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ε-CONTINUITY AND MONOTONE OPERATIONS

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We prove constructively in the sense of Bishop that a monotone, ε-continuous operation from $[0,1]$ into a metric space is 2ε-uniformly continuous. We derive a suitable version of Brouwer's fan theorem.

1. Introduction. Zaslavskii [**ABR**, Theorem 7.3] gives an example of a real valued function on $[0,1]$ which is continuous at each computable point but which fails to be uniformly continuous. Zaslavskii [**ABR**, Theorem 7.14] and Mandelkern [**MND1, MND2**] show constructively in the Russian and Bishop sense, respectively, that a *monotone*, continuous, real valued function on $[0,1]$ is uniformly continuous. In this paper, we weaken the hypothesis of continuity to ε-continuity, generalize the definition of monotone so that the map can be into any metric space, and consider (non-extensional) operations instead of functions. We prove constructively [**BSH**] that a monotone, ε-continuous operation from $[0,1]$ into a metric space is 2ε-uniformly continuous. Delimiting examples show that the 2ε in the conclusion is best possible.

2. Valuated fans. The *binary fan* F consists of all finite or empty strings from $\{0,1\}$. Denote a string $a \in F$ by $a_1a_2 \cdots a_n$ where $a_i \in \{0,1\}$, and the empty string by $\varnothing$. The *length* $|a|$ of a is the cardinality n of the string a. The *descendants* of string a are strings containing a as an initial segment. The *immediate descendants* of a are $a0 = a_1a_2 \cdots a_n0$ and $a1 = a_1a_2 \cdots a_n1$. Note $\varnothing 0 = 0$ and $\varnothing 1 = 1$. A *branch* B is the set of initial segments of a countable string $B_1B_2 \cdots$ from $\{0,1\}$. We shall write $B \sim B_1B_2 \cdots$.

A *valuated fan* F is the binary fan together with a function V mapping F into the set N of non-negative integers. A valuation is *sub-additive* if $V(a) \geq V(a0) + V(a1)$, for all $a \in F$.

A valuation is *branch bounded* if for any branch B of F there is an integer n so that if $a \in B$ and $|a| \geq n$, then $V(a) = 0$. A valuation is *bounded* if there is an integer m so that if $a \in F$ and $|a| \geq m$, then $V(a) = 0$.

The valuated *fan generated by* $a \in F$ consists all descendents of a but with their initial segments a deleted; the valuation is the induced valuation.

We now arrive at a theorem implied by Brouwer's fan theorem [**HTG**] but that is valid in the sense of Bishop [**BSH**].

PROPOSITION 1. *Every branch bounded, sub-additive valuation on the binary fan is bounded.*

Proof. We induct on the value $V(\varnothing)$. If $V(\varnothing) = 0$ we are done, so let $V(\varnothing) > 0$. Construct a branch B starting at $\varnothing$ by induction. If $a \in B$ and $V(a1) = V(\varnothing)$, then append $a1$ to B. Otherwise append $a0$. Since F is sub-additive, if $a \in F$ and $V(a) = V(\varnothing)$ then $a \in B$. Since F is branch bounded, there is an integer n so that if $|a| \geq n$ and $a \in B$, then $V(\varnothing) = 0$. Hence if $|a| \geq n$ then $V(a) < V(\varnothing)$. Construct the 2^n fans F_i generated by those $a \in F$ with $|a| = n$. In each, the induced value $V_i(\varnothing)$ is strictly less than $V(\varnothing)$ in F. By induction, each is bounded: There are integers m_i such that if $|b| \geq m_i$ and $b \in F_i$, then the induced value $V_i(b) = 0$. Hence if $a \in F$ and $|a| \geq n + \max\{m_i\}$, then $V(a) = 0$. □

3. Assigning valuations. In this section we show how an operation from $[0, 1]$ induces a valuation on the binary fan. To each $a = a_1 a_2 \cdots a_n \in F$ assign the diadic rationals

$$.a = \sum a_k 2^{-k} = .a_1 a_2 \cdots a_n \quad \text{(binary)},$$
$$.a^+ = .a + 2^{-|a|},$$

and the interval $I(a) = [.a, .a^+]$. To each branch $B \sim B_1 B_2 \cdots$ assign the real number

$$.B = \sum B_k 2^{-k} = .B_1 B_2 \cdots \quad \text{(binary)}.$$

Note that if $a \in B$, then $.B \in I(a)$.

DEFINITION. We denote two subsets of $[0, 1]$ by

$$B[0,1] = \{x \in [0,1] \,|\, x \text{ has an explicit binary representation}\},$$

and

$$D[0,1] = \{x \in [0,1] \,|\, x \text{ has a terminating binary representation}\}.$$

Note that $x \in B[0, 1]$ iff $x \geq d$ or $x \leq d$ for every diadic rational $d \in D[0, 1]$. □

DEFINITION. Let f be an operation on $B[0, 1]$ into a metric space M, d and $\varepsilon > 0$. Fix one value of $f(x)$ for each $x \in D[0, 1]$. A *valuation* on the binary fan *induced by* f, ε is assigned so that

$$V(a) = P \text{ implies } P - 2^{-2|a|} < \rho(a) < P + 1 - 2^{-2|a|-1},$$

where $\rho(a) = \varepsilon^{-1}d(f(.a^+), f(.a))$. □

Note that if $\rho(a) > P - 2^{-2|a|-1}$, then $V(a) \geq P$, and if $\rho(a) < 1 - 2^{-2|a|}$, then $V(a) = 0$.

4. Monotone operations. In this section we consider what valuation is induced on the binary fan by a monotone operation into a metric space. The notion of "between" replaces "order" in the definition of monotone.

DEFINITION. Let M, d be a metric space. A point $x \in M$ is *between* a and $b \in M$ if

$$d(a, x) + d(x, b) = d(a, b).$$

In addition if x is distinct from a and b, then x is *strictly between* a and b. □

The notion of "between" has been discussed by Blumenthal [**BLM**]; his use of "between" corresponds to our usage of "strictly between". We distinguish the present notions of "between" and "strictly between" in the next definition:

DEFINITION. An operation f from a metric space M_1 to a metric space M_2 is *monotone* if whenever x is strictly between a and $b \in M_1$, then $f(x)$ is between $f(a)$ and $f(b)$. □

LEMMA 1. *If x and y are between a and b then $d(x, y) \leq d(a, b)$.*

Proof. Let x and y be between a and b. Thus, adding

$$d(a, z) + d(z, b) = d(a, b)$$

for z equal to x and z equal to y, we obtain

$$2d(x, y) \leq d(a, x) + d(a, y) + d(b, x) + d(b, y) = 2d(a, b). \quad □$$

The next lemmas and a counterexample stated without proof show how *order* and *between* are related on the real line.

LEMMA. *If x is between distinct points a and b and not strictly between them, then $x = a$ or $x = b$.* □

LEMMA. *A real number x is (strictly) between a and $b \in R$ if $(a < x < b)$ $a \leq x \leq b$, or if $(a > x > b)$ $a \geq x \geq b$.* □

LEMMA. *If x is strictly between a and $b \in R$, then $a < x < b$ or $b < x < a$.* □

COUNTEREXAMPLE. If x between a and $b \in R$ implies $a \leq x \leq b$ or $a \geq x \geq b$, then for all $a \in R$, either $a \geq 0$ or $a \leq 0$. □

A real valued function which is monotone in the present sense need not be increasing or decreasing.

LEMMA. *If f is a monotone operation on $S \subset R$ to a metric space M, d and $a \leq x < b$ are in S with $d(f(a), f(x)) + d(f(x), f(b)) > d(f(a), f(b))$, then $x = a$.* □

Next we show that a monotone operation on $[0, 1]$ induces a sub-additive valuation on the binary fan.

PROPOSITION 2. *If f is a monotone operation from $B[0, 1]$ to a metric space M, d and $\varepsilon > 0$, then the valuation induced by f, ε is sub-additive.*

Proof. Now $|a0| = |a1| = |a| + 1$, so

$$V(a0) - 2^{-2|a|-2} < \rho(a0) \quad \text{and} \quad V(a1) - 2^{-2|a|-2} < \rho(a1).$$

Noting that monotonicity of f implies that $\rho(a) = \rho(a0) + \rho(a1)$, we find

$$V(a0) + V(a1) - 2^{-2|a|-1} < \rho(a).$$

Hence $V(a) \geq V(a0) + V(a1)$ and the valuation is sub-additive. □

5. ε-continuous operations.

In this section we turn our attention to what valuation on the binary fan is induced by an ε-continuous operation.

DEFINITION. An operation f from a metric space M_1, d_1 to a metric space M_2, d_2 is *ε-continuous* if for some $\varepsilon' < \varepsilon$ then for every $x \in M_1$ there is a $\delta > 0$ such that whenever $y \in M_1$ and $d_1(x, y) < \delta$, then $d_2(f(x), f(y)) < \varepsilon'$. □

Note that if $\varepsilon' < \varepsilon'' < \varepsilon$ then f is also ε''-continuous. Furthermore if $x = y$ then $d_2(f(x), f(y)) < \varepsilon'$.

PROPOSITION 3. *If f is an $\varepsilon/2$-continuous operation from $B[0,1]$ to a metric space M, d then the valuation induced by f, ε is branch bounded.*

Proof. Let B be a branch of the binary fan F. Choose $\varepsilon' < \varepsilon/2$ and $\delta > 0$ such that if $y \in B[0,1]$ and $|y - .B| < \delta$, then $d(f(y), f(.B)) < \varepsilon'$. Pick n so that $2^{-n} < \delta$ and $2\varepsilon' < \varepsilon(1 - 2^{-2n})$, and let $a \in B$ with $|a| \geq n$. Now $.B \in I(a)$ so $|.a - .B| < \delta$ and $|.a^+ - .B| < \delta$. Hence

$$d(f(.a^+), f(.a)) \leq d(f(.a^+), f(.B)) + d(f(.B), f(.a)) < 2\varepsilon' < \varepsilon(1 - 2^{-2|a|}),$$

and thus $V(a) = 0$. □

6. ε-uniformly continuous operations. In this section we prove that a monotone, ε-continuous operation on $[0,1]$ is 2ε-uniformly continuous.

DEFINITION. An operation f is *ε-uniformly continuous* from a metric space M_1, d_1 to a metric space M_2, d_2 if there is $\varepsilon' < \varepsilon$ and $\delta > 0$ such that whenever $x, y \in M_1$ and $d_1(x, y) < \delta$, then $d_2(f(x), f(y)) < \varepsilon'$.

THEOREM. *If f is a monotone operation from* $[0,1]$ *into a metric space M, d and $\varepsilon/2$-continuous on* $B[0,1]$ *then f is ε-uniformly continuous on* $[0,1]$.

Proof. Choose $\varepsilon' < \varepsilon$ so that f is also $\varepsilon'/2$-continuous on $B[0,1]$. Let the binary fan F have the valuation induced by f, ε'. By Proposition 3 the valuation is branch bounded, and by Propositions 1 and 2, the valuation is bounded. Hence, there is an m so that if $a \in F$ and $|a| \geq m$, then $V(a) = 0$.

Consider the finite set $S = \{1\} \cup \{.a \mid a \in F \text{ and } |a| = m\}$. By $\varepsilon'/2$-continuity, choose δ in $(0, 2^{-m})$ such that if $x \in [0,1]$, $z \in S$, and $|x - z| < \delta$, then $d(f(x), f(z)) < \varepsilon'/2$. Suppose that $x, y \in [0,1]$ and $|x - y| < \delta/3$. Either $|x - z| < \delta/2$ for some $z \in S$, or $|x - z| > \delta/3$ for each $z \in S$. In the former case $|y - z| < 5\delta/6$, and

$$d(f(x), f(y)) \leq d(f(x), f(z)) + d(f(z), f(y)) < \varepsilon' < \varepsilon.$$

In the latter case, pick $a \in F$ with $|a| = m$, such that x and y are strictly between $.a$ and $.a^+$; then by Lemma 1 and $V(a) = 0$:

$$d(f(x), f(y)) \leq d(f(.a), f(.a^+)) < \varepsilon' < \varepsilon. \qquad \square$$

7. Delimiting examples. The theorem is valid with $[0,1]$ replaced by $B[0,1]$. The first example shows that the result stated in the theorem is sharp with regard to ε. There is no obvious constructive example, so we give a classical one.

EXAMPLE 1 (Classical). The classical function

$$f(x) = \begin{cases} \varepsilon'/2, & \text{for } x > 1/2, \\ 0, & \text{for } x = 1/2, \\ -\varepsilon'/2, & \text{for } x < 1/2 \end{cases}$$

is monotone and $\varepsilon/2$-continuous for all $\varepsilon > \varepsilon' > 0$, but is not ε'-uniformly continuous. □

The next example shows that there are constructive ε-continuous operations which are neither continuous nor functions.

EXAMPLE 2 (Constructive). Let $x = .x_1x_2x_3 \cdots \in B[0,1]$. The operation

$$g(x) = \varepsilon'(x_1 - 1/2)$$

is ε-continuous and ε-uniformly continuous on $B[0,1]$ for any $\varepsilon > \varepsilon' > 0$.

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