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THE BOUNDARY REGULARITY OF THE SOLUTION OF THE $\bar{\partial}$-EQUATION IN THE PRODUCT OF STRICTLY PSEUDOCONVEX DOMAINS

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Let D be a strictly pseudoconvex domain in $\mathbf{C}^n$. We prove that for every $\bar{\partial}$-closed differential $(0, q)$-form f, $q \geq 1$, with coefficients of class $\mathscr{C}^\infty(D \times D)$, and continuous in the set $\bar{D} \times \bar{D} \setminus \Delta(D)$, the equation $\bar{\partial} u = f$ admits a solution u with the same boundary regularity properties. As an application, we prove that certain ideals of analytic functions in strictly pseudoconvex domains are finitely generated.

1. Introduction. Let D be a bounded strictly pseudoconvex domain in $\mathbf{C}^n$ with $\mathscr{C}^2$ boundary. It is known ([2], Theorem 2) that given a $(0, q)$-form f in D with coefficients of class $\mathscr{C}^\infty(D \times D)$ and continuous in $\bar{D} \times \bar{D}$, such that $\bar{\partial} f = 0$, $q = 1, \ldots, 2n$, there exists a $(0, q-1)$-form u in $D \times D$ such that the coefficients of u are also of class $\mathscr{C}^\infty(D \times D)$ and continuous in $\bar{D} \times \bar{D}$, and such that $\bar{\partial} u = f$.

In this paper, using the results from [2], and the method of [6], we prove the following theorem:

THEOREM 1. *Let D be a bounded strictly pseudoconvex domain in $\mathbf{C}^n$ with $\mathscr{C}^2$ boundary. Set $Q = (\bar{D} \times \bar{D}) \setminus \{(z, z) | z \in \partial D\}$. Suppose that f is a $(0, q)$ $\bar{\partial}$-closed differential form with coefficients in $\mathscr{C}^\infty(D \times D) \cap \mathscr{C}(Q)$. Then there exists a $(0, q-1)$-form u with coefficients in $\mathscr{C}^\infty(D \times D) \cap \mathscr{C}(Q)$, such that $\bar{\partial} u = f$.*

As an application, we prove a following theorem on the existence of the decomposition operators in some spaces of holomorphic functions in the product domain $D \times D$: Let D and Q be as above. Denote by $A_Q(D \times D)$ the space of all functions holomorphic in $D \times D$, which are continuous in Q. Let $(A_Q)_0(D \times D)$ be the subspace of $A_Q(D \times D)$, consisting of all functions which vanish on $\Delta(D)$, the diagonal in $D \times D$.

THEOREM 2. *Let $g_1, \ldots, g_N \in (A_Q)_0(D \times D)$ satisfy the following properties*: (i) $\{(z, s) \in Q | g_1(z, s) = \cdots = g_N(z, s) = 0\} = \Delta(D)$; (ii) for every $z \in D$, the germs at (z, z) of the functions g_i, $i = 1, \ldots, N$, generate the ideal of germs at (z, z) of holomorphic functions which vanish on $\Delta(D)$. Then for every $f \in (A_Q)_0(D \times D)$ there exist functions $f_1, \ldots, f_N \in A_Q(D \times D)$, such that $f = \sum_{i=1}^N g_i f_i$.

This theorem is an improvement of several results, obtained previously by different authors. Namely, Ahern and Schneider proved in [1], that if $f \in A(D)$, then there exist functions $f_i(z, s) \in A_Q(D \times D)$, such that

$$f(z) - f(s) = \sum_{i=1}^{n} (z_i - s_i) f_i(z, s), \qquad z, s \in D.$$

Øvrelid showed in [5], that if $s \in D$ is fixed and $g_1, \ldots, g_N \in A(D)$ are such that $\{ z \in \overline{D} | g_1(z) = \cdots = g_N(z) = 0\} = \{s\}$ and the germs of the functions g_i at s generate the ideal of germs at s of holomorphic functions which vanish at s, then every $f \in A(D)$ with $f(s) = 0$ can be written in the form

$$f(z) = \sum_{i=1}^{N} g_i(z) f_i(z), \qquad z \in D,$$

for some $f_i \in A(D)$. In [4], the validity of Theorem 2 was shown in the special case, when $D = U$—the unit disc in $\mathbf{C}$—and under the additional assumption, that there exists a neighborhood V of $\Delta(\partial U)$ in $\overline{U} \times \overline{U}$ such that $g_1, \ldots, g_N$ have no zeros in $V \cap (Q \setminus \Delta(U))$. The proof given in [4] is different from that in the present paper.

It could seem unnatural to omit the boundary diagonal $\Delta(\partial D)$ from study. However, when $f \in A(D \times D)$ and $f|_{\Delta(\overline{D})} \equiv 0$, $g_i(z, s) = z_i - s_i$, $i = 1, \ldots, n$, and

$$f(z, s) = \sum_{i=1}^{n} (z_i - s_i) f_i(z, s),$$

then, as in [1],

$$\frac{\partial f}{\partial z_k}(z, s) = f_k(z, s) + \sum_{i=1}^{n} (z_i - s_i) \frac{\partial f_i}{\partial z_k}(z, s),$$

and so, setting $s = z$, we obtain

$$\frac{\partial f}{\partial z_k}(z, z) = f_k(z, z), \qquad z \in D;$$

therefore, the functions f_i need not be in $A(D \times D)$, even if $f \in A(D \times D)$. In the sequel we will always assume that the considered domains are bounded. We will also use the following notations:

Given a domain $D \subset \mathbf{C}^n$, we denote by $\mathcal{O}(D)$ the space of holomorphic functions in D, and by $A(D)$ the algebra of all functions holomorphic in D and continuous in $\overline{D}$.

If $F(D)$ is a function space in the domain D, and $q = 1, 2, \ldots, F_{0q}(D)$ denotes the space of all differential forms of type $(0, q)$ with coefficients in $F(D)$.

Given a set X, $\Delta(X)$ is a diagonal in the Cartesian product $X \times X$.

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2. The solution of the $\bar{\partial}$-equation. In this section we give the proof of Theorem 1. Let $D \subset \mathbf{C}^n$ be a strictly pseudoconvex domain, with the defining function σ, i.e. σ is of class $\mathscr{C}^2$ and strictly plurisubharmonic in some neighborhood $\tilde{D}$ of $\bar{D}$, $D = \{z \in \tilde{D} | \sigma(z) < 0\}$, and $d\sigma(z) \neq 0$ for $z \in \partial D$. For $\varepsilon > 0$, set $\tau_\varepsilon(z, w) = \sigma(z) + \sigma(w) - \varepsilon|z - w|^2$, $(z, w) \in \tilde{D} \times \tilde{D}$. Then, if ε is sufficiently close to zero, the domain $G_\varepsilon = \{(z, w) \in \tilde{D} \times \tilde{D} | \tau_\varepsilon(z, w) < 0\}$ is strictly pseudoconvex in $\mathbf{C}^{2n}$ with the defining function τ_ε. Moreover, $\bar{D} \times \bar{D} \subset \bar{G}_\varepsilon$, and $\partial(\bar{D} \times \bar{D}) \cap \partial G_\varepsilon = \Delta(\partial D)$; therefore $Q \subset G_\varepsilon$ (we recall that $Q = (\bar{D} \times \bar{D}) \setminus \Delta(\partial D)$). It follows, that if $t < 0$ is sufficiently close to 0, the sets $G_{\varepsilon,t} = \{(z, w) \in \tilde{D} \times \tilde{D} | \tau_\varepsilon(z, w) < t\}$ are strictly pseudoconvex with $\mathscr{C}^2$ boundary, and $G_{\varepsilon,t} \subset G_{\varepsilon,t'} \subset G_\varepsilon$ for $t < t' < 0$. Set $E_{\varepsilon,t} = G_{\varepsilon,t} \cap (D \times D)$.

We want to apply [2], Theorem 2 to the domains $E_{\varepsilon,t}$. Note first, that if we define the mappings $\chi^i\colon \mathbf{C}^n \times \mathbf{C}^n, \to \mathbf{C}^n$, $i = 1, 2$, and $\chi^3\colon \mathbf{C}^n \times \mathbf{C}^n \to \mathbf{C}^n \times \mathbf{C}^n$ by $\chi^1(z, w) = z$, $\chi^2(z, w) = w$, and $\chi^3(z, w) = (z, w)$, and set, for fixed $\varepsilon > 0$ and $t < 0$, $\rho_1 = \rho_2 = \sigma$, and $\rho_3 = \tau_\varepsilon - t$, then

$$E_{\varepsilon,t} = \left\{ (z, w) \in \tilde{D} \times \tilde{D} | \rho_i(\chi^i(z, w)) < 0,\ i = 1, 2, 3 \right\}.$$

Therefore, $E_{\varepsilon,t}$ is a pseudoconvex polyhedron in the sense of [2]. We must also verify, that $E_{\varepsilon,t}$ satisfies the assumptions (C) and (CR) from [2], p. 523. Set

$$\operatorname{grad}_{\mathbf{C}} f = {}^t\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}, \frac{\partial f}{\partial w_1}, \ldots, \frac{\partial f}{\partial w_n} \right)$$

and

$$\operatorname{grad}_{\mathbf{R}} f = {}^t\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial w_n}, \frac{\partial f}{\partial \bar{z}_1}, \ldots, \frac{\partial f}{\partial \bar{w}_n} \right).$$

The condition (C) says, that for every ordered subset $A \subset \{1, 2, 3\}$, $A = \{\alpha_1, \ldots, \alpha_s\}$, the number $m_A = \operatorname{rank}(\operatorname{grad}_{\mathbf{C}} \chi^{\alpha_1}, \ldots, \operatorname{grad}_{\mathbf{C}} \chi^{\alpha_s})$ is constant in the neighborhood of the set

$$S_A = \left\{ (z, w) \in \partial E_{\varepsilon,t} | \rho\alpha_1(\chi^{\alpha_1}(z, w)) = \cdots = \rho_{\alpha_s}(\chi^{\alpha}s(z, w)) = 0 \right\}.$$

This condition is trivially satisfied, since the mappings χ^i are linear. It rests to verify the condition (CR): For every pair of ordered subsets $A, B \subset \{1,2,3\}$, $A = \{\alpha_1, \dots, \alpha_s\}$, $B = \{\beta_1, \dots, \beta_t\}$, such that for every $\beta_i \in B$,

$$\text{(2.1)} \qquad \operatorname{rank}\left(\operatorname{grad}_{\mathbf{C}} \chi^{\beta_i}, \operatorname{grad}_{\mathbf{C}} \chi^{\alpha_1}, \dots, \operatorname{grad}_{\mathbf{C}} \chi^{\alpha_s}\right) > m_A,$$

it follows that

$$\text{(2.2)} \quad \operatorname{rank}\Big(\operatorname{grad}_{\mathbf{R}}\left(\rho_{\beta_1} \circ \chi^{\beta_1}\right), \dots, \operatorname{grad}_{\mathbf{R}}\left(\rho_{\beta_t} \circ \chi^{\beta_t}\right), \operatorname{grad}_{\mathbf{R}} \chi^{\alpha_1}, \dots, \operatorname{grad}_{\mathbf{R}} \chi^{\alpha_s}, \operatorname{grad}_{\mathbf{R}} \overline{\chi^{\alpha_1}}, \dots, \operatorname{grad}_{\mathbf{R}} \overline{\chi^{\alpha_s}}\Big) = t + 2m_A$$

in a neighborhood of the set $S_{A \cup B}$. Note that if $3 \in A$ or $A = \{1,2\}$, then $m_A = 2n$, and hence for any β,

$$\operatorname{rank}\left(\operatorname{grad}_{\mathbf{C}} \chi^{\beta}, \operatorname{grad}_{\mathbf{C}} \chi^{\alpha_1}, \dots, \operatorname{grad}_{\mathbf{C}} \chi^{\alpha_s}\right) = 2n = m_A,$$

and so (2.1) is not satisfied. On the other hand, if $A = \{1\}$ or $A = \{2\}$, one can show that for every $B \subset \{1,2,3\}$ such that $A \cap B = \varnothing$, (2.1) holds. Therefore, in all those cases, we should verify (2.2).

Consider first the case $A = \{1\}$ and $B = \{2,3\}$. Then

$$S_{123} = \left\{ (z,w) \in \tilde{D} \times \tilde{D} \mid z, w \in \partial D,\ -\varepsilon |z - w|^2 = t \right\},$$

and the matrix

$$\left(\operatorname{grad}_{\mathbf{R}}\left(\rho_2 \circ \chi^2\right), \operatorname{grad}_{\mathbf{R}}\left(\rho_3 \circ \chi^3\right), \operatorname{grad}_{\mathbf{R}} \chi^1, \operatorname{grad}_{\mathbf{R}} \overline{\chi^1}\right)$$

at a point $(z,w) \in \tilde{D} \times \tilde{D}$ has the form

$$\left[\begin{array}{c|c|ccc|ccc}
 & \sigma_1(z) - 2\varepsilon(\bar{z}_1 - \overline{w}_1) & 1 & & & & & \\
 & \vdots & & \ddots & & & & \\
 & \sigma_n(z) - 2\varepsilon(\bar{z}_n - \overline{w}_n) & & & 1 & & & \\
\hline
\sigma_1(w) & \sigma_1(w) - 2\varepsilon(\overline{w}_1 - \bar{z}_1) & & & & & & \\
\vdots & \vdots & & & & & & \\
\sigma_n(w) & \sigma_n(w) - 2\varepsilon(\overline{w}_n - \bar{z}_n) & & & & & & \\
\hline
 & \sigma_{\bar{1}}(z) - 2\varepsilon(z_1 - w_1) & & & & 1 & & \\
 & \vdots & & & & & \ddots & \\
 & \sigma_{\bar{n}}(z) - 2\varepsilon(z_n - w_n) & & & & & & 1 \\
\hline
\sigma_{\bar{1}}(w) & \sigma_{\bar{1}}(w) - 2\varepsilon(w_1 - z_1) & & & & & & \\
\vdots & \vdots & & & & & & \\
\sigma_{\bar{n}}(w) & \sigma_{\bar{n}}(w) - 2\varepsilon(w_n - z_n) & & & & & &
\end{array}\right]$$

where we have set $\sigma_i = \partial\sigma/\partial\zeta_i$ and $\sigma_{\bar{i}} = \partial\sigma/\partial\bar{\zeta}_i$. Since $z \neq w$ for $(z,w) \in S_{123}$, this is true also for some neighborhood of S_{123}. Moreover, $(\sigma_1(w),\dots,\sigma_n(w),\ \sigma_{\bar{1}}(w),\dots,\sigma_{\bar{n}}(w)) \neq 0$ for $w \in \partial D$, since $d\sigma(w) \neq 0$ there. Therefore, in order to prove that the above matrix has rank $2 + 2m_A = 2 + 2n$, it is sufficent to show, that the vectors

$$u = (\sigma_1(w),\dots,\sigma_n(w),\ \sigma_{\bar{1}}(w),\dots,\sigma_{\bar{n}}(w)) = (u_1, \bar{u}_1)$$

and

$$v = (\bar{w}_1 - \bar{z}_1,\dots,\bar{w}_n - \bar{z}_n,\ w_1 - z_1,\dots,w_n - z_n) = (v_1, \bar{v}_1),$$

are linearly independent (over $\mathbf{C}$) in some neighborhood of S_{123}. But if $z, w \in \partial D$ and $u = \alpha v$ for some $\alpha \in \mathbf{C}$, $\alpha \neq 0$, then $u_1 = \alpha v_1$ and $\bar{u}_1 = \alpha\bar{v}_1$. Hence α is real. Therefore the vectors $z - w$ and $v(w) = (\sigma_{\bar{1}}(w),\dots,\sigma_{\bar{n}}(w))$ (the normal vector to ∂D at w) are linearly dependent over $\mathbf{R}$, as the vectors in $\mathbf{R}^{2n}$. This is impossible, if $z, w \in \partial D$ and z is sufficiently close to w, i.e. if t is sufficiently near 0. Hence, if we choose t sufficiently close to 0, vectors u and v are linearly independent over $\mathbf{C}$, for (z,w) in some neighborhood of S_{123}, and thus the condition (CR) is satisfied.

In order to prove (2.2) in the case $A = \{1\}$ and $B = \{2\}$ (resp. $B = \{3\}$), it suffices to note, that $S_{12} = \{(z,w) \in \partial E_{\varepsilon,t} | \sigma(z) = \sigma(w) = 0\}$ and $(\sigma_1(w),\dots,\sigma_n(w),\ \sigma_{\bar{1}}(w),\dots,\sigma_{\bar{n}}(w)) \neq 0$ for w in a neighborhood of ∂D (resp. that since

$$S_{13} = \left\{ (z,w) \in \partial E_{\varepsilon,t} | \sigma(z) = 0,\ \sigma(w) - \varepsilon|z - w|^2 = t \right\},$$

then also

$$\begin{aligned}(\sigma_1(w) - 2\varepsilon(\bar{w}_1 - \bar{z}_1),\dots,\sigma_n(w) - 2\varepsilon(\bar{w}_n - \bar{z}_n),\\ \sigma_{\bar{1}}(w) - 2\varepsilon(w_1 - z_1),\dots,\sigma_{\bar{n}}(w) - 2\varepsilon(w_n - z_n)) \neq 0 \quad \text{for } (z,w)\end{aligned}$$

in some neighborhood of S_{13}, provided that ε and t are sufficiently close to 0).

The verification of the condition (CR) for $A = \{2\}$ is similar. We obtain therefore the following corollary, which is Theorem 2 from **[2]** in this special situation:

COROLLARY 2.1. *If D is as above, then there exist $\varepsilon > 0$ and $t_0 < 0$ such that for every t with $t_0 < t < 0$, for every $q = 1,\dots,2n$, for every $f \in \mathscr{C}^\infty_{0q}(E_{\varepsilon,t}) \cap \mathscr{C}_{0q}(\overline{E_{\varepsilon,t}})$ with $\bar{\partial}f = 0$, there exists $u \in \mathscr{C}^\infty_{0q-1}(E_{\varepsilon,t}) \cap \mathscr{C}^\infty_{0q-1}(\overline{E_{\varepsilon,t}})$ such that $\bar{\partial}u = f$.*

In the next part of the proof of Theorem 1 we apply a method used in [6]. Consider first the case $q \geq 2$. Let $f \in \mathscr{C}^\infty_{0q}(D \times D) \cap \mathscr{C}_{0q}(Q)$ with $\bar\partial f = 0$. Take a strictly increasing sequence $\{t_n\}_{n=1}^\infty$ of negative real numbers, such that $\lim_{n\to\infty} t_n = 0$, and $t_1 > t_0$. Set $E_n = E_{\varepsilon,t_n}$ for simplicity. We shall construct a sequence $\{u_n\}_{n=1}^\infty$ of differential forms such that

$$(2.3) \qquad u_n \in \mathscr{C}^\infty_{0q-1}(E_n) \cap \mathscr{C}_{0q-1}(\overline{E_n}), \quad \bar\partial u_n = f \text{ in } \overline{E_n}, \quad \text{and} \\ u_{n+1|\overline{E_{n-1}}} = u_{n|\overline{E_{n-1}}}.$$

Suppose that $u_1, \ldots, u_m$ are constructed. By Corollary 2.1, there exists $v \in \mathscr{C}^\infty_{0q-1}(E_{m+1}) \cap \mathscr{C}_{0q-1}(\overline{E_{m+1}})$ such that $\bar\partial v = f$ in $\overline{E_{m+1}}$. Then

$$\bar\partial(u_m - v) = 0 \quad \text{on } \overline{E_m}.$$

Hence, by Corollary 2.1, there exists $w \in \mathscr{C}^\infty_{0q-2}(E_m) \cap \mathscr{C}_{0q-2}(\overline{E_m})$ such that $\bar\partial w = u_m - v$. Let χ be a $\mathscr{C}^\infty$ function on $\mathbf{C}^{2n}$, such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $\overline{E_{m-1}}$, $\chi \equiv 0$ on $(\overline{D} \times \overline{D}) \setminus \overline{E_m}$. Then the form χw, extended trivially by 0, is in $\mathscr{C}^\infty_{0q-2}(E_{m+1}) \cap \mathscr{C}_{0q-2}(\overline{E_{m+1}})$, and

$$\bar\partial(\chi w) = (\bar\partial\chi)w + \chi(u_m - v) \in \mathscr{C}^\infty_{0q-1}(E_{m+1}) \cap \mathscr{C}_{0q-1}(\overline{E_{m+1}}).$$

Define u_{m+1} on $\overline{E_{m+1}}$ by $u_{m+1} = v + \bar\partial(\chi w)$. Then u_{m+1} satisfies (2.3). Since $\bigcup_{n=1}^\infty \overline{E_n} = Q$, the desired solution u is defined by setting $u = u_n$ on $\overline{E_n}$. Now let $q = 1$. We need some auxiliary approximation lemmas:

LEMMA 2.2. *Let D, ε, t_0 be as in Corollary 2.1. Let $t, t' \in \mathbf{R}$ satisfy the condition $t_0 < t' < t < 0$. Then there exists a neighborhood U of $\overline{E_{\varepsilon,t}}$ such that every function f holomorphic in a neighborhood of $\overline{E_{\varepsilon,t'}}$ can be approximated on $\overline{E_{\varepsilon,t'}}$ by functions holomorphic in U.*

Proof. By Theorems 4.3.2 and 4.3.4 of [3], it is sufficient to find a neighborhood U of $\overline{E_{\varepsilon,t}}$ such that $(\overline{E_{\varepsilon,t'}})^\wedge_U = \overline{E_{\varepsilon,t'}}$, where $(\overline{E_{\varepsilon,t'}})^\wedge_U$ denotes the holomorphic convex hull of $\overline{E_{\varepsilon,t'}}$ in U. Fix t'' such that $t < t'' < 0$, and let $D_\eta = \{z \in \tilde{D} | \sigma(z) < \eta\}$. If $\eta > 0$ is sufficiently small, then $(\overline{D})^\wedge_{D_\eta} = \overline{D}$, and hence

$$(2.4) \qquad (\overline{D} \times \overline{D})^\wedge_{D_\eta \times D_\eta} = \overline{D} \times \overline{D}.$$

Moreover,

$$(2.5) \qquad (\overline{G_{\varepsilon,t'}})^\wedge_{G_{\varepsilon,t''}} = \overline{G_{\varepsilon,t'}}.$$

Set $U = (D_\eta \times D_\eta) \cap G_{\varepsilon,t''}$. Then U is a neighborhood of $\overline{E_{\varepsilon,t}}$, and it follows from (2.4) and (2.5) that

$$(\overline{E_{\varepsilon,t'}})^\wedge_U = ((\overline{D} \times \overline{D}) \cap \overline{G_{\varepsilon,t'}})^\wedge_U = (\overline{D} \times \overline{D}) \cap \overline{G_{\varepsilon,t'}} = \overline{E_{\varepsilon,t'}}.$$

LEMMA 2.3. *Let D, ε, t_0, t and t' be as in Lemma 2.2. Then every function $f \in A(E_{\varepsilon,t})$ can be uniformly approximated on $\overline{E_{\varepsilon,t'}}$ by functions which are holomorphic in a neighborhood of $\overline{E_{\varepsilon,t'}}$.*

Proof. We prove first one result on the separation of singularities:

LEMMA 2.4. *Given $\varepsilon > 0$, there exists $N \in \mathbf{N}$ and the strictly pseudoconvex domains $D_i \subset \mathbf{C}^n$, $i = i, \ldots, N$, such that $D \subset D_i$, $\operatorname{diam}(\partial D \setminus D_i) < \varepsilon$, and such that for every $f \in A(E_{\varepsilon,t})$ there exist functions $L_i f \in A((D_i \times D) \cap G_{\varepsilon,t})$, such that $f = \sum_{i=1}^{N} L_i f$.*

Proof. While $\overline{D}$ is compact, there exist a positive integer N, and points $z_1, \ldots, z_N \in \partial D$, such that $\partial D \subset \bigcup_{i=1}^{N} B(z_i, \varepsilon/4)$. Let $f \in A(E_{\varepsilon,t})$. Choose a function $\varphi_1 \in \mathscr{C}^\infty(\mathbf{C}^n)$, such that $0 \le \varphi_1 \le 1$, $\varphi_{1|\partial D \cap B(z_1,\varepsilon/2)} = 1$, $\varphi_{1|\partial D \setminus B(z_1, 3\varepsilon/4)} = 0$. Set $w_1 = f\bar{\partial}\varphi_1$. Then there exist strictly pseudoconvex domains D_1 and D_1' in $\mathbf{C}^n$ such that $D \cup (\partial D \setminus B(z_1, \varepsilon)) \subset D_1$, $D \cup (\partial D \cap \overline{B(z_1, \varepsilon/4)}) \subset D_1'$, $D_1'' = D_1 \cup D_1'$ is strictly pseudoconvex, and w_1, extended trivially by 0, is in

$$\mathscr{C}_{01}^\infty\big((D_1'' \times D) \cap G_{\varepsilon,t}\big) \cap \mathscr{C}_{01}\big((\overline{D}_1'' \times \overline{D}) \cap \overline{G_{\varepsilon,t}}\big),$$

and $\bar{\partial} w = 0$ there. Moreover, if D_1'' is sufficiently close to D, then the domain $(D_1'' \times D) \cap G_{\varepsilon,t}$ satisfies the assumptions (C) and (CR) from **[2]**. Therefore, by **[2]**, Theorem 2, there exists $\alpha_1 \in \mathscr{C}^\infty((D_1'' \times D) \cap G_{\varepsilon,t}) \cap \mathscr{C}((\overline{D_1''} \times \overline{D}) \cap \overline{G_{\varepsilon,t}})$ such that $\bar{\partial}\alpha_1 = w_1$. Set

$$L_1 f = \varphi_1 f - \alpha_1, \quad L_1' f = (1 - \varphi_1) f + \alpha_1.$$

Then

$$L_1 f \in A\big((D_1 \times D) \cap G_{\varepsilon,t}\big), \qquad L_1' f \in A\big((D_1' \times D) \cap G_{\varepsilon,t}\big),$$

and $f = L_1 f + L_1' f$.

Suppose that for $k < N - 1$ we have constructed the strictly pseudoconvex domains $D_1, \ldots, D_k$ and D_k' in $\mathbf{C}^n$ such that $D \cup (\partial D \setminus B(z_i, \varepsilon)) \subset D_i$, $i = 1, \ldots, k$, and $D \cup (\partial D \cap \bigcup_{i=1}^{k} \overline{B(z_i, \varepsilon/4)}) \subset D_k'$, and the functions $L_1 f, \ldots, L_k f$ and $L_k' f$ such that $L_i f \in A((D_i \times D) \cap G_{\varepsilon,t})$, $i = 1, \ldots, k$, $L_k' f \in A((D_k' \times D) \cap G_{\varepsilon,t})$, and $f = \sum_{i=1}^{k} L_i f + L_k' f$. Choose a function $\varphi_{k+1} \in \mathscr{C}^\infty \mathbf{C}^n)$ such that $0 \le \varphi_{k+1} \le 1$, $\varphi_{k+1} = 1$ on $\partial D \cap B(z_{k+1}, \varepsilon/4)$, and $\varphi_{k+1} = 0$ on $\partial D \setminus B(z_{k+1}, 3\varepsilon/4)$. Set $w_{k+1} = (L_k' f)\bar{\partial}\varphi_{k+1}$. Then there exist strictly pseudoconvex domains D_{k+1} and $D_{k+1}' \subset \mathbf{C}^n$, such that

$$D_k' \cup \big(\partial D \setminus B(z_{k+1}, \varepsilon)\big) \subset D_{k+1}, \quad D_k' \cup \big(\partial D \cap \overline{B(z_{k+1}, \varepsilon/4)}\big) \subset D_{k+1}',$$
$$D_{k+1}'' = D_{k+1} \cup D_{k+1}'$$

is strictly pseudoconvex, the domain $(D''_{k+1} \times D) \cap G_{\varepsilon,t}$ satisfies the assumptions (C) and (CR) from [2], the form w_{k+1} extended trivially by 0, is in

$$\mathscr{C}^{\infty}_{01}\big((D'_{k+1} \times D) \cap G_{\varepsilon,t}\big) \cap \mathscr{C}_{01}\left(\overline{D''_{k+1}} \times \overline{D} \cap \overline{G_{\varepsilon,t}}\right),$$

and $\bar{\partial} w_{k+1} = 0$ there. By [2], Theorem 2, there exists

$$\alpha_{k+1} \in \mathscr{C}^{\infty}\big((D''_{k+1} \times D) \cap G_{\varepsilon,t}\big) \cap \mathscr{C}\left(\left(\overline{D''_{k+1}} \times \overline{D}\right) \cap \overline{G_{\varepsilon,t}}\right)$$

such that $\bar{\partial}\alpha_{k+1} = w_{k+1}$. Set

$$L_{k+1}f = \varphi_{k+1}L'_k f - \alpha_{k+1}, \quad L'_{k+1}f = (1 - \varphi_{k+1})L'_k f + \alpha_{k+1}.$$

Then

$$L_{k+1}f \in A\big((D_{k+1} \times D) \cap G_{\varepsilon,t}\big), \quad L'_{k+1}f \in A\big((D'_{k+1} \times D) \cap G_{\varepsilon,t}\big),$$

$$f = \sum_{i=1}^{k+1} L_i f + L'_{k+1}f, \quad D \cup (\partial D \setminus B(z_i, \varepsilon)) \subset D_i, \qquad i = 1, \dots, k+1,$$

and

$$D \cup \left(\partial D \cap \bigcup_{i=1}^{k+1} \overline{B(z_i, \varepsilon/4)}\right) \subset D'_{k+1}.$$

After $N - 1$ steps, we obtain the decomposition $f = \sum_{i=1}^{N-1} L_i f + L'_{N-1}f$. It remains to put $D_N = D'_{N-1}$ and $L_N f = L'_{N-1}f$.

We return to the proof of Lemma 2.3. We can choose N, ε and the domains D_i in Lemma 2.4 in such a way, that there exist $t_1 \in \mathbf{R}$ with $t' < t_1 < t$ (where t and t' are as in the assumption of Lemma 2.3), $\delta > 0$, and $v_1, \dots, v_N \in \mathbf{C}^n$, such that for every $i = 1, \dots, N$, for every s such that $0 < s < \delta$, the set $\{(z + sv_i, w) | (z, w) \in (D_i \times \overline{D}) \cap G_{\varepsilon,t}\}$ contains $(\overline{D} \times \overline{D}) \cap \overline{G_{\varepsilon,t_1}}$. Therefore, if $f \in A(E_{\varepsilon,t})$ and $f = \sum_{i=1}^{N} L_i f$ is the decomposition of f according to Lemma 2.4, then the functions $g_{i,s}(z, w) = L_i f(z - sv_i, w)$ are defined in the set $(D'_s \times \overline{D}) \cap \overline{G_{\varepsilon,t_1}}$, where D'_s is some neighborhood of $\overline{D}$, and $g_{i,s} \to f$ uniformly on $(\overline{D} \times \overline{D}) \cap \overline{G_{\varepsilon,t_1}}$ as $s \to 0$. Given a function $g_{i,s}$, we can apply the decomposition procedure, described in Lemma 2.4, but now with respect to the second group of variables, and with respect to the domain $(D'_s \times D) \cap G_{\varepsilon,t_1}$. We obtain then for some $N' = N'(g_{i,s})$ the decomposition $g_{i,s} = \sum_{j=1}^{N'} L_j g_{i,s}$, where for $j = 1, \dots, N'$, $L_j g_{i,s} \in A((D'_s \times D_j) \cap G_{\varepsilon,t_1})$, and $D_j \subset \mathbf{C}^n$ is a strictly pseudoconvex domain such that $D \subset D_j$ and $\operatorname{diam}(\partial D \setminus D_j) < \varepsilon$. We choose then t_2 with $t' < t_2 < t_1$ and shift the functions $L_j g_{i,s}$ similarly as above, but now with respect to the second group of variables, in order to approximate every function $L_j g_{i,s}$ uniformly on $(\bar{D} \times \bar{D}) \cap \overline{G_{\varepsilon,t_2}}$ by functions of the form $h_{i,j,s,r}(z,w) = L_j g_{i,s}(z, w - ru_j)$, $u_j \in \mathbf{C}^n$, $r > 0$, defined in a set $(D'_s \times D''_r) \cap \overline{G_{\varepsilon,t_2}}$, where D''_r is some neighborhood of $\bar{D}$. Since $\overline{E_{\varepsilon,t'}} \subset (\bar{D} \times \bar{D}) \cap \overline{G_{\varepsilon,t_2}}$, we obtain the conclusion of Lemma 2.3.

Having proved Lemmas 2.2 ane 2.3, we can finish the proof of Theorem 1 for $q = 1$. Choose two sequences $\{t_n\}$ and $\{s_n\}$ of negative real numbers, such that $t_n, s_n \to 0$ as $n \to \infty$, $t_0 < s_1$, and $s_n < t_n < s_{n+1}$, $n = 1, 2, \ldots$. Set $E_n = E_{\varepsilon,t_n}$, $F_n = E_{\varepsilon,s_n}$. We shall construct a sequence of functions $\{u_n\}$ such that $u_n \in \mathscr{C}^\infty(E_n) \cap \mathscr{C}(\overline{E_n})$, $\bar{\partial} u_n = f$ on $\overline{E_n}$, and the uniform norm $\|u_{n+1} - u_n\|_{\bar{F}} \le 2^{-n}$. Suppose that $u_1, \ldots, u_m$ are constructed. By [2], Theorem 2, there exists $v \in \mathscr{C}^\infty(E_{m+1}) \cap \mathscr{C}(\overline{E_{m+1}})$, such that $\bar{\partial} v = f$ in $\overline{E_{m+1}}$. Then $u_m - v \in A(E_m)$. By Lemma 2.3 there exists a function w holomorphic in a neighborhood of $\overline{F_m}$, such that

$$\|w - (u_m - v)\|_{\overline{F_m}} \le 2^{-(m+1)}.$$

By Lemma 2.2 there exists a neighborhood U of $\overline{E_{m+1}}$ and a function t holomorphic in U, such that $\|t - w\|_{\bar{F}} \le 2^{-(m+1)}$. Let $u_{m+1} = t + v$ on $\overline{E_{m+1}}$. Then $u_{m+1} \in \mathscr{C}^\infty(E_{m+1}) \cap \mathscr{C}(\overline{E_{m+1}})$, $\bar{\partial} u_{m+1} = f$ on $\overline{E_{m+1}}$, and $\|u_{m+1} - u_m\|_{\overline{F_m}} \le 2^{-m}$. Since $u_{m+1} - u_m$ is holomorphic in E_m, it follows that the sequence $\{u_n\}$ converges to the function $u \in \mathscr{C}^\infty(D \times D) \cap \mathscr{C}(Q)$, such that $\bar{\partial} u = f$.

3. The decomposition in the algebra $A_Q(D \times D)$. We prove here Theorem 2. The method of the proof is that used by Øvrelid [5], therefore we give only the necessary modifications. It follows from the assumptions that at every point $(z, s) \in D \times D$, the germs at (z, s) of the functions g_i generate the ideal of germs at (z, s) of holomorphic functions vanishing on $\Delta(D)$. Therefore, by [3], Theorem 7.2.9, for every $f \in (A_Q)_0(D \times D)$ there exist functions $(Rf)_1, \ldots, (Rf)_N \in \mathcal{O}(D \times D)$ such that $f = \sum_{i=1}^N g_i (Rf)_i$. Let $N_i = \{(z,s) \in Q \setminus \Delta(D) | g_i(z,s) = 0\}$, $i = 1, \ldots, N$. Since the sets N_i are closed in $\mathbf{C}^{2n} \setminus \Delta(\bar{D})$, there exist functions $\tilde{\varphi}_i \in \mathscr{C}^\infty(\mathbf{C}^{2n} \setminus \Delta(\bar{D}))$ such that $0 \le \tilde{\varphi}_i \le 1$, $\sum_{i=1}^N \tilde{\varphi}_i = 1$, and $\tilde{\varphi}_i$ vanishes in a neighborhood of N_i in $\mathbf{C}^{2n} \setminus \Delta(\bar{D})$, $i = 1, \ldots, N$.

Choose $\varphi_0 \in \mathscr{C}^\infty(\mathbf{C}^{2n} \setminus \Delta(\partial D))$, such that $0 \le \varphi_0 \le 1$, $\varphi_0 = 1$ in some neighborhood W of $\Delta(D)$ in $D \times D$, and $\varphi_0 = 0$ in a neighborhood of $\partial(D \times D) \setminus \Delta(\partial D)$. Set $\varphi_i = (1 - \varphi_0)\tilde{\varphi}_i$, and define $(Sf)_i = \varphi_0 (Rf)_i + \varphi_i f / g_i$, $i = 1, \ldots, N$. Then $\sum_{i=1}^N g_i (Sf)_i = f$ in Q. Choose the neighborhoods W_1 and W_2 of $\Delta(D)$ in $D \times D$ such that $\overline{W_1} \subset W$ and $\overline{W_2} \subset W_1$

(the closures in Q), and let φ be a function in $\mathscr{C}^\infty(\mathbf{C}^{2n} \setminus \Delta(\partial D))$ such that $0 \leq \varphi \leq 1$, $\varphi = 1$ outside W_1, and $\varphi = 0$ in W_2. Set $L_r = \{u \in \mathscr{C}^\infty_{0r}(D \times D) \cap \mathscr{C}_{0r}(Q) | \bar{\partial} f \in \mathscr{C}_{0r}(Q)\}$. Define L^s_r, $0 \leq r, s$, the operators $\bar{\partial}$ and P_r on L^s_r similarly as in [5], and let $M^s_r = \{k \in L^s_r | k|_{W_1} \equiv 0\}$, and $k_0 = \sum_{i=1}^N \varphi \tilde{\varphi}_i / g_i \otimes e_i \in L^1_0$ (here $e_1, \ldots, e_N$ is some basis of $\mathbf{C}^N$). Using then Theorem 1 in place of Lemma 1 from [5], we end the proof similarly as in [5] (of course, after the suitable change of notations, according to that given above; in particular, the functions g'_i and h_i from the final part of the proof of [5], Theorem 1 should be replaced by (Sf_i) and f_i respectively).

Note. The operator $f \to (f_1, \ldots, f_N)$ from Theorem 2 is in general neither linear nor continuous. Nevertheless, if $n = 1$ (i.e. $D \subset \mathbf{C}$), then every $f \in (A_Q)_0(D \times D)$ can be represented as $f(z, s) = (z - s)(Rf)(z, s)$ with (uniquely determined) $Rf \in A_Q(D \times D)$, and the mapping $f \to Rf$ is linear and continuous (where $A_Q(D \times D)$ is equipped with the topology of uniform convergence on compact subsets of Q). Moreover, by Theorem 2, the function $z - s$ can be decomposed with respect to the functions g_i in the form $z - s = \sum_{i=1}^N g_i h_i$ with some $h_i \in A_Q(D \times D)$. Therefore, setting $f_i = (Rf)h_i$, $i = 1, \ldots, N$, we obtain the continuous and linear operator

$$(A_Q)_0(D \times D) \ni f \to ((Rf)h_1, \ldots, (Rf)h_N) \in \left[A_Q(D \times D)\right]^N,$$

such that

$$f = \sum_{i=1}^N g_i f_i. \tag{3.1}$$

We obtain therefore the full generalization of Theorem 2 in [4]. Similarly, if $D \subset \mathbf{C}^n$ is strictly pseudoconvex with $\mathscr{C}^2$ boundary, then, by a theorem of Ahern and Schneider [1], every function

$$f \in A_0(D \times D) = \left\{f \in A(D \times D) | f|_{\Delta(D)} = 0\right\}$$

can be decomposed with respect to the functions $z_1 - s_1, \ldots, z_n - s_n$ into $f(z, s) = \sum_{i=1}^n (z_i - s_i)\tilde{f}_i(z, s)$ with some functions $\tilde{f}_i \in A_Q(D \times D)$, and the operator $A_0(D \times D) \ni f \to (\tilde{f}_1, \ldots, \tilde{f}_n) \in [A_Q(D \times D)]^n$ is linear and continuous. Applying Theorem 2 to the functions $z_i - s_i$, $i = 1, \ldots, n$, and proceeding as above, we obtain the linear and continuous operator $A_0(D \times D) \ni f \to (f_1, \ldots, f_N) \in [A_Q(D \times D)]^N$, satisfying (3.1).

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