\dim V^G = \dim V^{\mathbf{Z}_2} - \left(\frac{\dim V - \dim V^{\mathbf{Z}_p}}{2}\right).$$

In [**3**], K. H. Dovermann and Ted Petrie show that for actions of D_p (and more generally any non p-group) on a homotopy sphere one cannot expect to find a numerical relation between the various dimensions of the fixed sets (in particular for smooth actions of D_p one cannot expect the Artin relation to hold). However, in [**8**, Thm. 1.3], E. Straume has shown that the Artin relation does hold, (mod 2). Specifically,

THEOREM ([**8**, Thm. 1.3.]): *If X is a mod $2p$ homology n-sphere (i.e., $X \sim_{2p} S^n$) with an action of $D_p = G$ and $X^{\mathbf{Z}_p} \sim_p S^l$, $X^{\mathbf{Z}_2} \sim_2 S^m$ then $\chi(X^G) = \chi(S^d)$ where*

$$d \equiv m - \left(\frac{n-l}{2}\right) \pmod 2.$$

In this paper we will generalize the Straume's result, and hence the Artin relation, considerably. Suppose G is a finite group which is an extension of an odd order p-group P by a cyclic 2-group $Q = \mathbf{Z}_{2^k}$; $P \rightarrowtail G \twoheadrightarrow Q$. We will call such groups G, "p-elementary", though this is not quite standared. Such G are always semi-direct products (Schur-Zassenhaus Lemma) via a homomorphism $\mathbf{Z}_{2^k} \xrightarrow{\phi} \mathrm{Aut}(P)$. If G acts on $X \sim_{2p} S^n$, we have:

THEOREM 1. *There exists a sequence of subgroups $e = P_m \lhd P_{m-1} \lhd \cdots \lhd P_1 \lhd P_0 = P$ and a corresponding sequence of non-negative integers $k_1 \leq k_2 \leq \cdots \leq k_m$ such that $\chi(X^G) = \chi(S^d)$ where*

$$d \equiv n(\mathbf{Z}_{2^k}) - \left(\sum_{i=1}^{m} \frac{n(P_i) - n(P_{i-1})}{2^{k-k_i}}\right) \pmod 2.$$

It should be noted here that the sequence of subgroups can be selected so that each factor group P_{i-1}/P_i is an irreducible representation of $\mathbf{Z}_{2^k}$ over the field $\mathbf{Z}_p$. If this is done, then by a Jordan-Hölder type theorem the length m is unique. Also, the subgroups P_i and the integers k_i depend entirely on the group structure of G. It isn't difficult to verify that a p-group with an action of $\mathbf{Z}_{2^k}$ has a decomposition similar to the above; we have taken pains in Lemma 2 below to ensure that one exists of an especially nice type. Also, we should regard Theorem 1 as a generalization of the situation for linear representations (see the remark following §3).

I would like to express sincere thanks to the referee, whose comments resulted in substantial improvements.

1. Irreducible representations of $\mathbf{Z}_{2^k}$ over $\mathbf{Z}_p$ (p odd prime). In this section we want to determine the irreducible representations of $\mathbf{Z}_{2^k}$ over $\mathbf{Z}_p$. The necessary results are contained in Lemma 1.

From now on the cyclic group $\mathbf{Z}_{2^j}$ will be written $C(j)$. If λ is a 2^j root of -1 in $\mathbf{Z}_p$, $\mathbf{Z}_p^{\lambda}$ denotes the one-dimensional representation of $C(j+1)$ given by multiplication by λ (this includes the case $\lambda = -1$, corresponding to $j = 0$ and $C(1)$, in which case we write $\mathbf{Z}_p^{-}$).

For any m such that $1 \leq m \leq k$, one can consider the induced representation of $C(m)$ over $\mathbf{Z}_p$, $\mathrm{Ind}_{C(1)}^{C(m)}(\mathbf{Z}_p^-)$, which we write as ρ_m. As a vector space over $\mathbf{Z}_p$, ρ_m has dimension 2^{m-1} and a generator of $C(m)$ acts on a basis $\{a_i\}_{i=1}^{2^{m-1}}$ by $a_i \to a_{i+1}$ if $i < 2^{m-1}$ while $a_{2^{m-1}} \to -a_1$. In general, if G is any group, $H \leq K \leq G$ are subgroups and V is a representation of H over some field, then induction is transitive, i.e. $\mathrm{Ind}_H^G(V) = \mathrm{Ind}_K^G(\mathrm{Ind}_H^K(V))$. Also if $V = V_1 + V_2$ then $\mathrm{Ind}_H^G(V) = \mathrm{Ind}_H^G(V_1) + \mathrm{Ind}_H^G(V_2)$. (For more information on induced representations, see [7; Chapter 7]).

Now there is a one-to-one (up to similarity) correspondence between faithful irreducible representations of $C(k)$ over $\mathbf{Z}_p$ and the irreducible factors of $x^{2^{k-1}+1}$ (consider the characteristic polynomial of a generator of $C(k)$). Our main concern here, therefore, will be to understand the factorisation of $x^{2^{k-1}+1}$ over $\mathbf{Z}_p$. Given any irreducible factor of $x^{2^{k-1}+1}$, note that if α is a root then the companion matrix in $\mathbf{Z}_p(\alpha)$ provides a representation of $C(k)$ which is faithful, irreducible, and such that the generator of $C(k)$ has the given factor as its characteristic polynomial. In the following lemma, evidently (a) is well-known—a proof is included for completeness.

LEMMA 1. (a) *The irreducible factors of* $x^{2^{k-1}+1}$ *all have the same degree* d *and that degree is the order of* p (mod 2^k), *i.e.* $p^d \equiv 1$ (mod 2^k).

(b) *If* $k = 1$ *or if* $k > 1$ *and* $p \equiv 1$ (mod 4) *then all the irreducible, faithful representations of* $C(k)$ *over* $\mathbf{Z}_p$ *are either* 1-*dimensional or are induced up from a* 1-*dimensional representation of a proper subgroup, all of the same* dim $= 2^{k-l}$.

(c) *If* $k > 1$ *and* $p \equiv 3$ (mod 4) *then all the faithful irreducible representations of* $C(k)$ *over* $\mathbf{Z}_p$ *are either* 2-*dimensional or are induced up from a* 2-*dimensional representation of a proper subgroup, all of the same* dim $= 2^{k-l+1}$.

Proof. (a) Let $g(x)$ be an irreducible factor of degree d of $x^{2^{k-1}} + 1$. Then $g(x)$ is the minimal polynomial for a primitive 2^k root of 1, say α. Consider the splitting field of $x^{p^d} - x$, which is just $\mathbf{Z}_p(\alpha)$ (since the degree of g is d). Thus $\alpha^{p^d - 1} = 1$, so that $2^k \mid p^d - 1$ (since α is a primitive root of 1). Now let $\hat{d}$ be any natural number such that $2^k \mid p^{\hat{d}} - 1$. We claim that $d \leq \hat{d}$, establishing (a). Let F be the splitting field of $x^{p^{\hat{d}}} - x$ and let $\phi\colon F \to F$ be the generator of the Galois group over $\mathbf{Z}_p$ given by $y \to y^p$. Suppose $\alpha, \phi(\alpha), \ldots, \phi^n(\alpha)$ are all distinct where $1 \leq n \leq \hat{d} - 1$ and consider the polynomial $h(x) = \prod_{i=0}^n (x - \phi^i(\alpha))$. The coefficients are symmetric functions in the $\phi^i(\alpha)$ and are fixed by ϕ hence belong to $\mathbf{Z}_p$.

Since $h(\alpha) = 0$, it follows that $d \leq n + 1 \leq \hat{d}$. Thus $d =$ degree of g is the order of $p \pmod{2^k}$.

(b) If $k = 1$, the only faithful, irreducible representation of $C(1)$ is $\mathbf{Z}_p^-$. So, we will assume that $k > 1$ and that $p \equiv 1 \pmod 4$. Let l be the largest integer such that $p \equiv 1 \pmod{2^l}$. If $k \leq l$, then $p \equiv 1 \pmod{2^k}$ and part (a) implies that any faithful irreducible representation of $C(k)$ has dimension 1, and these are given by multiplication by a 2^{k-1} root of -1, λ. These are the representations $\mathbf{Z}_p^\lambda$. If $k > l$ and $f(x)$ is an irreducible factor of $x^{2^{l-1}} + 1$ (deg f is 1, say $f(x) = x - \lambda$) then $g(x) = f(x^{2^{k-l}})$ has degree the order of $p \pmod{2^k}$, is a factor of $x^{2^{k-1}} + 1$ and is irreducible. On the other hand the characteristic polynomial of $\mathrm{Ind}_{C(l)}^{C(k)}(\mathbf{Z}_p^\lambda)$ is $x^{2^{k-l}} - \lambda$ (note that this representation has dimension 2^{k-l}).

(c) If $k > 1$ and $p \equiv 3 \pmod 4$, let l be the largest integer such that $p \equiv -1 \pmod{2^{l-1}}$. If $k \leq l$ then $p \equiv -1 \pmod{2^{k-1}}$ and $p^2 \equiv 1 \pmod{2^k}$. Thus any irreducible factor of $x^{2^{k-1}} + 1$ has degree 2 and so the dimension of the corresponding representation is 2. If $k > l$ and $f(x)$ is an irreducible factor of $x^{2^{l-1}} + 1$ (of degree 2) then $g(x) = f(x^{2^{k-l}})$ has degree the order of $p \pmod{2^k}$, is a factor of $x^{2^{k-1}} + 1$ and is irreducible. However, the characteristic polynomial of $\mathrm{Ind}_{C(l)}^{C(k)}(V)$ is $g(x)$ where V is a two-dimensional representation corresponding to $f(x)$ (note that this representation has dimension 2^{k-l+1}). This completes the proof of the lemma.

2. Normal chief series for p-elementary groups.

A normal chief series for a p-group P is a normal series whose adjacent quotients are elementary abelian. When P comes equipped with an automorphism ϕ of period 2^k (as in the present case, via conjugation) we would like to find a ϕ invariant normal chief series. We will call a representation of $C(k)$ over $\mathbf{Z}_p$ "homocyclic" if it decomposes into irreducible subrepresentations each having the same kernel.

LEMMA 2. *A p-group P with an automorphism ϕ of period 2^k has a ϕ invariant normal chief series whose adjacent quotients P_{i-1}/P_i are homocyclic representations of $C(k)$ with kernels $C(k_i)$, and $k_i \leq k_{i+1}$.*

Proof. For any p-group, P, the characteristic subgroup $P'P^p$ (P' is the commutator subgroup, P^p is generated by all pth powers) is called the Frattini subgroup, $\hat{P}$. $P/\hat{P}$ is elementary abelian and representatives in P of generators of $P/\hat{P}$ will generate P. Moreover, $\hat{P} = e$ iff P is elementary abelian (see [**5**; Ch. 5, Thm. 1.1]).

Set $P_0 = P$, consider the projection π: $P_0 \to P_0/\hat{P}_0$ and suppose that the representation of $C(k)$ on $P_0/\hat{P}_0$ decomposes into $V_1 \oplus \bar{V}_1$, where V_1 is the sum of all irreducible summands having the same, minimal kernel among the kernels appearing on $P_0/\hat{P}_0$, say $C(k_1)$. Now let $P_1 \lhd P_0$ be $\pi^{-1}(\bar{V}_1)$. Then on P_0/P_1, $C(k)$ acts with kernel $C(k_1)$. Consider $P_1/\hat{P}_1$ and write $P_1/\hat{P}_1$ as $V_2 \oplus \bar{V}_2$, where again V_2 is the sum of all irreducible summands with minimal kernel, say $C(k_2)$. $k_2 \geq k_1$ because generators for $P_1/\hat{P}_1$ lift to generators for P_1 and $C(k_1)$ acts trivially on P_0 hence on P_1 by [**5**; Thm. 1.4]. Let $P_2 = \pi^{-1}(\bar{V}_2)$ where π: $P_1 \to P_1/\hat{P}_1$. This process can be continued until a P_j is found such that $\hat{P}_j = e$. But then P_j is elementary abelian and certainly P_j can continue to be decomposed in this way. Thus we have a normal series

$$e = P_m \lhd P_{m-1} \lhd \cdots \lhd P_1 \lhd P_0 = P$$

such that $C(k)$ acts on P_{i-1}/P_i with $C(k_i)$ and $k_i \leq k_{i+1}$, $i = 1, 2, \ldots, m$.

3. Special cases and the Main Theorem. If G acts on a mod-p homology sphere X, we wish to compare the degree, δ_{X^P}, of a generator of $C(k)$ acting on X with the degree, δ_{X^P}, of the generator on X^P (the induced action since $P \lhd G$). The following lemma is central and is a modification of a key result of [**8**, compare Prop. 1.1].

LEMMA 3. *Suppose G is a semidirect product of an elementary abelian p-group P and a cyclic 2-group $C(k)$ such that the action of $C(k)$ on P (by conjugation) has kernel $C(m)$ and is irreducible. If G acts on a* mod-p *homology n-sphere X then the degrees δ_X and δ_{X^P} are related as follows*:

$$\delta_X = (-1)^{\varepsilon}\delta_{X^P} \quad \textit{where } \varepsilon = \frac{n - n(P)}{2^{k-m}}.$$

Proof. Proceeding exactly as in [**8**, loc. cit.], we consider the relative fibration $(X, Z) \to (X_P, Z_P) \xrightarrow{\pi} BP$, where $Z = X^P \sim_p S^r$. There is the spectral sequence of this relative fibering with E_2-term given by $E_2^{i,j} = H^i(BP) \otimes H^j(X, Z)$ (coefficients in Z_p). If d: $E_2^{0,n} \to E_2^{n-r,r+1}$ (where $r = n(P)$) is the transgression then $d(x) = A \otimes \delta z$ where x generates $H^n(X)$ and z generates $H^r(Z)$. If rank P is 1 then $A = t^{(n-r)/2}$, where t generates $H^2(BP)$. If rank $P > 1$ then recall the Borel identity, $n - r = \Sigma\,(n(H) - r)$, with sum on all corank 1 subgroups H in P. Suppose there are exactly s corank 1 subgroups $H_1, \ldots, H_s$ such that $n(H_i) - r > 0$. Letting $r_i = n(H_i)$, there are elements $w_1, \ldots, w_s \in H^2(BP)$ and an $a \in H^0(BP)$ such that $A = aw_1^{d_1}w_2^{d_2} \cdots w_s^{d_s}$ where $d_i = (r_i - r)/2$ (see [**6**, Thm. 2]).

Since P is an irreducible representation of $C(k)$ (let α be a generator) with kernel $C(m)$, P has either dimension 1 (if either $k-m=1$ or if $k-m>1$ and $p\equiv 1 \pmod 4$ with $k-m\le l$, where l is as defined in the proof of Lemma 1 (b) and depends only on p), or has dimension 2^{k-m-l} (resp. $2^{k-m-l+1}$) (if $p\equiv 1 \pmod 4$, resp. $p\equiv 3 \pmod 4$).

Now, just as in [**8**], $C(k)$ acting by conjugation on P determines an action of $C(k)$ on the fibration (and so on the spectral sequence) as follows. Define

$$\phi\colon EG\times X\to EG\times X \quad \text{by } \phi(e,x)=(e\alpha,\alpha^{-1}x)$$

where α generates $C(k)$. For $g\in P$, we have $\phi(g(e,x))=\psi(g)\phi(e,x)$ (ψ is the automorphism of P defined by $\alpha^{-1}g\alpha=\psi(g)$). Thus we have an action on the fibration (since $EG\simeq EP$):

$$\begin{array}{ccc} (X,Z) & \xrightarrow{\alpha} & (X,Z) \\ \downarrow & & \downarrow \\ (X_P,Z_P) & \xrightarrow{\tilde{\alpha}} & (X_P,Z_P) \\ \downarrow & & \downarrow \\ BP & \xrightarrow{\bar{\alpha}} & BP \end{array}$$

$\bar{\alpha}\colon BP\to BP$ is induced by $\psi\colon P\to P$. If P has dimension 1, $\bar{\alpha}^*(t)=\lambda t$, $\psi\colon P\to P$ is multiplication by λ, a 2^{k-m-1} root of -1 and t generates $H^2(BP)$. If the dimension of P is larger than 1, the action of $C(k)$ on the collection of subgroups $\{H_1,\dots,H_s\}$ must be considered (and the corresponding action on $w_1,\dots,w_s$). First of all, if $p\equiv 1 \pmod 4$, $\alpha^{2^{k-m-1}}$ acts on P by multiplication on the basis elements by λ, a 2^{l-1} root of -1 and no smaller power of α leaves the H_i invariant (smaller powers are represented by even dimensional irreducible subrepresentations). If $p\equiv 3 \pmod 4$, since there are no roots of -1 in $\mathbf{Z}_p$, the smallest power of α leaving the H_i invariant is $\alpha^{2^{k-m-1}}$ (this is just multiplication by -1). Therefore the members of $\{H_1,\dots,H_s\}$ are permuted, each one in a orbit of size 2^{k-m-l} (if $p\equiv 1 \pmod 4$) or size 2^{k-m-1} (if $p\equiv 3 \pmod 4$). This observation has several consequences. If H_i and H_j are in the same orbit, $(n(H_i)-r)/2=(n(H_j)-r)/2$ and it follows from the Borel Identity that 2^{k-m-l} ($p\equiv 1 \pmod 4$) or 2^{k-m-1} ($p\equiv 3 \pmod 4$) divides $(n-r)/2$. Now consider the class $aw_1^{d_1}\cdots w_s^{d_s}$. It follows from [**6**; Thm. 2; Lemma 3] that if $w_{i_1},\dots,w_{i_{2^{k-m-l}}}$ ($p\equiv 1 \pmod 4$) or $w_{i_1},\dots,w_{i_{2^{k-m-1}}}$ ($p\equiv 3 \pmod 4$) are in the same orbit, the classes are permuted, say $w_{i_j}\to w_{i_{j+1}}$ and $w_{i_{2^{k-m-l}}}\to\lambda w_{i_1}$ (λ a 2^{l-1} root of -1 and $p\equiv 1 \pmod 4$) (or $w_{i_{2^{k-m-l}}}\to -w_{i_1}$

if $p \equiv 3 \pmod 4$)). Under $\bar{\alpha}^*$ the class $aw_1^{d_1} \cdots w_s^{d_s}$ is sent to $\lambda^\varepsilon aw_1^{d_1} \cdots w_s^{d_s}$ (or $(-1)^\varepsilon aw_1^{d_1} \cdots w_s^{d_s}$) where $\varepsilon = (n-r)/2^{k-m-l+1}$ (or $(n-r)/2^{k-m}$ if $p \equiv 3 \pmod 4$)).

Consider now the commutative diagram (from the E_2-term):

$$\begin{array}{ccc} H^n(X, Z) & \xrightarrow{\alpha^*} & H^n(X, Z) \\ \downarrow d & & \downarrow d \\ H^{n-r}(BP) \otimes H^{r+1}(X, Z) & \xrightarrow{\bar{\alpha}^* \otimes \alpha^*} & H^{n-r}(BP) \otimes H^{r+1}(X, Z). \end{array}$$

We have:

$$\begin{aligned} d(\alpha^* x) &= \delta_X(A \otimes \delta z) = (\bar{\alpha}^* \otimes \alpha^*)(A \otimes \delta z) \\ &= \lambda^\varepsilon \delta_{X^P}(A \otimes \delta z) \quad \left(\text{or } (-1)^\varepsilon \delta_{X^P}(A \otimes \delta z)\right) \end{aligned}$$

where

$$\varepsilon = \begin{cases} (n-r)/2^{k-m-l+1} & \text{if } p \equiv 1 \pmod 4, \\ (n-r)/2^{k-m} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Thus

$$\delta_X = \lambda^\varepsilon \delta_{X^P} \quad \left(\text{or } (-1)^\varepsilon \delta_{X^P}\right).$$

Since each of δ_X, δ_{X^P} is ± 1, it follows that if $k - m \leq l$,

$$2^{k-m-1} \mid (n-r)/2$$

while if $k - m > l$,

$$2^{l-1} \mid (n-r)/2^{k-m-l+1}$$

(all of this only when $p \equiv 1 \pmod 4$)).

Finally we have,

$$\delta_X = (-1)^\varepsilon \delta_{X^P} \quad \text{where } \varepsilon = (n-r)/2^{k-m}.$$

This completes Lemma 3.

We can now prove an analogue of [**8**, Thm. 1.3]. Suppose G is a semidirect product of a p-group P and $C(k)$. Also, suppose that G acts on a $\mathbf{Z}_p$-homology n-sphere X.

LEMMA 4. *There is a sequence of subgroups $e = P_m \lhd P_{m-1} \lhd \cdots \lhd P_1 \lhd P_0 = P$ and a corresponding sequence of non-negative integers $k_1 \leq k_2 \leq \cdots \leq k_m$ such that if δ_X and δ_{X^P} denote, respectively, the degrees of a* *generator α of $C(k)$ on X, X^P then*

$$\delta_X = (-1)^{\varepsilon}\delta_{X^P}$$

where

$$\varepsilon = \sum_{i=1}^{m} \frac{n(P_i) - n(P_{i-1})}{2^{k-k_i}}.$$

Proof. This now follows directly from Lemmas 2 and 3 applied to the P_{i-1}/P_i action on X^{P_i}, where a normal series is obtained as in Lemma 2 and a refinement made so that adjacent quotients are irreducible.

The proof of the following is now clear.

THEOREM 1. *If G is a semidirect product as above, acting on a* mod-$2p$ *homology n-sphere X, then $\chi(X^G) = \chi(S^d)$ where*

$$d \equiv n(\mathbf{Z}_{2^k}) - \left(\sum_{i=1}^{m} \frac{n(P_i) - n(P_{i-1})}{2^{k-k_i}} \right) \pmod 2$$

where the P_i and k_i are as in Lemma 4.

Proof. From a well-known result of Floyd ([**4**]), $\chi(X^G)$ is the Lefschetz number of a generator of $\mathbf{Z}_{2^k}$ acting on X^P. One can easily verify that (from Lemma 4),

$$\delta_{X^P} = (-1)^{n - n(\mathbf{Z}_{2^k}) + \varepsilon}.$$

Since $n + n(P)$ is even,

$$\chi(X^G) = 1 + (-1)^{n(\mathbf{Z}_{2^k}) - \varepsilon}.$$

This completes the proof of Theorem 1.

COROLLARY. *If G and X are as in Theorem 1 and, moreover, G is a direct product then*

$$\chi(X^G) = \chi(X^{\mathbf{Z}_{2^k}}).$$

Proof. The reader may check that in this case the sum term appearing in the conclusion is 0 mod 2 (this is easy to see via Lemma 3). Note that this corollary is also easily obtained from a well-known result of Floyd (see [**1**; Ch. III, Th. 4.4.]).

REMARK. Suppose G is an extension of an elementary abelian p-group P by a cyclic 2-group $\mathbf{Z}_{2^k}$, $P \rightarrowtail G \twoheadrightarrow \mathbf{Z}_{2^k}$ and $\psi\colon \mathbf{Z}_{2^k} \to \mathrm{Aut}(P)$ has kernel

$\mathbf{Z}_{2^m}$. If V is a real representation of G then we have:

$$\dim V^G \equiv \dim V^{\mathbf{Z}_{2^k}} - \left(\frac{\dim V - \dim V^P}{2^{k-m}}\right) \pmod{2}.$$

This can be verified by considering the real irreducible representations of G, which originate from complex irreducible representation which in turn are induced up from complex irreducible representations of the subgroup $P \times \mathbf{Z}_{2^m}$. If those complex irreducible representations of G, for which both P and $\mathbf{Z}_{2^m}$ act nontrivially, are compared with those for which P acts nontrivially but $\mathbf{Z}_{2^m}$ acts trivially, the congruence above can be derived. It should also be noted that if $m = 0$ then the above congruence is actually an equality (for more information see [**7**; Chapters 7, 8 and 13]).

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