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WEAK*-CLOSED COMPLEMENTED INVARIANT SUBSPACES OF $L_\infty(G)$ AND AMENABLE LOCALLY COMPACT GROUPS

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One of the main results of this paper implies that a locally compact group G is amenable if and only if whenever X is a weak*-closed left translation invariant complemented subspace of $L_\infty(G)$, X is the range of a projection on $L_\infty(G)$ commuting with left translations. We also prove that if G is a locally compact group and M is an invariant W^*-subalgebra of the von Neumann algebra $\mathrm{VN}(G)$ generated by the left translation operators l_g, $g \in G$, on $L_2(G)$, and $\Sigma(M) = \{g \in G; l_g \in M\}$ is a normal subgroup of G, then M is the range of a projection on $\mathrm{VN}(G)$ commuting with the action of the Fourier algebra $A(G)$ on $\mathrm{VN}(G)$.

1. Introduction. Let G be a locally compact group and $L_\infty(G)$ be the algebra of essentially bounded measurable complex-valued functions on G with pointwise operations and essential sup norm. Let X be a weak*-closed left translation invariant subspace of $L_\infty(G)$. Then X is *invariantly complemented* in $L_\infty(G)$ if X admits a left translation invariant closed complement, or equivalently, X is the range of a continuous projection on $L_\infty(G)$ commuting with left translations.

H. Rosenthal proved in [**13**] that if G is an abelian locally compact group and X is a weak*-closed translation invariant complemented subspace of $L_\infty(G)$, then X is invariantly complemented in $L_\infty(G)$. Recently Lau [**11**, Theorem 3.3] proved that a locally compact group G is left amenable if and only if every left translation invariant weak*-closed subalgebra of $L_\infty(G)$ which is closed under conjugation is invariantly complemented. Note that if T is the circle group, then the Hardy space H_∞ is a weak*-closed translation invariant subalgebra of $L_\infty(T)$ and *not* (invariantly) complemented (see [**15**] and Corollary 4).

In [**20**, Lemma 4], Y. Takahashi proved that if G is a compact group, then any weak*-closed complemented left translation invariant subspace of $L_\infty(G)$ is invariantly complemented. However, there is a gap in Takahashi's adaptation of Rosenthal's argument (see Zentralblatt für Mathematik 1982: 483.43002). It should be observed that Rosenthal's original argument in [**13**, Theorem 1.1] is valid only for locally compact groups G which is amenable as discrete (for example when G is solvable). Indeed it follows from [**21**, Theorem 16] that under Martin's Axiom, if P is a bounded projection of $L_\infty(G)$ onto $\mathbf{C}$ (which is a weak*-closed and left translation invariant subspace of $L_\infty(G)$), the functions $x \to \langle l_{x^{-1}} P l_x f, h\rangle = \langle P l_x f, h\rangle$, where $f \in L_\infty(G)$ and $h \in L_1(G)$, is in general bounded but not measurable even when G is compact.

In §3 of this paper, we generalize Rosenthal's result to all amenable locally compact groups (and thus giving a correct proof of Takahashi's Lemma 4 in [**20**] for all compact groups). More precisely, our Theorem 1 implies that a locally compact group G is amenable if and only if whenever X is a weak*-closed translation invariant complemented subspace of $L_\infty(G)$, X is invariantly complemented. Furthermore (Corollary 4), if G is compact, then X is even the range of a weak*-weak* continuous projection which commutes with left translations. Also in this case, $L_\infty(G)$ has a unique left invariant mean (for example when $G = \mathrm{SO}(n, \mathbf{R})$, $n \geq 5$) if and only if every bounded projection of $L_\infty(G)$ into $L_\infty(G)$ which commutes with left translations is weak*-weak* continuous.

Our proof of Theorem 1 depends heavily on a recent result of Losert and Rindler [**12**] on the existence of an asymptotically central unit in $L_1(G)$ of an amenable locally compact group.

Finally in §4 we give a non-commutative analogue of Lau's result [**11**, Theorem 3.3]. We prove that (Theorem 4) if M is an invariant W^*-subalgebra of the von Neuman algebra $\mathrm{VN}(G)$ generated by the left translation operators $\{l_g;\ g \in G\}$ on $L_2(G)$ of a locally compact group G and $\Sigma(M) = \{g \in G;\ l_g \in M\}$ is a normal subgroup of G, then M is invariantly complement. However, we do not know if the normality condition on $\Sigma(M)$ may be dropped or not unless $\Sigma(M)$ is compact or open.

2. Preliminaries. If E is a Banach space, then E^* denotes its continuous dual. Also if $\phi \in E^*$ and $x \in E$, then the value of ϕ at x will be written as $\phi(x)$ or $\langle \phi, x\rangle$.

Throughout this paper, G denotes a locally compact group with a fixed left Haar measure. Let $C(G)$ denote the Banach algebra of bounded continuous complex-valued functions on G with the supremum norm, and let $C_0(G)$ be the closed subspace of $C(G)$ consisting of all functions in $C(G)$ which vanish at infinity. The Banach spaces $L_p(G)$, $1 \leq p \leq \infty$, are as defined in [7]. If f is a complex-valued function defined locally almost everywhere on G, and if $a, t \in G$, then $(l_a f)(t) = f(a^{-1}t)$ and $(r_a f)(t) = f(ta)$ whenever this is defined. We say that G is *amenable* if there exists

$m \in L_\infty(G)^*$ such that $m \geq 0$, $\|m\| = 1$ and $m(l_a f) = m(f)$ for which $f \in L_\infty(G)$ and $a \in G$ (m is called a *left invariant mean*). Amenable locally compact groups include all compact groups and all solvable groups. However, the free group on two generators is not amenable (see [**4**]).

For $g \in G$, the corresponding inner automorphism induces a map τ_g' on $L_\infty(G)$ by $\tau_g' f(x) = f(gxg^{-1})$. The adjoint map τ_g on $L_1(G)$ is given by $\tau_g\phi(x) = \phi(g^{-1}xg)\Delta(g)$, where Δ is the Haar modulus function of G. This can also be written as $\tau_g\phi = \delta_g * \phi * \delta_{g^{-1}}$, where δ_g stands for the Dirac measure concentrated at $g \in G$ (convolution as defined in [**7**]). A net $\{u_\alpha\}$ in $L_1(G)$ is called an *approximate unit* if $\lim_\alpha\|u_\alpha * \phi - \phi_1\|_1 = \lim_\alpha\|\phi * u_\alpha - \phi\|_1 = 0$ for all $\phi \in L_1(G)$. The net $\{u_\alpha\}$ is said to be *asymptotically central* if $\lim_\alpha\|u_\alpha\|^{-1}\|\tau_g u_\alpha - u_\alpha\| = 0$ for all $g \in G$. The following result of Losert and Rindler is the key to the proof of one of our main results:

LEMMA 1 ([**12**, *Theorem* 3]). *Let G be an amenable locally compact group, then $L_1(G)$ has an asymptotically central approximate unit $\{u_\alpha\}$ with $\|u_\alpha\| \leq 1$.*

3. Subspaces of $L_\infty(G)$. A *left Banach G-module X* is a Banach space X which is left G-module such that

(i) $\|s \cdot x\| \leq \|x\|$ for all $x \in X$, $s \in G$.

(ii) for all $x \in X$, the map $s \to s \cdot x$ is continuous from G into X.

In this case, we define for each $f \in X^*$, $s \in G$, $x \in X$

$$\langle f \cdot s, x\rangle = \langle f, s \cdot x\rangle.$$

Define also $\langle f \cdot \mu, x\rangle = \int\langle f, s \cdot x\rangle\, d\mu(s)$, $\mu \in M(G)$, $f \in X^*$, $x \in X$, where $M(G)$ is the space of (complex, bounded) Radon measures on G. Then $f \cdot \mu \in X^*$, $f \cdot \mu = f \cdot s$ if $\mu = \delta_s$ and $(f \cdot \mu_1) \cdot \mu_2 = f \cdot (\mu_1 * \mu_2)$ for $\mu_1, \mu_2 \in M(G)$.

A subspace $L \subseteq X^*$ is called *G-invariant* if $L \cdot s \subseteq L$ for all $s \in G$.

LEMMA 2. *Let L be a weak*-closed subspace of X^*. Then L is G-invariant if and only if $L \cdot \phi \subseteq L$ for each $\phi \in L_1(G)$.*

Proof. Suppose that L is G-invariant and $\phi \in L_1(G)$, $\phi \geq 0$ and $\|\phi\|_1 = 1$. Define $\Phi(f) = \int f(t)\phi(t)\, dt$, $f \in C(G)$. Then Φ is a positive functional on $C(G)$ with norm one. Hence there exists a net $\{m_\alpha\}$ in $C(G)^*$ such that each m_α is a convex combination of point evaluations and m_α converges to Φ in the weak* topology of $C(G)^*$. If $m_\alpha = \sum_{i=1}^n \lambda_i p_{s_i}$, where $p_s(h) = h(s)$, $h \in C(G)$, $s \in S$, and $f \in L$, then $f \cdot m_\alpha = \sum_{i=1}^n \lambda_i f \cdot s_i$ converges to $f \cdot \phi$ in the weak*-topology of X^*. Hence $f \cdot \phi \in L$.

Conversely, if $L \cdot \phi \subseteq L$ for each $\phi \in L_1(G)$ and $s \in G$, let $m \in L_\infty(G)^*$ such that m extends $p_s \in C(G)^*$ and $\|m\| = \|p_s\| = 1$. Then $m \geq 0$. Hence there exists a net $\{\phi_\alpha\} \subseteq L_1(G)$, $\phi_\alpha \geq 0$, $\|\phi_\alpha\|_1 = 1$, such that $\{\phi_\alpha\}$ converges to m in the weak* topology of $L_\infty(G)^*$. Consequently, if $f \in L$, then $f \cdot \phi_\alpha$ converges in the weak* topology of X^* to $f \cdot s$.

A left Banach G-module X is called *non-degenerate* if the closed linear span of $\{g \cdot x;\ g \in G,\ x \in X\}$ is X.

THEOREM 1. *Let G be a locally compact group. Then G is amenable if and only if whenever X is a non-degenerate left Banach G-module and L is a weak*-closed G-invariant subspace of X which is complemented in X, then there exists a projection Q of X^* onto L such that $Q(f \cdot s) = Q(f) \cdot s$ for all $s \in G$, $f \in X^*$.*

Proof. If G is amenable, there exists an asymptotically central approximate unit $\{u_\alpha\}$ in $L_1(G)$, $\|u_\alpha\| \leq 1$ (Lemma 1). Let m be an invariant mean on $L_\infty(G)$. For each $s \in G$, $f \in X^*$, put $P_{\alpha,s}(f) = O(f \cdot (u_\alpha * \delta_s)) \cdot (\delta_{s^{-1}} * u_\alpha)$. By Lemma 2, $P_{\alpha,s}$: $X^* \to L$ and $\|P_{\alpha,s}\| \leq \|P\|$. For each fixed α, $f \in X^*$, $x \in X$, the function $s \to \langle x, P_{\alpha,s}(f)\rangle$ is bounded and continuous. Hence we may define the mean P_α of the family $\{P_{\alpha,s}\}_{s \in G}$ by

$$\langle x, P_\alpha f\rangle = m\{s \to \langle x, P_{\alpha,s}(f)\rangle\}.$$

Then P_α: $X^* \to L$ (since L is weak*-closed and if $x \in X$ is annihilated by L, then $\langle x, P_\alpha f\rangle = 0$ by Lemma 2), and $\|P_\alpha f\| \leq \|P\|$. Finally define $Q(f) = \text{weak* } \lim_\alpha P_\alpha(f)$. Again Q: $X^* \to L$, $\|Q\| \leq \|P\|$. For $f \in L$, $f \cdot (u_\alpha * \delta_s) \in L$. Hence $(P_{\alpha,s})(f) = f \cdot (u_\alpha * u_\alpha)$. Now $\{u_\alpha * u_\alpha\}$ is also an approximate unit in $L_1(G)$. Since X is non-degenerate, Cohen's factorization theorem [**8**, 32.26] implies that each y in X has the form $\phi \cdot x$, $x \in X$, $\phi \in L_1(G)$. Hence

$$\langle f \cdot u_\alpha * u_\alpha - f, y\rangle = \langle f, (u_\alpha * u_\alpha) \cdot (\phi \cdot x) - \phi \cdot x\rangle \to 0$$

i.e. $P_{\alpha,s}(f) = f$.

Now for each $t \in G$

$$\begin{aligned} P_{\alpha,s}(f \cdot t) - P_{\alpha,ts}(f) \cdot t = {} & P\big(f \cdot t \cdot (u_\alpha * \delta_{t^{-1}} * \delta_{ts})\big) \cdot (\delta_{s^{-1}} * u_\alpha) \\ & - P\big(f \cdot (u_\alpha * \delta_{ts})\big) \cdot (\delta_{s^{-1}} * u_\alpha) \\ & + P\big(f \cdot (u_\alpha * \delta_{ts})\big) \cdot \big(\delta_{(ts)^{-1}} * \delta_t * u_\alpha * \delta_{t^{-1}} * \delta_t\big) \\ & - P\big(f \cdot (u_\alpha * \delta_{ts})\big)\big(\delta_{(ts)^{-1}} * u_\alpha * \delta_t\big). \end{aligned}$$

Hence

$$\|P_{\alpha,s}(f \cdot t) - P_{\alpha,s}(f) \cdot t\| \le 2\|P\| \, \|f\| \, \|\delta_t * u_\alpha * \delta_{t^{-1}} - u_\alpha\|$$

and this estimate carries over to $\|P_\alpha(f \cdot t) - P_\alpha(f) \cdot t\|$ by invariance of m. Since we assume $\|\delta_t * u_\alpha * \delta_{t^{-1}} - u_\alpha\| \to 0$, we get $Q(f \cdot t) = Q(f) \cdot t$.

The converse follows as in the proof of Theorem 3.3 in [**11**] by considering $X = L_1(G)$ and $(s \cdot \phi)(t) = \phi(s^{-1}t)$, $s \in G$, $t \in G$, $\phi \in L_1(G)$. Then if $f \in L_\infty(G)$, $(f \cdot s)(t) = f(st) = (l_{s^{-1}} f)(t)$.

Let Z be a locally compact Hausdorff space. Consider a jointly continuous action $G \times Z \to Z$. Assume that Z has a quasi-invariant measure ν. For each $s \in G$, define $\chi_s(E) = \nu(s^{-1}E)$. Then $\nu_s \ll \nu$. Hence there is a locally ν-integrable Radon Nikodym derivative $(d\nu_s/d\nu)$ such that $\nu_s = (d\nu_s/d\nu) \cdot \nu$. Also $L_1(Z, \nu)$ is a non-degenerate Banach left G-module (see [**5**, Lemma 2.3]): $s \cdot \phi = \delta_s * \phi$, $s \in G$, $\phi \in L_1(Z, \nu)$ where $(\delta_s * \phi)(\xi) = (d\nu_s/d\nu)(\xi)(s^{-1}\xi)$ defined ν-a.e. on Z. Hence Theorem 1 implies:

COROLLARY 1. *Let G be a locally compact group. Then G is amenable if and only if for any locally compact Hausdorff space Z and jointly continuous action $G \times Z \to Z$ such that Z has a quasi-invariant measure, then any weak*-closed G-invariant complemented subspace of $L_\infty(Z, \nu)$ is invariantly complemented.*

REMARK. Theorem 1 also implies Lemma 3.1 of [**13**] for $L_p(G)$, $1 < p < \infty$, and Theorem 4.1 of [**11**].

If H is a closed subgroup of a locally compact group, then there exists a non-trivial quasi-invariant measure ν on the coset space $G/H = \{xH;\ x \in G\}$ which is essentially unique. Write $L_\infty(G/H) = L_\infty(G/H, \nu)$.

COROLLARY 2. *Let G be a locally compact group. Then G is amenable if and only if every weak*-closed complemented invariant subspace of*

$L_\infty(G/H)$, H a closed subgroup of G, is the range of a projection on $L_\infty(G/H)$ which commutes with translation.

COROLLARY 3. *Let G be an amenable locally compact group and X be a weak*-closed left translation invariant subspace of $L_\infty(G)$. Then X is the range of a weak*-weak* continuous projection on $L_\infty(G)$ commuting with left translation if and only if $X \cap C_0(G)$ is weak*-dense in X.*

Proof. This follows from Corollary 2 and Lemma 5.2 of [**11**].

COROLLARY 4. *Let G be a locally compact group. Then G is compact if and only if G has the following property*:

($*$) Whenever X is a weak*-closed complemented left translation invariant subspace of $L_\infty(G)$, there exists a weak*-weak* continuous projection from $L_\infty(G)$ onto X commuting with left translations.

Proof. If G is compact, property ($*$) follows from Corollary 2, and Lemma 2.1, Lemma 5.2 of [**11**]. Conversely, if ($*$) holds, then apply the property to the one-dimensional subspace $X = \mathbf{C}$. It follows that there exists $\phi \in L_1(G)$, $\phi \geq 0$, $\phi(1) = 1$ such that $\phi(l_s f) = \phi(f)$ for all $f \in L_\infty(G)$, $s \in G$. In particular, G is compact.

A bounded linear operator T from $L_\infty(G)$ into $L_\infty(G)$ is said to commute with convolution from the left if $T(\phi * f) = \phi * T(f)$ for all $\phi \in L_1(G)$ and $f \in L_\infty(G)$. In this case, T also commutes with left translations i.e. $T(l_s f) = l_s T(f)$ for all $s \in G$ (see [**10**, Lemma 2]).

LEMMA 3. *If T is a weak*-weak* continuous linear operator from $L_\infty(G)$ into $L_\infty(G)$ and T commutes with left translations, then T also commutes with convolutions from the left.*

Proof. Let $\phi \in L_1(G)$, $\phi \geq 0$ and $\|\phi\|_1 = 1$. Let $\phi_\alpha = \sum_{i=1}^n \lambda_i \delta_{s_i}$ be a net of convex combinations of point measures on G such that $\int f(t)\, d\phi_\alpha(t)$ converges to $\int f(t)\, d\phi(t)$ for each $f \in C(G)$. Hence if $h \in L_\infty(G)$, then the net

$$\langle \phi_\alpha * h, k \rangle = \langle k * \tilde{h}, \phi_\alpha \rangle \to \langle k * \tilde{h}, \phi \rangle = \langle \phi * h, k \rangle$$

for each $k \in L_1(G)$ ($\tilde{h}(t) = h(t^{-1})$). Consequently,

$$T(\phi * h) = \lim_\alpha T(\phi_\alpha * h) = \lim_\alpha \phi_\alpha * T(h) = \phi * T(h).$$

LEMMA 4 **[10]**. *If G is compact, then any bounded linear operator T from $L_\infty(G)$ into $L_\infty(G)$ which commutes with convolution from the left is weak*-weak* continuous.*

Proof. This is proved in **[10**, Theorem 2][1]. We give here a different proof. Indeed if $\phi \in L_1(G)$, then $\phi = \phi_1 * \phi_2$, $\phi_1, \phi_2 \in L_1(G)$ by Cohen's factorization theorem. Hence if $f \in L_\infty(G)$, then

$$\langle T^*(\phi), f\rangle = \langle \phi_1 * \phi_2, T(f)\rangle = \langle \phi_2, \tilde{\phi}_1 * T(f)\rangle$$
$$= \langle \phi_2, T(\tilde{\phi}_1 * f)\rangle = \langle T^*(\phi_2), \tilde{\phi}_1 * f\rangle = \langle \phi_1 \odot T^*(\phi_2), f\rangle$$

i.e. $T^*(\phi) = \alpha_1 \odot T^*(\phi_2)$, where $\odot$ is the Arens product defined on the second conjugate algebra $L_\infty(G)^* = L_1(G)^{**}$. Since G is compact, $L_1(G)$ is an ideal in $L_\infty(G)^*$ (see **[6]**). Hence $T^*(\phi) \in L_1(G)$, i.e. T is weak*-weak* continuous.

PROPOSITION 1. *Let G be a compact group. The following are equivalent*:

(a) *$L_\infty(G)$ has a unique left invariant mean.*

(b) *If E is a finite dimensional G-invariant subspace of $L_\infty(G)^*$ (i.e $l_s^* E \subseteq E$ for all $s \in G$) such that the map $s \to l_s^*\psi$ of G into E is continuous, then $E \subseteq L_1(G)$.*

(c) *Any bounded (projection) linear operator T from $L_\infty(G)$ into $L_\infty(G)$ which commutes with left translations is weak*-weak* continuous.*

(d) *Any bounded (projection) linear operator T from $L_\infty(G)$ into $L_\infty(G)$ which commutes with left translation also commutes with convolution from the left.*

Proof. (a) $\Rightarrow$ (b). Consider a continuous representation π of G on E defined by $\pi(s)(m) = l_{s^{-1}}^* m$, $s \in G$, $m \in E$. Since E is finite dimensional, there exists an inner product $\langle\, ,\rangle$ on E such that π is unitary. We may further assume that π is irreducible. Let $\{\psi_1, \ldots, \psi_n\}$ be an orthonormal basis of E. Write $e_{ij}(s) = \langle \pi(s)\psi_j, \psi_i\rangle$ for the coefficients of π. For $g \in L_\infty(G)$, $\psi \in L_\infty(G)^*$, define $\psi \cdot g \in L_\infty(G)^*$ by $\langle \psi \cdot g, f\rangle = \langle \psi, gf\rangle$, $f \in L_\infty(G)$. Then for any $f, g \in L_\infty(G)$, $\psi \in L_\infty(G)^*$, we have

$$\langle f, l_s^*(\psi \cdot g)\rangle = \langle l_s f, \psi \cdot g\rangle = \langle g \cdot (l_s f), \psi\rangle$$
$$= \langle l_s((l_{s^{-1}} g) \cdot f), \psi\rangle = \langle f, (l_s^*\psi) \cdot (l_{s^{-1}} g)\rangle.$$

[1] The converse to Theorem 2 in **[10]** was omitted in print. It is stated on page 352.

Consequently $l_s^*(\psi \cdot g) = (l_s^*\psi) \cdot (l_{s^{-1}}g)$. Furthermore, observe that

$$l_s^*\psi_i = \sum_{j=1}^{n} e_{ji}(s^{-1})\psi_j, \qquad l_{s^{-1}}e_{ik} = \sum_{l=1}^{n} e_{li}(s^{-1})e_{lk}.$$

Since π is unitary, $\sum_i e_{ji}(x)\overline{e_{li}(x)} = \delta_{jl}$. Put $\phi_k = \sum_{i=1}^{n} \psi_i \cdot \bar{e}_{ik}$ ($^-$ denotes the complex conjugate). Then

$$\begin{aligned} l_s^*\phi_k &= \sum_i (l_s^*\psi_i) \cdot (l_{s^{-1}}(\bar{e}_{ik})) \\ &= \sum_{i,j,l} e_{ji}(s^{-1})\psi_j\overline{e_{li}(s^{-1})} \cdot \overline{e_{lk}} = \sum_j \psi_j \bar{e}_{jk} = \phi_k \end{aligned}$$

for all $s \in G$. By assumption, $\phi_k \in L_1(G)$. Finally

$$\sum_k \phi_k \cdot e_{lk} = \sum_i \psi_i\Big(\sum_k \bar{e}_{ik} \cdot e_{lk}\Big) = \psi_l$$

and $\phi \cdot f \in L_1(G)$ whenever $\phi \in L_1(G)$, $f \in L_\infty(G)$. Hence $\psi_l \in L_1(G)$ for all $l = 1, 2, \ldots, n$.

(b) $\Rightarrow$ (c). Since G is compact, it follows that $T(\phi * f) = \phi * T(f)$ for all $\phi \in L_1(G)$, $f \in C(G)$. If $\phi \in L_1(G)$ such that $\{l_s^*\phi;\ s \in G\}$ belongs to a finite-dimensional G-invariant subspace of $L_\infty(G)^*$, then the same is true for $T^*\phi$. Hence $T^*\phi \in L_1(G)$ by (b). Since elements of this type are dense in $L_1(G)$, $T^*(L_1(G)) \subseteq L_1(G)$ i.e. T is weak*-weak* continuous.

That (c) $\Rightarrow$ (d) follows from Lemma 3.

(d) $\Rightarrow$ (a). If $L_\infty(G)$ has more than one left invariant mean, then there exists a left invariant mean m such that $m \notin L_1(G)$. Now define $T(f) = m(f) \cdot 1$, $f \in L_\infty(G)$. Then T is a projection of $L_\infty(G)$ into $L_\infty(G)$ commuting with left translations. But T does not commute with convolution by Lemma 4.

REMARK. As known (see [**3**], [**15**] and [**16**]) if G is a nondiscrete compact abelian group (or more generally, G is amenable as discrete), then $L_\infty(G)$ has more than one left invariant mean. However, if $n \geq 5$, and $G = \mathrm{SO}(n, \mathbf{R})$, then $L_\infty(G)$ has a unique left invariant mean (see [**14**] and [**17**] for more details).

4. Subspaces of $\mathrm{VN}(G)$. Let $P(G)$ be the continuous positive definite functions on G (see [**6**]). If H is a closed subgroup of G, let

$$P_H = \{\phi \in P(G);\ \phi(g) = 1 \text{ for all } g \in H\}$$

Then P_H is a subsemigroup of $P(G)$.

LEMMA 5. *If H is a closed normal subgroup of G, $g \notin H$, there exists $\phi \in P_H$ such that $\phi(g) = 0$.*

Proof. Consider the quotient group G/H and let $\psi \in P(G/H)$ such that $\psi(gH) = 0$ and $\psi(H) = 1$. Define $\phi = \psi \circ \pi$, where π is the canonical mapping of G onto G/H. Then $\phi \in P(G)$, $\phi(h) = 1$, for all $h \in H$ and $\phi(g) = 0$ (see [**2**, p. 199]).

Let $\mathrm{VN}(G)$ denote the von Neumann algebra generated by the left translation operators l_g, $g \in G$, on $L_2(G)$. Then the predual of $\mathrm{VN}(G)$ may be identified with $A(G)$, a subalgebra of $C_0(G)$ with pointwise multiplication, consisting of all functions ϕ of the form $\phi(g) = \int h(g^{-1}t)\overline{k(t)}\,dt$, $h, k \in L_2(G)$. Furthermore, $A(G)$ with the predual norm is a semi-simple commutative Banach algebra and a closed two sided ideal of $B(G)$, the linear span of $P(G)$. There is a natural action of $A(G)$ on $\mathrm{VN}(G)$ defined by $\langle \phi \cdot x, \psi \rangle = \langle x, \phi\psi \rangle$, $x \in \mathrm{VN}(G)$. When G is commutative, then $A(G)$ and $\mathrm{VN}(G)$ are isometrically isomorphic to $L_1(\hat{G})$ and $L_\infty(\hat{G})$ respectively (where $\hat{G}$ is the dual group of G) and the action of $A(G)$ on $\mathrm{VN}(G)$ corresponds to convolution of functions in $L_1(\hat{G})$ and $L_\infty(\hat{G})$. (see [**2**] for more details.)

A subspace M of $\mathrm{VN}(G)$ is called *invariant* if $\phi \cdot x \in M$ for all $\phi \in A(G)$, $x \in M$. Define

$$\sum(M) = \{g \in G; \, l_g \in M\}.$$

If M is an invariant W^*-subalgebra of $\mathrm{VN}(G)$, then $\Sigma(M) = H$ is a non-empty closed subgroup of G and $M = N_H$, the ultraweak closure of the linear span of $\{l_g;\, g \in H\}$ in $\mathrm{VN}(G)$ (see [**18**, Theorems 6 and 8]).

LEMMA 6. *Let M be an invariant W^*-subalgebra of* $\mathrm{VN}(G)$ *such that $\Sigma(M) = H$ is a normal subgroup of G. Then $M = \{x \in \mathrm{VN}(G);\, \phi \cdot x = x$ for all $\phi \in P_H\}$.*

Proof. Let $N = \{x \in \mathrm{VN}(G);\, \phi \cdot x = x$ for all $\phi \in P_H\}$. Then N is weak*-closed, invariant and $N \supseteq N_H = M$ (since $\phi \cdot l_g = \phi(g) l_g = l_g$ for $\phi \in P_H$, $g \in H$). Now if $g \in G$ and $l_g \in N$, then $\phi(g) = 1$ for all $\phi \in P_H$. In particular $\Sigma(M) \subseteq H$ by Lemma 4. Hence if $x \in N$, then $\operatorname{supp}(x) \subseteq \Sigma(N) \subseteq H$ by Proposition 4.4 [**2**]. Consequently, $x \in N_H$ by Theorem 3 [**19**].

The following implies one direction of Theorem 3.3 [**11**] when G is abelian:

THEOREM 2. *Let M be an invariant W^*-subalgebra of* $\mathrm{VN}(G)$ *such that $\Sigma(M) = H$ is a normal subgroup of G. Then there exists a continuous projection P of* $\mathrm{VN}(G)$ *onto M such that $P(\phi \cdot x) = \phi \cdot P(x)$ for all*

$\phi \in A(G)$ *and* $x \in \mathrm{VN}(G)$. *In particular, M admits a closed complement which is also invariant.*

Proof. By Lemma 6, $M = \{x \in \mathrm{VN}(G);\ \phi \cdot x = x$ for all $\phi \in P_H\}$. For each $x \in \mathrm{VN}(G)$, let K_x denote the weak*-closed convex hull of $\{\phi \cdot x;\ \phi \in P_1(G)\}$, where $P_1(G) = \{\phi \in P(G);\ \phi(e) = 1\}$, and $\langle \phi \cdot x, \psi\rangle = \langle x, \phi\psi\rangle$, $\psi \in A(G)$. Then K_x is a weak*-closed subset of $\mathrm{VN}(G)$. For each $\psi \in P_H$, let $T_\psi\colon K_x \to K_x$ be defined by $T_\psi(y) = \psi \cdot y$, $y \in K_x$. Then T_ψ is weak*-weak* continuous and affine. Since P_H is a commutative semigroup, an application of the Markov-Kakutani fixed point theorem ([**1**, p. 456]) shows that $M \cap K_x$ is nonempty for each $x \in \mathrm{VN}(G)$. By Theorem 2.1 in [**9**], there exists a projection P from $\mathrm{VN}(G)$ onto M and P commutes with any weak*-weak* continuous operator from M into M which commutes with $\{T_\psi;\ \psi \in P_H\}$. Hence $P(\phi \cdot x) = \phi \cdot P(x)$ for each $\phi \in A(G)$, $x \in \mathrm{VN}(G)$.

REMARK. Lemma 5 (hence Lemma 6 and Theorem 2) holds for any compact subgroup (see Eymard [**2**, Lemma 3.2]) and any open subgroup H of G (see Hewitt and Ross [**8**, 32.43]) without normality.

REFERENCES

[1] N. Dunford and J. T. Schwartz, *Linear Operators* 1, Interscience Publishers, Inc., New York, 1957.

[2] P. Eymard, *L'algèbe de Fourier d'un groupe localement compact*, Bull. Soc. Math. France, **92** (1964), 181–236.

[3] E. Granirer, *Criteria for compactness and for discreteness of locally compact amenable groups*, Proc. Amer. Math. Soc., **40** (1973), 615–624.

[4] F. P. Greenleaf, *Invariant Means on Topological Groups and Their Applications*, Van Nostrand (1969).

[5] ______, *Amenable actions of locally compact groups*, J. Funct. Anal., **4** (1969), 295–315.

[6] M. Grosser, *$L_1(G)$ as an ideal in its second dual space*, Proc. Amer. Math. Soc., **73** (1979), 363–364.

[7] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I*, Springer-Verlag, 1963.

[8] ______, *Abstract Harmonic Analysis II*, Springer-Verlag, 1970.

[9] A. T. Lau, *Semigroup of operators on dual Banach spaces*, Proc. Amer. Math. Soc., **54** (1976), 393–396.

[10] ______, *Operators which commute with convolutions on subspaces of $L_\infty(G)$*, Colloquium Mathematicum, **39** (1978), 351–359.

[11] ______, *Invariantly complemented subspaces of $L_\infty(G)$ and amenable locally compact groups*, Illinois J. Math., **26** (1982), 226–235.

[12] V. Losert and H. Rindler, *Asymptotically Central Functions and Invariant Extensions of Dirac Measure*, Probability Measures on Groups VII, Proceedings Oberwolfach 1983, Springer Lecture Notes Math. **1069** (1984), 368–378.

[13] H. P. Rosenthal, *Projections onto translation invariant subspace of $L^p(G)$*, Memoirs Amer. Math. Soc., **63** (1966).
[14] J. Rosenblatt, *Uniqueness of invariant means for measure-preserving transformations*, Trans. Amer. Math. Soc., **265** (1981), 623–636.
[15] W. Rudin, *Invariant means on L^∞*, Studia Mathematica, **44** (1972), 219–227.
[16] J. D. Stafney, *Arens multiplication and convolution*, Pacific J. Math., **14** (1964), 1423–1447.
[17] D. Sullivan, *For $n > 3$ there is only one finitely additive rotationally invariant measure on the n-sphere defined on all Lebesgue measurable subsets*, Bull. Amer. Math. Soc., **4** (1981), 121–123.
[18] M. Takesaki and N. Tatsuuma, *Duality and subgroups*, Ann. of Math., **93** (1971), 344–364.
[19] _____, *Duality and subgroups II*, J. Funct. Anal., **11** (1972), 184–190.
[20] Y. Takahashi, *A characterization of certain weak*-closed subalgebras of $L_\infty(G)$*, Hokhaido Math. J., **11** (1982), 116–124.
[21] M. Talgrand, *Closed convex hull of set of measurable functions, Riemann-measurable functions and measurability of translations*, Ann. Inst. Fourier, Grenoble, **32** (1982), 39–69.

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