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TRIDIAGONAL MATRIX REPRESENTATIONS OF CYCLIC SELF-ADJOINT OPERATORS

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A bounded cyclic self-adjoint operator C, defined on a separable Hilbert space, can be represented as a tridiagonal matrix with respect to the basis generated by a cyclic vector. If the main diagonal entries are zeros, C may be regarded as the real part of a weighted shift operator. Define J to be the corresponding imaginary part and it follows that $CJ - JC = -2iK$ where K is a diagonal operator. The main purpose of this paper is to show that if the subdiagonal entries converge to a non-zero limit and if K is of trace class then C has an absolutely continuous part.

1. Introduction. Let C be a bounded self-adjoint operator on an infinite dimensional Hilbert space $\mathscr{H}$. If C has a cyclic vector ϕ then the Gram-Schmidt orthogonalization process applied to $\{\phi, C\phi, C^2\phi,\dots\}$ provides a basis $\{\phi_n\}$ for $\mathscr{H}$ which will make the matrix of C tridiagonal:

$$(1.1)\qquad C = \begin{bmatrix} b_1 & a_1 & 0 & 0 & \cdots \\ a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}, \qquad a_n > 0.$$

It is then easily shown (see [4]) that $\phi_n = P_n(C)\phi_1$ with the polynomials $\{P_n\}$ satisfying the following:

$$(1.2)\qquad P_1(\lambda) = 1, \qquad P_2(\lambda) = \frac{\lambda - b_1}{a_1},$$

$$P_n(\lambda) = \frac{(\lambda - b_{n-1})P_{n-1}(\lambda) - a_{n-2}P_{n-2}(\lambda)}{a_{n-1}}, \qquad (n > 2).$$

If $C = \int \lambda \, dE_\lambda$ and $\mu(\beta) = \|E(\beta)\phi_1\|^2$ for any Borel set β, then C can be represented as a multiplication operator on $L^2(\mu)$. The polynomials $\{P_n(\lambda)\}$ form an orthonormal basis for this space.

In the following, results on orthogonal polynomials are combined with the techniques of operator theory to study the relation between the constants in the tridiagonal matrix representation of the operator and its spectrum. Throughout it will be assumed that the main diagonal elements in (1.1) are zero. In this case C may be regarded as the real part of the shift operator T defined by $T\phi_n = 2a_n\phi_{n+1}$. Letting J be the corresponding imaginary part it can be shown that $CJ - JC = -2iK$ where K is a diagonal operator with diagonal entries $a_1^2, -a_1^2 + a_2^2, -a_2^2 + a_3^2, \ldots$. The main purpose of this paper is to study the spectrum of C, and, in particular, to establish the existence of an absolutely continuous part, under the assumption that K is of trace class.

Several results in the literature should be noted. In case the sequence $\{a_n\}$ is increasing the operator C is the real part of a hyponormal operator and the existence of an absolutely continuous part follows from results due to Putnam [**8**]. Commutator techniques are used in [**4**] to analyze the spectrum if the sequence $\{a_n\}$ decreases monotonically to a non-zero limit. Both of these special cases apply to the study of phase operators which are associated with the phase of the harmonic oscillator. (See [**6**], [**3**].) Also, as indicated above, the constants in the matrix representation of C appear in the recursion formula for a system of orthogonal polynomials. Systems of orthogonal polynomials and the corresponding distribution functions (or weight functions) have been studied by Nevai, among others, under various assumptions on the coefficients in the recursion formula. The results obtained by Nevai in [**7**] identify the absolutely continuous part of C if $\Sigma|a_n - \frac{1}{2}| < \infty$. Again this condition implies that K is of trace class. These facts motivated the main result to be presented. The remainder of the paper studies the properties of the sequence $\{a_n\}$ under the assumption that the absolutely continuous part of C is trivial.

2. Eigenvalues. It is shown in Stone [**10**] that $Cx = \lambda x$ if and only if $\sum_{n=1}^{\infty} |P_n(\lambda)|^2 < \infty$. If λ is an eigenvalue then $x = \Sigma x_n \phi_n$ with $x_n = P_n(\lambda)$. Also it is shown in [**4**] that if $Cx = \lambda x$ then

$$(2.1) \qquad \langle Kx, x\rangle = a_1^2 P_1^2(\lambda) + \sum_{n=2}^{\infty} \left(-a_{n-1}^2 + a_n^2\right) P_n^2(\lambda) = 0,$$

and if

$$S_N(\lambda) = a_1^2 P_1^2(\lambda) + \sum_{n=2}^{N} \left(-a_{n-1}^2 + a_n^2\right) P_n^2(\lambda)$$

then

$$(2.2) \quad S_N(\lambda) = \left[a_{N-1}P_{N-1}(\lambda) - \frac{\lambda}{2}P_N(\lambda)\right]^2 + \left(a_N^2 - \frac{\lambda^2}{4}\right)P_N^2(\lambda).$$

These facts are needed for the first result along with the following notation: $f^+(t) = f(t)$ if $f(t) \geq 0$ and $f^+(t) = 0$ if $f(t) < 0$; $f^-(t) = |f(t)|$ if $f(t) \leq 0$ and $f^-(t) = 0$ if $f^-(t) > 0$.

THEOREM 1. *If* $\Sigma_{n=2}^{\infty}(-a_{n-1}^2 + a_n^2)^- < \infty$ *then* C *has no eigenvalues in* $(-2\alpha, 2\alpha)$ *where* $\alpha = \liminf a_n$.

Proof. Assume that C has an eigenvalue λ in the interval $(-2\alpha, 2\alpha)$. Choose M such that $n > M$ implies that

$$a_n^2 > \frac{1}{2}\left(\frac{\lambda^2}{4} + \alpha^2\right) \quad \text{and} \quad \sum_{M+1}^{\infty}\left(-a_n^2 + a_n^2\right)^- < \frac{1}{4}\left(\alpha^2 - \frac{\lambda^2}{4}\right).$$

Choose N such that $P_N^2(\lambda) = \max_{n>M} P_n^2(\lambda)$. Then for $l > N$

$$\begin{aligned} S_l(\lambda) &\geq S_N(\lambda) - \frac{1}{4}\left(\alpha^2 - \frac{\lambda^2}{4}\right)P_N^2(\lambda) \\ &= \left[a_{N-1}P_{N-1}(\lambda) - \frac{\lambda}{2}P_N(\lambda)\right]^2 + \left(a_N^2 - \frac{\lambda^2}{4}\right)P_N^2(\lambda) \\ &\quad - \frac{1}{4}\left(\alpha^2 - \frac{\lambda^2}{4}\right)P_N^2(\lambda) \\ &\geq \frac{1}{4}\left(\alpha^2 - \frac{\lambda^2}{4}\right)P_N^2(\lambda). \end{aligned}$$

If $P_N(\lambda) \neq 0$ then λ cannot be an eigenvalue. The same is true if $P_N(\lambda) = 0$ and $S_N(\lambda) \neq 0$. If $P_N(\lambda) = 0$ and $S_N(\lambda) = 0$ then $P_{N-1}(\lambda) = 0$ and the recursion formula (1.2) implies that $P_n(\lambda) = 0$ for $1 \leq n \leq N - 1$ which contradicts the fact that $P_1(\lambda) = 1$.

3. Absolute continuity. If $C = \int \lambda \, dE_\lambda$, let $\mathscr{H}_{ac}(C)$ denote the set of elements x in $\mathscr{H}$ for which $\|E_\lambda x\|^2$ is an absolutely continuous function of λ. It is known that $\mathscr{H}_{ac}(C)$ is a subspace that reduces C. The restriction of C to this subspace is called the absolutely continuous part of C. The following lemma, due to Putnam [**8**, see the proof of Theorem 2.2.4], is needed for the next result.

LEMMA. *If* C *and* J *are bounded self-adjoint operators defined on* $\mathscr{H}$ *and if* $CJ - JC = 2iK$ *then for any interval* Δ *and any* x *in* $\mathscr{H}$,

$$|\langle KE(\Delta)x, E(\Delta)x\rangle| \leq \|J\| \, |\Delta| \, \|E(\Delta)x\|^2,$$

where $|\Delta|$ *denotes the length of* Δ.

It will now be assumed that $\lim a_n = \frac{1}{2}$. Since it is well known that the spectrum of the real part of the unilateral shift operator is exactly the interval $[-1, 1]$, Weyl's Theorem guarantees that the spectrum of C contains the interval $[-1, 1]$ and that any points in the spectrum outside of this interval must be eigenvalues.

THEOREM 2. *If* $\lim a_n = \frac{1}{2}$ *and if* $\sum |-a_{n-1}^2 + a_n^2| < \infty$ *then* C *has a non-trivial absolutely continuous part whose spectrum contains the interval* $(-1, 1)$.

Proof. Given $k > 1$ choose M such that $n > M$ implies that

$$\left|a_n^2 - \frac{1}{4}\right| < \frac{1}{16k} \quad \text{and} \quad \sum_{M+1}^{\infty} \left|-a_{n-1}^2 + a_n^2\right| < \frac{1}{8k}.$$

Consider the operator C_k whose first M weights are $\frac{1}{2}$, with the remaining weights equal to those of C. It will first be shown that the spectrum of the absolutely continuous part of C_k contains the interval $(-1 + 1/k, 1 - 1/k)$. Toward this end let Δ be a subinterval of $(-1 + 1/k, 1 - 1/k)$. Several cases need to be considered. Let $\mu(\Delta) = \|E(\Delta)\phi_1\|^2$, and note that

$$\langle KE(\Delta)\phi_1, E(\Delta)\phi_1\rangle = a_1^2\left|\int_\Delta P_1\, d\mu\right|^2 + \sum_{n=2}^{\infty}\left(-a_{n-1}^2 + a_n^2\right)\left|\int_\Delta P_n\, d\mu\right|^2.$$

Case I. Suppose $\int_\Delta P_n^2\, d\mu \leq \mu(\Delta) = \int_\Delta P_1^2\, d\mu$ for each n. Then

$$\begin{aligned}|\langle KE(\Delta)\phi_1, E(\Delta)\phi_1\rangle| &\geq a_1^2|\mu(\Delta)|^2 - \mu(\Delta)\sum\left|-a_{n-1}^2 + a_n^2\right|\int_\Delta P_n^2\, d\mu. \\ &\geq |\mu(\Delta)|^2\frac{1}{4}\left(1 - \frac{1}{2k}\right).\end{aligned}$$

Case II. Suppose $\int_\Delta P_n^2\, d\mu \leq \mu(\Delta)$ except for a finite number of n. Then there exists N such that $\int_\Delta P_N^2\, d\mu = \max_n \int_\Delta P_n^2\, d\mu$. It follows that

$$|\langle KE(\Delta)\phi_1, E(\Delta)\phi_1\rangle| \geq a_1^2\left|\int_\Delta d\mu\right|^2 + \mu(\Delta)\sum_{n=2}^{N}\left(-a_{n-1}^2 + a_n^2\right)\int_\Delta P_n^2\,d\mu$$
$$-\mu(\Delta)\sum_{n=2}^{N}\left(-a_{n-1}^2 + a_n^2\right)^+\int_\Delta P_n^2\,d\mu$$
$$-\sum_{N+1}^{\infty}\left|-a_{n-1}^2 + a_n^2\right|\left|\int_\Delta P_n\,d\mu\right|^2.$$

It then follows from (2.2) that

$$|\langle KE(\Delta)\phi_1, E(\Delta)\phi_1\rangle|$$
$$\geq \mu(\Delta)\left[\int_\Delta\left[a_{N-1}P_{N-1} - \frac{\lambda}{2}P_N\right]^2 d\mu + \int_\Delta\left(a_N^2 - \frac{\lambda^2}{4}\right)P_N^2\,d\mu\right]$$
$$-\frac{1}{8k}\mu(\Delta)\int_\Delta P_N^2\,d\mu$$
$$\geq \frac{1}{16k}|\mu(\Delta)|^2.$$

Case III. Suppose there exists a subsequence $\{P_{n_j}\}$ such that $\int_\Delta P_{n_j}^2\,d\mu \geq \mu(\Delta)$. Choose L sufficiently large such that

$$\sum_{L+1}^{\infty}\left|-a_{n-1}^2 + a_n^2\right|\left|\int_\Delta P_n\,d\mu\right|^2 \leq \mu(\Delta)\sum_{L+1}^{\infty}\left|-a_{n-1}^2 + a_n^2\right|$$
$$\leq \frac{1}{32k}|\mu(\Delta)|^2.$$

Choose P_M such that $M > L$ and $\int_\Delta P_M^2\,d\mu \geq \mu(\Delta)$. Choose P_N such that $\int_\Delta P_N^2\,d\mu = \max_{1\leq n\leq M}\int_\Delta P_n^2\,d\mu$. Then

$$|\langle KE(\Delta)\phi_1, E(\Delta)\phi_1\rangle|$$
$$\geq a_1^2\left|\int_\Delta P_1\,d\mu\right|^2 + \mu(\Delta)\sum_{n=2}^{N}\left(-a_{n-1}^2 + a_n^2\right)\int_\Delta P_n^2\,d\mu$$
$$-\mu(\Delta)\sum_{n=2}^{N}\left(-a_{n-1}^2 + a_n^2\right)^+\int_\Delta P_n^2\,d\mu$$
$$-\sum_{N+1}^{M}\left|-a_{n-1}^2 + a_n^2\right|\left|\int_\Delta P_n\,d\mu\right|^2 - \sum_{M+1}^{\infty}\left|-a_{n-1}^2 + a_n^2\right|\left|\int_\Delta P_n\,d\mu\right|^2.$$

Hence

$$\begin{aligned}
&|\langle KE(\Delta)\phi_1, E(\Delta)\phi_1\rangle| \\
&\quad \geq \left[\int_\Delta \left(a_{N-1}P_{N-1} - \frac{\lambda}{2}P_N\right)^2 d\mu + \int_\Delta \left(a_N^2 - \frac{\lambda^2}{4}\right)P_N^2\, d\mu\right]\mu(\Delta) \\
&\qquad - \frac{1}{8k}\mu(\Delta)\int_\Delta P_N^2\, d\mu - \frac{1}{32k}|\mu(\Delta)|^2 \\
&\quad \geq \frac{1}{32k}|\mu(\Delta)|^2.
\end{aligned}$$

In any case, therefore,

$$|\langle KE(\Delta)\phi_1, E(\Delta)\phi_1\rangle| \geq \frac{1}{32k}|\mu(\Delta)|^2.$$

This result and the previous lemma imply that $|\mu(\Delta)| \leq 32k\|J\|\,|\Delta|$. If β is a Borel subet of $(-1 + 1/k, 1 - 1/k)$ of Lebesgue measure zero, then for any $\varepsilon > 0$ there exists a pairwise disjoint sequence of intervals $\{\Delta_j\}$ such that $\Delta_j \subset (-1 + 1/k, 1 - 1/k)$, $\beta \subset U\Delta_j$ and $\Sigma|\Delta_j| < \varepsilon$. Then $\mu(\beta) \leq \Sigma_j \mu(\Delta_j) \leq 32k\|J\|\Sigma|\Delta_j|$ and it readily follows that $\mu(\beta) = 0$. Hence C_k has a non-trivial absolutely continuous part whose spectrum contains the interval $(-1 + 1/k, 1 - 1/k)$. Recall that C and C_k differ only in a finite number of weights. Thus C is a trace class perturbation of C_k and it follows from the Kato-Rosenblum Theorem that the spectrum of the absolutely continuous part of C must contain $(-1 + 1/k, 1 - 1/k)$ for each k as was to be shown.

4. Singular operators. It was shown in [**2**] that the condition $\liminf a_n > 0$ is necessary for the existence of a non-trivial absolutely continuous part for C. Assume now that C is a cyclic self-adjoint operator for which $\mathscr{H}_{\text{ac}}(C)$ is trivial. Must it follow that $\liminf a_n = 0$? It will be shown below that the problem can be simplified. Sufficient conditions will then be presented for an affirmative answer. Throughout it will be assumed that $\|C\| = 1$ so that $\text{sp}(C) \subset [-1, 1]$.

For any self-adjoint operator C the orthogonal complement of $\mathscr{H}_{\text{ac}}(C)$ can be decomposed as $\mathscr{H}_p(C) \oplus \mathscr{H}_{\text{sc}}(C)$ with $\mathscr{H}_p(C)$ generated by the eigenvectors of C. In [**1**] Cary and Pincus have shown that if $\mathscr{H}_{\text{sc}}(C) = \mathscr{H}$ then there exists a cyclic vector y such that $C + y \otimes y$ is diagonal. It is easily seen that the matrix of C with respect to the basis obtained by orthonormalizing $\{y, Cy, C^2y, \ldots\}$ differs from the matrix $D = C + y \otimes y$ with respect to the same basis only in the first diagonal element. It is reasonable, therefore, to consider cyclic self-adjoint diagonal operators. Several results follow.

In case C is compact, so that the spectrum of C has exactly one limit point, it is easily shown that $\lim a_n = 0$.

THEOREM 3. *If the spectrum of C is symmetric about the origin with exactly two limit points then* $\liminf a_n = 0$. *In particular, if C is invertible then* $\lim a_{2k} = 0$ *and if C is not invertible then* $\lim a_{2k+1} = 0$.

Proof. Assume, without loss of generality, that 1 and -1 are the limit points of the spectrum. From the matrix representation $a_n = \int \lambda P_n P_{n+1}\, d\mu$. Since P_{2k}/λ is a polynomial of degree $(2k-2)$, $\int P_{2k}/\lambda\, P_{2k+1}\, d\mu = 0$. Hence $a_{2k} = \int(\lambda^2 - 1)P_{2k}/\lambda\, P_{2k+1}\, d\mu$. If the spectrum of C, which is the support of the measure μ, is bounded away from the origin then Holders Inequality and the fact that the sequence $\{P_k\}$ converges pointwise to zero on the spectrum of C imply that $\lim a_{2k} = 0$.

Now assume that C is not invertible. The recursion formula and an induction argument show that $a_n^2 + a_{n+1}^2 = \int \lambda^2 P_{n+1}^2\, d\mu$. Hence

$$a_{2k}^2 = \int \lambda^2 \left(P_{2k}^2 - P_{2k-1}^2 + P_{2k-2}^2 - \cdots\right) d\mu$$

$$= \int (\lambda^2 - 1)\left(P_{2k}^2 - P_{2k-1}^2 + P_{2k-2}^2 - \cdots\right) d\mu.$$

Let $X(\lambda)$ be the characteristic function of the singleton set $\{0\}$. Then

$$\mu(\{0\}) = \int X^2\, d\mu = \sum_{n=1}^{\infty} \left|\int X(\lambda) P_n(\lambda)\, d\mu\right|^2 = \sum_{n=1}^{\infty} \left|P_n(0)\mu(\{0\})\right|^2.$$

If $\mu(\{0\}) \neq 0$ then $\sum_{n=1}^{\infty} P_n^2(0) = 1/\mu(\{0\})$. Since $P_{2k}(0) = 0$ for $k = 1, 2, \ldots$, it follows that

$$a_{2k}^2 = 1 + \int_{\lambda \neq 0} (\lambda^2 - 1)\left(P_{2k}^2 - P_{2k-1}^2 + P_{2k-2}^2 - \cdots\right) d\mu.$$

Using the Christoffel-Darboux summation formula [**11**, Theorem 3.2.2]

$$\sum_{k=1}^{n} P_k(\xi) P_k(\lambda) = a_n \frac{P_n(\xi)P_{n+1}(\lambda) - P_{n+1}(\xi)P_n(\lambda)}{\lambda - \xi}$$

with $\xi = -\lambda$, it can be shown that the integral on the right converges to zero. Hence $\lim_{k \to \infty} a_{2k}^2 = 1$ and since $\lim_{k\to\infty}(a_{2k}^2 + a_{2k+1}^2) = 1$ it follows that $\lim a_{2k+1}^2 = 0$ as was to be shown. □

THEOREM 4. *If the spectrum of C has a finite number of limit points then* $\liminf a_n = 0$.

Proof. Suppose $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the limit points of the spectrum of C. Let $\pi(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_k)$. Then $\pi(\lambda)P_n(\lambda)$ can be expressed as a linear combination of $P_1, P_2, \ldots, P_{n+k}$ with the coefficient of P_{n+k} equal to $a_n a_{n+1} \cdots a_{n+k-1}$. Hence $\int \pi(\lambda) P_n P_{n+k}\, d\mu = a_n \cdots a_{n+k-1}$. Let $\{\xi_i\}$ be the support of the measure μ. Then given $\varepsilon > 0$ there exists M such that for $i > M$, $|\pi(\xi_i)| < \varepsilon/2$. Also there exists N such that for $n > N$, $|\int_E \pi(\lambda) P_n P_{n+k}\, d\mu| < \varepsilon/2$ where $E = \{\xi_1, \xi_2, \ldots, \xi_M\}$. For $n > N$, $|\int \pi(\lambda) P_n P_{n+k}\, d\mu| = a_n a_{n+1} \cdots a_{n+k-1} < \varepsilon$. Thus the sequence $\{a_n a_{n+1} \cdots a_{n+k-1}\}$ converges to zero and hence $\liminf a_n = 0$ as was to be shown. $\square$

THEOREM 5. *If for each n there exists a set of* $2n$ *points in* $[-1, 1]$ *such that all but a finite number of the atoms of* μ *are within* ε_n *of this set, with* $\lim(\varepsilon_n)^{1/n} = 0$, *then* $\liminf a_n = 0$.

Proof. Fix n and let $M = \{\xi_1, \xi_2, \ldots, \xi_{2n}\}$ be the corresponding set. If A denotes the set of atoms within ε_n of M and if B denotes the remaining finite set of atoms, then for any k,

$$\begin{aligned}(a_k a_{k+1} \cdots a_{k+2n-1})^{1/2n} &= \left[\int_A (\lambda - \xi_1) \cdots (\lambda - \xi_{2n}) P_k(\lambda) P_{k+2n}(\lambda)\, d\mu \right.\\ &\qquad \left. + \int_B (\lambda - \xi_1) \cdots (\lambda - \xi_{2n}) P_k(\lambda) P_{k+2n}^{(\lambda)}\, d\mu\right]^{1/2n}.\end{aligned}$$

Choose k sufficiently large so that $|P_k(\lambda)| < \varepsilon_n$ for any λ in B. It then follows from Holder's Inequality that

$$(a_k a_{k+1} \cdots a_{k+2n-1})^{1/2n} \leq (2^{2n} \cdot \varepsilon_n + 2^{2n} \cdot \varepsilon_n)^{1/2n} \leq 4\varepsilon_n^{1/2n}$$

which implies that $\liminf a_n = 0$. $\square$

Theorem 5 can be used to construct an example of a diagonal operator whose spectrum has an infinite number of limit points and yet $\liminf a_n = 0$.

EXAMPLE. The aim is to define an atomic measure μ on $[-1, 1]$ so that the limit points of the support of the measure are 1, -1, $1 - 1/2^{n^2}$, $-1 + 1/2^{n^2}$ $(n = 1, 2, \ldots)$. To this end define the positive atoms by the following scheme:

$$\frac{1}{2} + \frac{1}{4};\ \frac{1}{2} + \frac{1}{8},\ 1 - \frac{1}{2^4} + \frac{1}{16};\ \frac{1}{2} + \frac{1}{32},\ 1 - \frac{1}{2^4} + \frac{1}{64},\ 1 - \frac{1}{2^9} + \frac{1}{128};$$

etc. Define the negative atoms symmetrically and assign weights to these points so that $\mu[-1,1] = 1$. If $Cf(\lambda) = \lambda f(\lambda)$ for f in $L^2(\mu)$ then C has a tridiagonal matrix representation with respect to the basis $\{P_n(\lambda)\}$ and by Theorem 5 $\liminf a_n = 0$. (Note that the conditions of the Theorem are satisfied with $\varepsilon_n = 1/2^{n^2}$.)

The previous results depend on the placement of the atoms of the discrete measure μ. The final result of this section depends on the rate of convergence of the measure on the sequence of eigenvalues.

THEOREM 6. *If* $\{\lambda_k\}$ *is the set of eigenvalues for the cyclic diagonal operator* C *and if* $[\sum_{k>n} \mu(\lambda_k)]^{1/2n}$ *converges to zero then* $\liminf a_n = 0$.

Proof. Let $A_n = \bigcup_{k>n}\{\lambda_k\}$. Then

$$(a_1 a_2 \cdots a_n)^{1/n} = \left(\int_{A_n} (\lambda - \lambda_1)\cdots(\lambda - \lambda_n) P_{n+1}\, d\mu\right)^{1/n}$$

$$\leq 2\left(\int_{A_n} P_{n+1}^2\, d\mu\right)^{1/2n}\left(\int_{A_n} d\mu\right)^{1/2n} \leq 2\left[\sum_{k>n} \mu(\lambda_k)\right]^{1/2n}.$$

It follows that $\liminf a_n = 0$.

REMARK. It is of course possible to define an atomic measure on $[-1,1]$ so that the atoms are dense in $[-1,1]$ and the convergence condition of Theorem 6 fails. It would be interesting to know $\liminf a_n$ under these conditions.

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