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SPECTRAL DECOMPOSITION OF $L^2(N \setminus \mathrm{GL}(2), \eta)$

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Let G denote $\mathrm{GL}(2, F)$, where F is $\mathbf{R}$ or $\mathbf{C}$ or a p-adic field. Let η be a non-trivial character of the unipotent upper triangular group N in G. The object of this paper is to present an explicit spectral decomposition of $L^2(N \setminus G, \eta)$, the representation of G induced unitarily by η. This representation is well-known to be multiplicity-free and to be quasi-equivalent to the (right) regular representation of G.

Introduction. In §1 we give a 'cuspidal' characterization of the discrete spectrum of $L^2(N \setminus \mathrm{SL}(2, F), \eta)$. In §2 we prove a crucial duality formula, which later allows us to decompose the continuous spectrum. It also 'explains' the occurrence of ε'-factors in the measure giving the direct integral decomposition.

The scalar product of §3 is originally due, in the p-adic case, to H. Jacquet. Here we suitably modify and extend his (unpublished) work and also treat the archimedean case. We have tried to present a unified approach.

This work was done as a part of my doctoral thesis **[11]** at Columbia. Besides the obvious debt to my advisor then, H. Jacquet, I would like to mention the strong influence of R. Godement's paper ([**3**]) on the spectral analysis of modular functions, and thank P. Sally for his interest and critical comments. I would like to thank the referee for his helpful remarks which led to simplifications of some of the proofs. Thanks are also due to Miss M. Murray and Mrs. Anne Wolfsheimer for their excellent typing of this manuscript.

0. Terminology. Fix a non-trivial character ψ of F^+. Let dx be the self-dual measure on F with respect to ψ and let $|\ \ |$ be the normalized absolute value. When F is non-archimedean, let $\mathfrak{O}$, $\mathfrak{p}$, v, π and $\mathbf{F}_q$ respectively denote the ring of integers, maximal ideal, valuation, uniformizer and residue field. On F^2 we will always take the product measure induced by the self-dual measure dx on F. We will denote by U_F the group $\{x \in F \mid |x| = 1\}$.

Let

$$G = \mathrm{GL}(2, F), \quad N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \subset G,$$

$$A = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \middle| a_1, a_2 \in F^* \right\}, \quad B = NA,$$

$$Z = \left\{ \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \middle| t \in F^* \right\}, \quad P = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\}, \quad \text{and}$$

$$K = \begin{cases} \mathrm{GL}_2 \mathfrak{O}, & \text{if } F \text{ is } p\text{-adic} \\ O(2), & \text{if } F = \mathbf{R} \\ U(2), & \text{if } F = \mathbf{C}. \end{cases}.$$

Denote by η the character of N given by: $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \mapsto \psi(x)$. Then $L^2(N \backslash G, \eta)$ consists of (classes of) measurable functions f on G which are square-integrable mod N and satisfy:

$$f(ng) = \eta(n) f(g), \quad \text{for almost all } (n, g) \text{ in } N \times G.$$

Let $G^1 = \mathrm{SL}(2, F)$. For every subgroup H of G, let H^1 (resp. H_K) denote $H \cap G^1$ (resp. $H \cap K$).

Let δ: $A \to \mathbf{R}_+^*$ be the module A defined by: $\delta(a) = |a_1/a_2|$, for

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

Let d^*a denote the product measure on $A \xrightarrow{\approx} (F^*)^2$, $a \mapsto (a_1, a_2)$, so that $d^*a = d^*a_1 \cdot d^*a_2$.

Fix a Haar measure dk on K such that dk coincides with d^*a on A_K. Then the quotient measure on $N \backslash G$ may be identified, via Iwasawa decomposition: $G = NAK$, with: $\delta(a)^{-1} \cdot d^*a dk$.

If h_1, h_2 are in $L^2(N \backslash G, \eta)$ (resp. $L^2(N \backslash G)$), the function: $g \mapsto h_1(g)\overline{h_2(g)}$ is invariant on the left by N. We set

$$\langle h_1, h_2 \rangle = \int_{N \backslash G} h_1(g) \overline{h_2(g)}\, dg.$$

For any locally compact topological group H, let $\hat{H}$ denote the Mackey dual of H, viz. the set of classes of inequivalent, irreducible, unitary representations of H. By abuse of notation, we will often use the same symbol for an element of $\hat{H}$ and for a representative of that element.

If S is a topological space, and if f_1, f_2 are two complex-valued functions on S, then we write: $f_1 \prec f_2$ if there exists a positive scalar C

such that $|f_1(x)| \leq C|f_2(x)|$, for all x in S. Sometimes we will also write: $f_1(x) \prec f_2(x)$ for all $x \in S$. In particular, $f_1 \prec 1$ implies that f_1 is bounded on S.

We write $f_1 \succ f_2$ if $f_2 \prec f_1$, and $f_1 \asymp f_2$ if we have simultaneously: $f_1 \prec f_2$ and $f_1 \succ f_2$.

We will always write, for a positive real number t, that $t \ll 1$ (resp. $t \gg 1$) if t is very close to 0 (resp. to $+\infty$).

1. The intertwining operator θ. We first recall the natural identification of the homogeneous space $N \setminus G^1$ with $F^2 - \{0\}$.

G^1 acts by right multiplication ρ on $F^2 = \{(x, y) | x, y \in F\}$. This action leaves $F^2 - \{0\}$ stable and is transitive on it. Furthermore, if e_1 denotes the vector $(0, 1)$ in F^2, then N is precisely the stabilizer of e_1 in G^1. This results in a right G^1-space analytic isomorphism:

$$(1.0) \qquad \begin{aligned} N \setminus G^1 &\tilde{\to} F^2 - \{0\}, \\ Ng &\mapsto \rho_g \cdot e_1 = e_1 \cdot g. \end{aligned}$$

Let $S(F^2)$ denote the space (cf. [7]) of Schwartz-Bruhat functions on F^2. It is stable under the right action ρ of G^1.

DEFINITION 1.1. For Φ in $S(F^2)$ and g in G^1, set:

$$f_\Phi(g) = \Phi(e_1 \cdot g).$$

Since the Haar measure on F^2 is invariant under the action of G^1, it must be, under the above identification, the invariant measure on $N \setminus G^1$. Hence we obtain:

LEMMA 1.2. $f_\Phi \in L^2(N \setminus G^1)$, *for every* $\Phi \in S(F^2)$.

DEFINITION 1.3.

$$S(N \setminus G^1) = \{ f_\Phi | \Phi \in S(F^2) \}.$$

Note that the map $\Phi \mapsto f_\Phi$ is injective and is equivariant with respect to the right action of G^1.

The following proposition is an immediate consequence of the remark preceding Lemma 1.2.

PROPOSITION 1.4. *The G^1-equivariant isomorphism*

$$S(F^2) \tilde{\to} S(N \setminus G^1), \quad \Phi \mapsto f_\Phi,$$

extends to a Hilbert space isomorphism of the unitary G^1-modules $L^2(F^2)$ and $L^2(N \setminus G^1)$.

DEFINITION 1.5. Let $\mathscr{L}$ denote the set of all lines in F^2 passing through the origin, and set:

$$S_0(F^2) = \left\{ \Phi \in S(F^2) \,|\, \int_L \Phi(z)\,dz = 0, \right.$$
$$\left. \forall \text{ line } L \text{ in } \mathscr{L} \text{ with Haar measure } dz \right\}.$$

PROPOSITION 1.6. *$S_0(F^2)$ is G^1-stable, and is dense in $L^2(F^2)$.*

Proof. The right action of G^1 on F^2 leads to an action of G^1 on the space $\mathscr{L}$ of all lines in F^2 as follows: Given L in $\mathscr{L}$, choose w in F^2 such that $L = Fw$. If g is in G^1, set: $\rho_g(L) = F \cdot (\rho_g w)$. It may be checked that $\rho_g(L)$ is independent of the choice of w.

For every Φ in $S(F^2)$ and for every L in $\mathscr{L}$ with Haar measure dz, set:

$$A(\Phi)(L, dz) = \int_L \Phi(z)\,dz.$$

And for every g in G^1, let dz_g be the Haar measure on $\rho_g(L)$ induced by dz. Then the G^1-stability of $S_0(F^2)$ follows from the fact:

$$A(\rho_g\Phi)(L, dz) = A(\Phi)(\rho_g(L), dz_g).$$

To prove the density of $S_0(F^2)$ in $L^2(F^2)$, it suffices to show that for every Φ in $C_c^\infty(F^2)$, and for any $\varepsilon > 0$, we can find a function Φ_ε in $C_c^\infty(F^2)$ such that $\Phi - \Phi_\varepsilon \in S_0(F^2)$ and such that $\|\Phi_\varepsilon\|_2 < \varepsilon$. Here $\| \; \|_2$ is the norm in $L^2(F^2)$.

Given t in F^*, define:

$$T_t(x) = |t|^{-1}\Phi(t^{-1}x), \quad \text{for all } x \text{ in } F^2.$$

Then it is straight-forward to see that $A(T_t(\Phi)) = A(\Phi)$ for all t and that $\|T_t(\Phi)\|_2 = \|\Phi\|_2$.

Since Φ has compact support, we can find a positive number r such that the support of $T_t(\Phi)$ is disjoint from the support of Φ when $|t| > r$. Furthermore, one sees that the functions $\{ T_{t^m}(\Phi) | m = 0, 1, 2, \dots \}$ all have disjoint support when $|t| > r$. Fix a t with $|t| > r$, and choose an integer k such that $\sqrt{k}\,\varepsilon > \|\Phi\|_2$. Set:

$$\Phi_\varepsilon = \frac{1}{k} \sum_{j=0}^{k-1} T_{t^j}(\Phi).$$

Then certainly $A(\Phi_\varepsilon) = A(\Phi)$. And since $\{T_{t^j}(\Phi)\}$ have disjoint support, they are orthogonal in $L^2(F^2)$. So we have:

$$\|\Phi_\varepsilon\|_2 = \frac{1}{k}\left(\sum_{j=1}^{k-1}\|T_{t^j}(\Phi)\|_2^2\right)^{1/2} = \frac{1}{k}\sqrt{k\|\Phi\|_2^2} < \varepsilon,$$

as desired.

DEFINITION 1.7. A function h on G^1 is *regular* mod N if

(i) h is K^1-finite;

(ii) $h(ng) = \eta(n)h(g)$, for all n in N and g in G^1;

(iii) for every compact set Ω in G, there exist a positive scalar C and a smooth function $\tilde{h}$ on Ω such that, for all g in Ω, we have:

(A) when F is non-archimedean:

$$h\left[\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}g\right] = \begin{cases} 0, & \text{if } |a| > C^{-1} \\ |a|^2\tilde{h}(g), & \text{if } |a| < C; \end{cases}$$

and

(B) when F is archimedean:

$$h\left[\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}g\right]: \begin{cases} \to 0 \text{ rapidly as } |a| \to \infty \\ \prec |a|^{\beta_F}\cdot\tilde{h}(g), \text{ as } |a| \to 0, \end{cases}$$

where $\beta_{\mathbf{R}} = 2$ and $\beta_{\mathbf{C}} = 3/2$.

DEFINITION 1.8. $R^1(\eta) = \{h\text{: regular on } G^1 \bmod N\}$.

$R^1(\eta)$ is stable under G if F is p-adic, and under $(\operatorname{Lie} G^1, K^1)$ if F is $\mathbf{R}$ or $\mathbf{C}$. (See Wallach [15] for a definition of $(\operatorname{Lie} G^1, K^1)$-modules.)

It is simple to check that:

$$R^1(\eta) \subset L^2(N\setminus G^1,\eta). \tag{1.9}$$

Since smooth functions on G^1 with compact support mod N belong to $R^1(\eta)$, this space is dense in $L^2(N\setminus G^1,\eta)$.

Let w denote the Weyl element $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1\end{smallmatrix}\right)$.

DEFINITION 1.10. For Φ in $S_0(F^2)$, set:

$$\theta_\Phi(g) = \int_N f_\Phi(wng)\overline{\eta(n)}\,dn, \quad \text{for all } g \text{ in } G^1.$$

It may be checked that this is well-defined. It will turn out that θ_Φ is even square-integrable mod N. More precisely, we will show:

PROPOSITION 1.11. $\theta_\Phi \in R^1(\eta)$, *for all* $\Phi \in S_0(F^2)$.

Proof. Let Φ be in $S_0(F^2)$. Then for all n in N and g in G, we have:

$$\begin{aligned}\theta_\Phi(ng) &= \int_N f_\Phi(wung)\bar{\eta}(u)\,du = \int_N f_\Phi(wug)\bar{\eta}(un^{-1})\,du \\ &= \eta(n)\theta_\Phi(g).\end{aligned}$$

Observe that, for every a in F^*, we have:

$$\theta_\Phi\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} = |a|^2\int_F \Phi(a, ax)\bar{\psi}(a^2x)\,dx.$$

Sending x to $a^{-1}x$, we get:

$$\theta_\Phi\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} = |a|\int_F \Phi(a, x)\bar{\psi}(ax)\,dx. \tag{1.12}$$

If F is p-adic, then Φ has compact support in F^2, and hence it is clear that

$$\theta_\Phi\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}$$

vanishes if $|a|$ is large. If F is archimedean, we have for every $M > 0$: $\int_{-M}^{M}\Phi(a, x)\bar{\psi}(ax)\,dx \to 0$ rapidly as $|x| \to \infty$. And we also have: $\Phi(a, x) \to 0$ rapidly as $|x| \to \infty$. These two facts imply (using (1.12)) that

$$\theta_\Phi\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} \to 0$$

rapidly as $|a| \to \infty$.

Now we look at the behavior when $|a|$ is near 0. We can write:

$$\int_F \Phi(a, x)\bar{\psi}(ax)\,dx = f_1(a) + f_2(a),$$

where

$$f_1(a) = \int_F (\Phi(a, x) - \Phi(0, x))\bar{\psi}(ax)\,dx,$$

and

$$f_2(a) = \int_F \Phi(0, x)(\bar{\psi}(ax) - 1)\,dx.$$

Assume that F is p-adic and that $|a| \ll 1$. Then as Φ is locally constant, $\Phi(a, x) = \Phi(0, x)$. Consequently $f_1(a) = 0$. Furthermore, $f_2(a)$ equals: $\int_C \Phi(0, x)(\bar{\psi}(ax) - 1)\,dx$, where C is a compact set in F outside which $\Phi(0, x)$ is zero. Since $\bar{\psi}$ is a locally constant function of F, we have: $\bar{\psi}(ax) = 1$ for x in C. Hence $f_2(a) = \int_F \Phi(0, x)\,dx$, which is zero since Φ

is in $S_0(F^2)$. Thus

$$\theta_\Phi \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

vanishes when $|a| \ll 1$.

Let us now turn to the archimedean case. For every Φ in $S(F^2)$, we set:

$$\Phi^{\sim}(t) = \int_F \Phi(0, x)\overline{\Psi}(tx)\,dx,$$

$$D_1\Phi(u, v) = \frac{\partial \Phi}{\partial u}, \quad \text{and} \quad D\Phi^{\sim}(t) = \frac{d\Phi^{\sim}}{dt}.$$

Then we can write: (with γ_a: a C^∞-path from 0 to a in F)

$$f_1(a) = \int_F \overline{\psi}(ax) \int_{\gamma_a} D_1\Phi(t, x)\,dt = \int_{\gamma_a} dt \int_F D_1\Phi(t, x)\overline{\psi}(ax)\,dx,$$

and $f_2(a) = \int_{\gamma_a} D\Phi^{\sim}(t)\,dt$, where dt is the restriction to γ_a of the natural 1-form on F. It is clear that both $\int_F D_1\Phi(t, x)\overline{\psi}(ax)\,dx$ and $D\phi^{\sim}(t)$ are both $O(1)$ when $|t| \le |a| \lessdot 1$. Hence we obtain (for $i = 1$ *and* $i = 2$):

$$f_i(a) = \begin{cases} O(|a|), & \text{if } F = R, \\ O(|a|^{1/2}), & \text{if } F = \mathbf{C}, \end{cases} \quad \text{when } |a| \to 0.$$

It then follows from (1.12) that

$$\theta_\Phi \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

has the desired asymptotic behavior when $|a| \to 0$.

REMARK 1.13. We have actually shown that in the non-archimedean case θ_Φ is even compactly supported mod N, for each Φ in $S_0(F^2)$.

DEFINITION 1.14.
(a) V_c = closure of $\{\theta_\Phi | \Phi \in S_0(F^2)\}$ in $L^2(N \setminus G, \eta)$
(b) ${}^0V$ = the orthocomplement of V_c in $L^2(N \setminus G, \eta)$.
These two spaces are clearly stable under G^1.

DEFINITION 1.15. For each h in $R^1(\eta)$, set:

$$h^N(g) = \int_N h(w^{-1}ng)\,dn.$$

PROPOSITION 1.16. *Let h be in $R^1(\eta)$. Then $\int_N h(w^{-1}ng)\,dn$ is convergent, and the resulting function h^N satisfies the following properties*:

(a) $h^N(ng) = h^N(g)$, *for all n in N and g in G*;

(b) h^N *is K^1-finite*;

(c) (i) *when F is p-adic*,

$$h^N\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} \quad \textit{is } 0\ (\textit{resp. } O(1)) \quad \textit{if } |a| \ll 1\ (\textit{resp. } |a| \gg 1);$$

and (ii) *when F is archimedean*,

$$h^N\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} \prec \begin{cases} \begin{Bmatrix} 1, & \textit{if } F = \mathbf{R} \\ |a|^{1/2}, & \textit{if } F = \mathbf{C}\end{Bmatrix}, & \textit{as } |a| \to \infty \\ |a|^2, & \textit{as } |a| \to 0\end{cases}$$

Using this proposition and the identification $N \backslash G^1 \cong F^2 - \{0\}$, the following result is easily deduced.

COROLLARY 1.17. $h^N \in L^2(N \backslash G^1)$, *for all $h \in R^1(\eta)$*.

Proof of Proposition 1.16. The convergence claim is easy to prove, and the parts (a) and (b) are clear. We give a proof of part (c):

(i) F: *p-adic*. Let h be in $R^1(\eta)$. Then using Iwasawa decomposition, we get:

$$\begin{aligned} (1.18)\quad h^N\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} &= |a|^2 \int_{\mathfrak{O}} h\left[\begin{pmatrix} a^{-1} & 0 \\ 0 & a\end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1\end{pmatrix} w^{-1}\right] dx \\ &+ |a|^2 \int_{|x|>1} h\left[\begin{pmatrix} a^{-1} & 0 \\ 0 & a\end{pmatrix}\begin{pmatrix} 1 & -x^{-1} \\ 0 & 1\end{pmatrix}\right. \\ &\qquad \left.\times \begin{pmatrix} -x^{-1} & 0 \\ 0 & -x\end{pmatrix}\begin{pmatrix} 1 & 0 \\ x^{-1} & 1\end{pmatrix}\right] dx. \end{aligned}$$

Let I_a and II_a respectively denote the first and second expression on the right of (1.18).

When $|a| \ll 1$, since h is regular mod N,

$$h\left[\begin{pmatrix} a^{-1} & 0 \\ 0 & a\end{pmatrix} k\right] = 0$$

for all k in K^1. Hence $\mathrm{I}_a = 0$ when $|a| \ll 1$.

When $|a| \gg 1$, we have: $|a^{-1}| \ll 1$ and since h is in $R^1(\eta)$ we see that the regularity of h implies that the integrand of I_a is $O(|a|^{-1})$. This times $|a|^2$ gives $O(1)$. Hence $\mathrm{I}_a = O(1)$ when $|a| \gg 1$, since $\mathrm{Vol}(\mathfrak{O}) < \infty$.

It remains to consider II_a. Clearly we have

$$\mathrm{II}_a = |a|^2 \int_{\mathfrak{P}} \psi(-a^2x) h\left[\begin{pmatrix} -a^{-1}x & 0 \\ 0 & -ax^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right]|x|^{-2}\,dx.$$

When $|a| \gg 1$, $|a^{-1}| \ll 1$ and $|a^{-1}x| < |a^{-1}| \ll 1$ for all x in $\mathfrak{p}$. So the integrand of II_a above is $\prec |a^{-2}x^2|$, and so:

$$\mathrm{II}_a \prec O(1), \quad \text{when } |a| \gg 1,$$

taking into account the fact that $\mathrm{Vol}(\mathfrak{p}) < \infty$.

Now let $|a| \ll 1$. There exist positive scalars C, D and $\delta \in \mathbf{C}$ such that: (for x in $\mathfrak{p}$)

(A) $$\operatorname{supp}:\left\{b \mapsto h\left[\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right]\right\} \subset \{|b| \leq C\},$$

and

(B) $$|b| \leq D \Rightarrow h\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} = \delta |b|^2.$$

Consequently, we get:

$$\mathrm{II}_a = |a|^2 \int_{|x| \leq C|a|} \psi(-a^{-1}x) h\left[\begin{pmatrix} -a^{-1}x & 0 \\ 0 & ax^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right]|x|^{-2}\,dx.$$

Since $|a| \ll 1$ and since $|x| \leq C|a|$, h is invariant on the right by $\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right)$. Thus we get using (A) and (B):

$$\begin{aligned}\mathrm{II}_a = \delta &\int_{|x| \leq D|a|} \psi(a^{-2}x)\,dx \\ &+ |a| \int_{D < |x| \leq C} \psi(-xa^{-1}) h\begin{pmatrix} -x & 0 \\ 0 & -x^{-1} \end{pmatrix}|x|^{-2}\,dx.\end{aligned}$$

Now

$$\int_{|x| \leq D|a|} \psi(-a^{-2}x)\,dx = |a| \int_{|x| \leq D} \psi(-a^{-1}x)\,dx = 0,$$

if $|a| \ll 1$.

Since h is K^1-finite, we can write:

$$h\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} = \sum_{\omega \in S} h_\omega(x).$$

for all x in F^*, where S is a finite subset of $\hat{U}_F$ and where h_ω satisfies:

$$h_\omega(ux) = \omega(u) h_\omega(x), \quad \text{for all } u \text{ in } U_F \text{ and } x \text{ in } F^*.$$

Consequently we obtain:

$$\int_{D<|x|\leq C} \psi(-xa^{-1}) h\begin{pmatrix} -x & 0 \\ 0 & -x^{-1} \end{pmatrix} |x|^{-2}\, dx$$

$$= \sum_{\{n | D < q^{-n} \leq C\}} q^{2n} \sum_{\omega \in S} h_\omega(-\pi^n) G(\omega, a^{-1}\pi^n),$$

where

$$G(\omega, t) = \int_{U_F} \omega(u) \bar{\psi}(tu)\, du, \quad \text{for } t \text{ in } F^*. \tag{1.19}$$

We will be done if we show that for each $\omega \in S$, and for each n such that $D < q^{-n} \leq C$, the *Gauss sum* $G(\omega, a^{-1}\pi^n)$ is zero when $|a| \ll 1$. This follows from the following:

Claim 1.20. For every ω in $\hat{U}_F$, there exists $l \in \mathbf{Z}$ such that

$$|t| > q^l \Rightarrow G(\omega, t) = 0.$$

Indeed, if $\omega = 1$ we have: $G(\omega, t) = \int_{\mathfrak{O}} \bar{\psi}(tx)\, dx - \int_{\mathfrak{p}} \bar{\psi}(tx)\, dx$. And if $\omega \neq 1$ with $\mathfrak{p}^r$ the largest ideal such that ω is trivial on $1 + \mathfrak{p}^r$, we see that:

$$G(\omega, t) = \sum_{U_F/(1+\mathfrak{p}^r)} \omega(u) \bar{\psi}(tu) \int_{\mathfrak{p}^r} \bar{\psi}(tux)\, dx.$$

If $p^{-n(\omega)} = \ker(\psi)$, we get by orthogonality of characters of compact groups that:

$$G(\omega, t) = 0, \quad \text{if } v(t) < -l_\omega$$

where

$$l_\omega = n(\psi) + 1 \text{ (resp. } n(\psi) + r) \quad \text{if } \omega = 1 \text{ (resp. } \omega \neq 1). \qquad \square$$

(ii) F: *archimedean*. We will first study the behavior of

$$h^N\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad \text{as } |a| \to 0.$$

Since h is K^1-finite, there exists an f in $C_c^\infty(G^1)$ such that $h = h * f^\vee$. Here $f^\vee$ denotes the function: $g \mapsto f(g^{-1})$, and $*$ denotes convolution product. It then follows that:

$$h^N\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = (h^N * f^\vee)\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \tag{1.21}$$

$$= \int_{N \backslash G^1} h^N\left[\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} g\right] \mu_f(g)\, dg,$$

where $\mu_f(g) = \int_{N \backslash G^1} f(ng)\, dn$.

Note that μ_f has compact support mod N. So there exists a compact set Ω_1 in $A^1K^1 \simeq N \setminus G^1$ such that:

$$h^N\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \iint_{\Omega_1} h^N\left[\begin{pmatrix} ab & 0 \\ 0 & a^{-1}b^{-1} \end{pmatrix} d\right] \cdot \mu_f\left[\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} k\right] \frac{d^*b\,dk}{|b|^2}.$$

Since $\mu_f \prec 1$, this yields:

$$\left| h^N\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right| \prec \iint_{\Omega_1} \left| h^N\left[\begin{pmatrix} ab & 0 \\ 0 & a^{-1}b^{-1} \end{pmatrix} k\right] \right| \cdot |b|^{-1}\, d^*b\, dk. \tag{1.22}$$

Let $\Omega_2(a)$ be the image of Ω_1 under the map:

$$\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} k \mapsto \begin{pmatrix} a^{-1}b & 0 \\ 0 & ab^{-1} \end{pmatrix} k.$$

Then (1.22) becomes:

$$\left| h^N\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right| \prec |a|^2 \cdot H(a), \tag{1.23}$$

where

$$H(a) = \iint_{\Omega_2(a)} \left| h^N\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} k \right| \cdot |b|^{-1}\, d^*b\, dk.$$

Observing that $H(a)$ is at least $O(1)$ for $|a|$ small, we get from (1.23):

$$\left| h^N\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right| \prec |a|^2 \quad \text{as } |a| \to 0.$$

It remains to show the asserted behavior of

$$h^N\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

as $|a| \to \infty$ in the archimedean situation. For this we use the Iwasawa decomposition: $G^1 = NA^1K^1$ and write:

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\bar{x}(1 - x\bar{x})^{1/2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (1 - x\bar{x})^{-1/2} & 0 \\ 0 & (1 - x\bar{x})^{1/2} \end{pmatrix} k_x,$$

where

$$k_x = \begin{pmatrix} \bar{x}(1 - x\bar{x})^{-1/2} & -(1 - x\bar{x})^{-1/2} \\ (1 - x\bar{x})^{-1/2} & x(1 - x\bar{x})^{-1/2} \end{pmatrix} \in K^1.$$

(Note that when $F = \mathbf{R}$, $x = \bar{x}$.)

Thus:

$$h^N\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = |a|^2 \int_F h\left[\begin{pmatrix} a^{-1}(1-x\bar{x})^{-1/2} & 0 \\ 0 & a(1-x\bar{x})^{1/2} \end{pmatrix} k_x w^{-1}\right] \cdot \bar{\psi}\left(a^{-2}\bar{x}(1-x\bar{x})^{-1/2}\right) dx.$$

Again, when $|a| \to \infty$, $|a|^{-1} \cdot |(1-x\bar{x})^{-1/2}| \to 0$, and the regularity of h implies:

$$h\left[\begin{pmatrix} a^{-1}(1-x\bar{x})^{-1/2} & 0 \\ 0 & a(1-x\bar{x})^{1/2} \end{pmatrix} k_x w^{-1}\right] \prec |a|^{-\beta_F} \cdot (1-x\bar{x})^{-\beta_F/2}.$$

Consequently,

$$h^N\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \prec |a|^{2-\beta_F} \int_F (1-x\bar{x})^{-\beta_F/2}\, dx \prec |a|^{2-\beta_F},$$

when $|a| \to \infty$, as desired. □

PROPOSITION 1.24. *If h is in $R^1(\eta)$, then*

$$h \in {}^0V \Leftrightarrow h^N = 0.$$

Proof. For every Φ in $S_0(F^2)$ and every h in $R^1(\eta)$, a formal manipulation yields:

$$\langle \theta_\Phi, h \rangle = \langle f_\Phi, h^N \rangle. \tag{1.25}$$

To make this rigorous we need only to check that: $\int_{G^1} f(wg)\overline{h(g)}\,dg$ is absolutely convergent. We may replace h by $|h|$ and assume that $h \geq 0$. Then the desired result follows from the asymptotic properties of f_Φ and h (cf. Proposition 1.16).

Since $S_0(F^2)$ is dense in $L^2(F^2)$, we see that $S_0(N \backslash G^1) = \{f_\Phi | \Phi \in S_0(F^2)\}$ is dense in $L^2(N \backslash G^1)$. Therefore any h in $R^1(\eta)$ is orthogonal to V_c = the closure of $\{\theta_\Phi | \Phi \in S_0(F^2)\}$ if and only if h^N is identically zero. □

DEFINITION 1.26.

$${}^0V_K = \{h \in {}^0V | h \text{ is } K^1\text{-finite}\}.$$

THEOREM 1.27.

(a) ${}^0V_K$ *is admissible*;

(b) ${}^0V_K = {}^0V \cap R^1(\eta)$, *i.e., every h in ${}^0V_K$ is regular* mod N; *and*

(c) ${}^0V_K$ *is precisely the space of all K^1-finite vectors in the discrete spectrum of $L^2(N \backslash G^1; \eta)$.*

REMARK 1.28. It is known (cf. [10], for example) that $L^2(N \setminus G; \eta)$ is a multiplicity-free representation. Hence every irreducible of G^1 occurring in ${}^0V_K$ will occur exactly once.

Proof of Theorem 1.27. We will show the admissbility (a) and regularity (b) of ${}^0V_K$ simultaneously.

We first decompose ${}^0V_K$ according to the right action as:

$$ {}^0V_K = \bigoplus_{(\sigma, H_\sigma) \in \widehat{K^1}} {}^0V_K(\sigma), $$

where each σ-isotypic space ${}^0V_K(\sigma)$ can be identified with

$$ \{ f\colon G^1 \to H_\sigma | f(ngk) = \eta(n)\sigma(k^{-1})f(g), \text{ for all } n \in N, $$
$$ g \in G^1 \text{ and } k \in K^1; \textit{ and } f \in L^2(N \setminus G^1/K) \}. $$

For every (σ, H_σ) in $\widehat{K^1}$ and for each character ω of $B_K^1 = B \cap K^1$, we let $H_\sigma(\omega)$ denote the subspace of $H_\sigma(\omega)$ consisting of vectors v such that $\sigma(b)v = \omega(b)v$ for all b in B_K^1. Then it can be seen (cf. [5]) that $\dim H_\sigma(\omega) \leq 1$, and that, for a given σ, $H_\sigma(\omega) = 0$ for all but finitely many characters ω of B_K^1. We get:

$$ {}^0V_K(\sigma) = \bigoplus_{\omega \in B_K^1} {}^0V_K(\sigma, \omega), $$

where

$$ {}^0V_K(\sigma, \omega) = \{ f \in {}^0V_K(\sigma) | f(gb) = \overline{\omega}(b) f(g), $$
$$ \text{for all } g \text{ in } G^1 \text{ and } b \text{ in } B_K^1 \}. $$

To show the admissibility of ${}^0V_K$, we need to prove that ${}^0V_K(\sigma, \omega)$ is finite-dimensional for every $(\sigma, H_\sigma) \in \widehat{K^1}$ and for every ω is $\widehat{B_K^1}$.

F: non-archimedean

The desired result in this case will follow immediately from:

LEMMA 1.29. *Fix* $(\sigma, H_\sigma) \in \hat{K}_1$ *and* $\omega \in \hat{B}_k^1$. *Then we can find a positive constant* $C = C(\sigma, \omega)$ *such that for every* h *in* ${}^0V_K(\sigma, \omega)$ *there exists a vector* v *in* $H_\sigma(\omega)$ *with the property*:

$$ h\left[\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right] = \begin{cases} |a|^2 \overline{\omega} \begin{pmatrix} a/|a| & 0 \\ 0 & |a|/a \end{pmatrix} v_0 & \text{if } |a| \leq C \\ 0 & \text{if } |a| \geq C^{-1}. \end{cases} $$

For every integer $r \geq 1$, let K_r^1 denote the principal congruence subgroup of level r, namely: $\{k \in K^1 | dk \equiv I \pmod{\mathfrak{p}^r}\}$. Let m be the smallest integer r such that $K_r^1 \subset \mathrm{Ker}(\sigma)$.

Let x be in $\mathfrak{p}^m$. Then, for every h in ${}^0V(\sigma, \omega)$, we have:

$$h\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = h\left[\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right], \quad \text{for all } a \text{ in } F^*.$$

But

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a^2x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix};$$

so

$$h\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \psi(a^2x)h\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \qquad \forall a \in F^*,\ x \in \mathfrak{p}^m.$$

Let $\mathfrak{p}^{-n(\psi)}$ be the largest ideal on which ψ is trivial. Then we get:

$$h\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = 0, \quad \text{if } 2v(a) < -m - n(\psi),$$

i.e., if $|a| > q^{m+n(\psi)/2}$.

Since the case when $H_\sigma(\omega) = 0$ is trivial, we will choose (σ, H_σ) and ω such that $H_\sigma(\omega) \neq \{0\}$. We will produce certain explicit special functions θ_Φ in $V_c(\sigma, \omega)$ so that h is forced to have the derived asymptotic property as $|a| \ll 1$ by virtue of h being orthogonal to $V_c(\sigma, \omega)$. For any right G^1-module X, the space $X(\sigma, \omega)$ is defined in the obvious way.

For any Φ in $S_0(F^2; \sigma, \omega)$, we have:

$$\begin{aligned} \theta_\Phi\begin{pmatrix} \pi^l & 0 \\ 0 & \pi^{-l} \end{pmatrix} &= q^{-l}\int_F \Phi(\pi^l, x)\bar{\psi}(\pi^l x)\,dx \\ &= q^{-l}\int_{\mathfrak{p}^l} \Phi\left[(0, \pi^l)\begin{pmatrix} 0 & -1 \\ 1 & \pi^{-l}x \end{pmatrix}\right]\bar{\psi}(\pi^l x)\,dx \\ &\quad + q^{-l}\int_{|x| > q^{-l}} \Phi\left[(0, x)\begin{pmatrix} 1 & 0 \\ \pi^l x^{-1} & 1 \end{pmatrix}\right]\bar{\psi}(\pi^l x)\,dx. \end{aligned}$$

We will choose Φ in such a way that the support of $\{t \mapsto \Phi[(0, t)k]\}$ is in $\{t^{-1} \in \mathfrak{p}^{m-l}\}$, uniformly for all k in K^1. Then the integral:

$$\int_{\mathfrak{p}^l} \Phi\left[(0, \pi^l)\begin{pmatrix} 0 & -1 \\ 1 & \pi^{-l}x \end{pmatrix}\right]\bar{\psi}(\pi^l x)\,dx$$

vanishes. Furthermore if $x^{-1} \in \mathfrak{p}^{m-l}$, then the matrix

$$\begin{pmatrix} 1 & 0 \\ \pi^l x^{-1} & 1 \end{pmatrix}$$

is in K_m^1, and so leaves Φ invariant (on the right). Thus we get:

$$(1.30)\quad \theta_\Phi\begin{pmatrix}\pi^l & 0\\ 0 & \pi^{-l}\end{pmatrix} = q^{-l}\int_{|x|>q^{m-l}}\Phi(0,x)\bar{\psi}(\pi^l x)\,dx = q^{-l}\cdot\Phi^{\sim}(\pi^l),$$

where $\Phi^{\sim}(u) = \int_F \Phi(0,x)\bar{\psi}(ux)\,dx$.

Let $\omega \neq 1$. Then the conductor of ω is $\mathfrak{p}^r$ with $r > 0$, i.e., $1+\mathfrak{p}^r$ is the largest subgroup of U_F on which ω is trivial. Choose a non-zero vector v_0 in $H_\sigma(\omega)$ and define for every $j \geq [3m/2] - n(\psi)$:

$$\phi_j(\pi^{-i}) = \delta_{i,j+r+n(\psi)}\cdot v_0.$$

Since every non-zero element of $F^2 - \{0\}$ can be written as $(0,\pi^{-i})k$ for some integer i and some k in K^1, we see that we get a unique function Φ_j in $S_0(F^2;\sigma,\omega)$ such that $\Phi_j(0,\pi^{-i})$. Then for every integer $l \geq [-m/2]$, we have:

$$l + j + r + n(\psi) > l + j + n(\psi) \geq m-1,$$

and thus:

$$\operatorname{Supp}(\phi_j) \subset \{\pi^l x^{-1} \in \mathfrak{p}^m\}.$$

So by (1.30), we get:

$$\begin{aligned}\theta_{\Phi_j}\begin{pmatrix}\pi^l & 0\\ 0 & \pi^{-l}\end{pmatrix} &= q^{-l+j+r+n(\psi)}\cdot v_0\int_{U_F}\bar{\psi}(\pi^{l-j-r-n(\psi)}\cdot u)\,\omega(u)\,du\\ &= \delta_{l,j}\cdot q^r\cdot \operatorname{Vol}(U_F).\end{aligned}$$

Now, since h is orthogonal to θ_{Φ_j}, we have for every $j \geq [3m/2] - n(\psi)$:

$$\begin{aligned}0 &= \int_{|a|\leq q^{m/2}}\left\langle h\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}, \theta_{\Phi_j}\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\right\rangle |a|^{-2}\,d^*a\\ &= \sum_{l\geq[-m/2]} q^{2l}\left\langle h\begin{pmatrix}\pi^l & 0\\ 0 & \pi^{-l}\end{pmatrix}, \theta_{\Phi_j}\begin{pmatrix}\pi^l & 0\\ 0 & \pi^{-l}\end{pmatrix}\right\rangle.\end{aligned}$$

Consequently we get:

$$q^{2j+r}\left\langle \phi\begin{pmatrix}\pi^j & 0\\ 0 & \pi^{-j}\end{pmatrix}, V_0\right\rangle\cdot \operatorname{Vol}(U_F) = 0,$$

$$\text{for every } j \geq [3m/2] - n(\psi).$$

Hence: $h\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix} = 0$, if $|a| \leq q^{n(\psi)-3m/2}$.

We now turn to the case when $\omega = 1$.

Choose for every $j > 3m/2 - n(\psi)$, a function $\Phi_j \in S_0(F^2; \sigma, 1)$ such that:

$$\Phi_j^{\sim}(\pi^l) = \begin{cases} q^{r+n(\psi)} \cdot v_0, & \text{if } l = j \\ -q^{r+n(\psi)-1} \cdot v_0, & \text{if } l = j+1 \\ 0, & \text{otherwise.} \end{cases}$$

Once again v_0 is some fixed non-zero vector in $H_\sigma(1)$.

$\left\langle h, \theta_{\Phi_j} \right\rangle = 0$ implies:

$$\left\langle h\begin{pmatrix} \pi^j & 0 \\ 0 & \pi^{-j} \end{pmatrix}, v_0 \right\rangle q^j = \left\langle h\begin{pmatrix} \pi^{j+1} & 0 \\ 0 & \pi^{-j-1} \end{pmatrix}, v_0 \right\rangle \cdot q^{j+2}$$

for every $j > 3m/2 - n(\psi)$.

Then we must have:

$$h\begin{pmatrix} \pi^j & 0 \\ 0 & \pi^{-j} \end{pmatrix} = -q^{-2j} \cdot v_0, \qquad \forall j > \frac{3m}{2} - n(\psi).$$

Since h is invariant under the right action of B_K^1, we get:

$$h\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = |a|^2 v_0, \quad \text{if } |a| \leq q^{n(\psi)-3m/2}.$$

This proves Lemma 1.29 and hence parts (a) and (b) of Theorem 1.27 in the non-archimedean case.

$F = \mathbf{R}$

For each integer n, let σ_n denote the irreducible module of SO(2) given by: $k_\theta \mapsto e^{in\theta}$, where

$$k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

It suffices to show that each ${}^0V_K(\sigma_n)$ is finite-dimensional. This will follow from the following explicit bound:

LEMMA 1.31.

$$\dim_{\mathbf{C}} {}^0V_K(\sigma_n) \leq \begin{cases} \max\left\{0, \dfrac{|n|-3}{2}\right\}, & \text{if } n \equiv 1\ (2) \\ \dfrac{|n|}{2}, & \text{if } n \equiv 0\ (2). \end{cases}$$

Proof. Choose for every $l \in \mathbf{Z}^+$, a function Φ_l in $S(F^2;\sigma_n)$ by:

$$\Phi_l(0,x) = P_l e^{-\pi x^2}, \quad \text{where } P_l = \sum_{j=0}^{l} c_{jl} x^{2j+m}, \ m = |n|,$$

such that $c_{ll} \neq 0$. Since $\mathbf{R}^2 - \{0\} = Y \cdot K^1$ with $Y = \{(0,x) | x \in \mathbf{R}^*\}$, every Φ in $S(F^2;\sigma_n)$ is determined by its restriction to Y.

For Φ_l to be in $S_0(\sigma_n)$, we need:

$$\sum_{j=0}^{l} c_{jl} \int_{\mathbf{R}} x^{2j+m} e^{-\pi x^2}\, dx = 0,$$

which is automatic if m is odd. When m is even, choose $\{c_{jl}\}$ such that

$$\sum_{0\le j\le l} c_{jl}\pi^{-j} \int_{\mathbf{R}_+^*} x^{j+(m+1)/2} e^{-x}\, d^*x = 0,$$

i.e., $\sum_{0\le j\le l} c_{jl}\pi^{-j}\Gamma(j+(m+1)/2) = 0$. We can do this for every $l \geq 0$. Set $\theta_l = \theta_{\Phi_l}$, and let $a \in \mathbf{R}^*$. Then we obtain:

$$\theta_l\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} = |a|^2 \int_{\mathbf{R}} \Phi_l\left[\left(0, a(1+x^2)^{1/2}\right)k_x\right] \bar{\psi}(a^2x)\, dx,$$

where

$$k_x = \begin{pmatrix} x(1+x^2)^{-1/2} & -(1-x^2)^{-1/2} \\ (1+x^2)^{-1/2} & x(1+x^2)^{-1/2}\end{pmatrix} \in \mathrm{SO}(2).$$

Thus $\theta_l\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}$ equals:

$$|a|^2 \sum_{j=0}^{l} c_{jl} \int_{\mathbf{R}} a^{2j+m}(1+x^2)^{j+m/2} e^{-\pi a^2(1+x^2)} \frac{(x\pm i)^m}{(1+x^2)^{m/2}} \bar{\psi}(a^2x)\, dx,$$

where $\pm$ denotes $\operatorname{sgn}(n)$, which becomes:

$$|a| e^{-\pi a^2} \sum_{0\le j\le l} c_{jl} \int_{\mathbf{R}} (x\pm ia)^{m+j}(x\mp ia)^j e^{-\pi x^2} \bar{\psi}(ax)\, dx,$$

as $x \to a^{-1}x$.

Let

$$Q_l(a) = \sum_{0\le j\le l} c_{jl} \sum_{p=0} \sum_{q=0} \binom{j}{p}\binom{m+j}{j} (\pm i)^{q-p-m} a^{m+2j-p-q} H_{p+q}(\sqrt{2\pi}a),$$

where H_{p+q} is the $(p+q)$th Hermite polynomial (see Magnus [9]). Then we obtain:

$$\theta_l\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} = |a| e^{-2\pi a^2} Q_l(a).$$

Let d_l be the coefficient of the leading term a^{m+2l} in Q_l. Then

$$d_l = c_{jl} \sum_{p=0}^{l} \sum_{q=0}^{m+l} \binom{l}{p} \binom{m+l}{q} (\pm i)^{q-p-m_2 p+q} (\sqrt{2\pi})^{p+q}$$

$$= c_{ll}(\pm i)^m \sum_{k=0}^{m+2l} \beta_{kl} (\pm\sqrt{-8\pi})^k,$$

where

$$\beta_{kl} = \sum_{p=0}^{\min(l,k)} (-i)^p \binom{l}{p} \binom{m+l}{k-p}.$$

Clearly, for every l, there exists a k such that $\beta_{kl} \neq 0$. And $c_{ll} \neq 0$ by assumption. Thus $d_l \neq 0$, for otherwise $\sqrt{\pi}$ would be algebraic! Therefore, the degree Q_l is precisely $m + 2l$.

Let $\phi \in {}^0V_K(\sigma_n)$. Then $\phi \perp \{\theta_l\}$, and we have

$$0 = \int_{\mathbf{R}^*} \phi \begin{pmatrix} 0 & 0 \\ 0 & a^{-1} \end{pmatrix} \overline{Q_l(a)} |a| e^{-2\pi a^2} |a|^{-2} \, d^x a$$

$$= \int_{\mathbf{R}} \tilde{\phi}(a) Q_l'(a) e^{-2\pi a^2} \, da,$$

where

$$\tilde{\phi}(a) = \begin{cases} \phi \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, & \text{if } a \neq 0 \\ 0 & \text{if } a = 0 \end{cases},$$

and $Q_l'(a) = a^{-2}\overline{Q_l(a)}$. Note that Q_l' is a polynomial of degree $m + 2l - 2$ if $m + 2l \geq 2$.

Let Z_m be the closure in $L^2(\mathbf{R})$ of the span of the union of $\{a^k e^{-\pi a^2} | k \in \mathbf{Z}_+, k - m \equiv 1(2)\}$ and

$$\left\{ Q_l'(a) e^{-2\pi a^2} \middle| \begin{array}{ll} l > 0 & \text{if } m\text{: even or } m = 1, \\ l \geq 0 & \text{if } m\text{: odd and } \geq 3 \end{array} \right\}.$$

Let $Z_m^\perp$ denote the ortho-complement of Z_m in $L^2(\mathbf{R})$.

Then

$$\dim_{\mathbf{C}} Z_m^\perp \leq \begin{cases} \max\{0, m - 3/2\} & \text{if } m \equiv 1\ (2) \\ \dfrac{m}{2} & \text{if } m \equiv 0\ (2). \end{cases}$$

Claim. $\tilde{\phi} \in L^2(R)$, for all $\phi \in {}^0V(\sigma_n)$.

We are done modulo this claim because then $\{\tilde{\phi} | \phi \in {}^0V(\sigma_n)\} \subset Z_m^{\perp}$ and $\phi \to \tilde{\phi}$ is injective.

To prove the claim, observe that since ϕ is K-finite, there exists $f \in C_c^\infty(G)$ such that $\phi = \phi * f^\vee$. Hence

$$\tilde{\phi}(a) = (\phi * f^\vee)\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \int_G \phi\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} g\right) f(g)\, dg$$

$$= \int_{N\setminus G} \phi\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} g\right) dg$$

$$\cdot \int_N \psi\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} n \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}\right) f(ng)\, dn.$$

There exists a compact set Ω in $N \setminus G \simeq AK$ such that the inner integral vanishes unless $g \in \Omega$. So

$$\tilde{\phi}(a) = \int_\Omega \phi\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} g\right) \hat{f}_g(a^2)\, dg,$$

where

$$\hat{f}_g(y) = \int_{\mathbf{R}} f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(xy)\, dx.$$

There exists a positive function h in $S(\mathbf{R})$ such that $|\hat{f}_g| \le h$, for $g \in \Omega$. This implies

$$|\phi(a)| \le h(a^2) \int_\Omega \left| \phi\left(\begin{pmatrix} ab & 0 \\ 0 & a^{-1}b^{-1} \end{pmatrix} k\right) \right| \cdot |b|^{-2}\, d^*b\, dk$$

$$\prec h(a^2) \int_\omega \left| \phi\begin{pmatrix} ab & 0 \\ 0 & a^{-1}b^{-1} \end{pmatrix} \right| |b|^{-2}\, d^*b,$$

where ω is a compact set in A

$$= h(a^2) \cdot |a|^2 \int_\omega \left| \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \right| \cdot |b|^{-2}\, d^*b \prec h(a) \cdot |a|^2.$$

This proves the claim.

Incidentally we have also shown the regularity mod N of functions in ${}^0V(\sigma_n)$.

$F = \mathbf{C}$

In this case the unitary irreducibles of $K^1 = \mathrm{SU}(2)$ are given by:

$$\sigma_m \colon K^1 \to \operatorname{Aut} H_m, \qquad m \in \mathbf{Z}_+,$$

where $H_m = \{$homogeneous polynomials of degree m in $\mathbf{C}[u, v]\}$, with u, v: indeterminates over $\mathbf{C}$.

The characters ω_n of

$$B_K^1 = \left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\right\}$$

are given by:

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \mapsto e^{in\theta}, \quad \text{for } n \in \mathbf{Z}.$$

It can be seen that $H_m(\omega_n) \neq 0$ iff $|n| \leq m$ and $m \equiv |n| \pmod 2$. Explicitly, $H_m(\omega_n)$ is the $\mathbf{C}$-span of $u^{m_1} \cdot v^{m_2}$, where

$$m_1 = \frac{m + |n|}{2} \quad \text{and} \quad m_2 = \frac{m - |n|}{2}.$$

We will be done if we show:

LEMMA 1.32.
(a) $\dim_{\mathbf{C}} {}^0V_K(\sigma_m, \omega_n) \leq m + 1$, *for all* $m \in \mathbf{Z}_+$, $n \in \mathbf{Z}$.
(b) *The functions in* ${}^0V_K(\sigma_m)$ *are regular on* G^1 mod N.

The proof of this lemma is very similar to that of Lemma 1.31 (the real case), and will be omitted.

It remains to prove part (c) of Theorem 1.27.

DEFINITION 1.33. $V_{\text{disc}} = \{K$-finite vectors in the discrete spectrum of $L^2(N \backslash G^1, \eta)\}$.

Since we have shown that ${}^0V_K$ is an admissible subspace of $L^2(N \backslash G^1, \eta)$, we have:

$$(1.34) \qquad {}^0V_K \subset V_{\text{disc}}.$$

In what follows we will show the reverse inclusion.

Let (π, H_π) be an irreducible unitary representation of G^1. Let $H_{\pi,K}$ denote the space of K-finite vectors in H_π. Then we get an admissible representation of G^1 (resp. $(\operatorname{Lie} G^1, K^1)$) on $H_{\pi,K}$ when F is p-adic (resp. archimedean).

DEFINITION 1.35. An irreducible, square-integrable G^1-module (π, H_π) has a *Whittaker model relative to* η iff $\operatorname{Hom}_X(H_{\pi,K}, V_{\text{disc}}) \neq \{0\}$, where

$$X = \begin{cases} G^1, & \text{if } F \text{ is } p\text{-adic} \\ (\operatorname{Lie} G^1, K^1), & \text{if } F \text{ is archimedean.} \end{cases}$$

For a definition of intertwining operators between (Lie G^1, K^1)-modules, see Wallach [[**15**]).

DEFINITION 1.36.

$$\Sigma(\eta) = \left\{ \begin{array}{r} \text{Irreducible, inequivalent, square-integrable } G^1\text{-modules} \\ \text{which have Whittaker models relative to } \eta \end{array} \right\}.$$

REMARK 1.37. It is known that the right regular representation $L^2(G)$ is a multiple of (the multiplicity-free representation) $L^2(N \setminus G, \eta)$. This is a theorem of B. Blackadar, a proof of which can be found in [**6**]. From this theorem it can be deduced that V_{disc} embeds in the discrete spectrum of $L^2(G^1)$. Thus we have the following decomposition:

$$V_{\text{disc}} \simeq \bigoplus_{\pi \in \Sigma(\eta)} \pi.$$

Let (π, H_π) be in $\Sigma(\eta)$. Then, since V_{disc} is multiplicity-free, $H_{\pi,K}$ embeds in V_{disc} in a *unique* manner.

DEFINITION 1.38. Let (π, H_π) be in $\Sigma(\eta)$.

(a) $\mathfrak{W}(\pi) =$ the unique component of V_{disc} isomorphic to $H_{\pi,K}$.

(b) A *Whittaker vector* (relative to η) associated to (π, H_π) is an element of $\mathfrak{W}(\pi)$.

Since we have the decomposition:

$$V_{\text{disc}} = \bigoplus_{(\pi, H_\pi) \in \Sigma(\eta)} \mathfrak{W}(\pi), \tag{1.39}$$

we need only to show that $\mathfrak{W}(\pi)$ is in ${}^0V_K$ for all (π, H_π) in $\Sigma(\eta)$. This will be achieved by the following:

PROPOSITION 1.40. *Let (π, H_π) be in $\Sigma(\eta)$. Then every Whittaker vector W (relative to η) associated to π is regular on $G^1 \bmod N$, and satisfies*: $W^N \equiv 0$.

Proof.

F: *p-adic*

Every discrete series representation of G^1 is either supercuspidal or special (cf. [4], [7] for definition). The proposition is a standard fact (cf. [7]) for any supercuspidal π in $\Sigma(\eta)$. Assume that π is special. Then we know from Godement ([4]) that every Whittaker vector W associated to π satisfies:

$$W\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{cases} 0, & \text{if } |a| \gg 1 \\ |a|^2, & \text{if } |a| \ll 1. \end{cases}$$

Hence W is regular mod N.

Suppose $W^N \neq 0$. Then the $\rho(G^1)$-span X of W^N is admissible, and every f in X is a function on G^1 satisfying:

$$f(ng) = f(g), \quad \text{for all } n \text{ in } N \text{ and } g \text{ in } G^1.$$

For each f in X, let $f|_{A^1}$ denote the restriction of f to A^1. In other words, $f|_{A^1}$ is the function on A^1 satisfying:

$$f|_{A^1}(a) = f(a), \quad \text{for every } a \text{ in } A^1.$$

Let $C^\infty(A^1)$ denote the space of locally constant functions on A^1. Note that since each f in X is K^1-finite, $f|_{A^1}$ is locally constant on A^1. So we get a map:

$$r\colon X \to C^\infty(A^1) \quad \text{with } r(f) = f|_{A^1}.$$

Let X_N denote the image of X under r, and let ρ_N denote the natural action of A^1 on X_N, given by:

$$\rho_N(a) f|_{A^1}(b) = f(ba), \quad \text{for all } a, b \text{ in } A^1.$$

It is clear that (ρ_N, X_N) is a smooth A^1-module.

Claim. X_N is an admissible A^1-module.

Indeed, we may choose (cf. Deligne [**2**], for example) a neighborhood basis $(K^1_m)_{m\geq 1}$ of 1 in G such that

(i) each K^1_m has an Iwahori factorization: $K^1_m = N_m \cdot A^1_m \cdot \overline{N}_m$, where $N_m = N \cap K^1_m$, $A^1_m = A^1 \cap K_m$, and $\overline{N}_m = \overline{N} \cap K^1_m$; and

(ii) $(A^1_m)_{m\geq 0}$ is a neighborhood basis of 1 in A^1 (consisting of compact subgroups). The claim will be proved if we show that $X_N^{A^1_m}$ is finite-dimensional, for every $m \geq 0$.

But for every m, the restriction map $r\colon X \to X_N$ gives rise to maps:

$$r\colon X^{K^1_m} \to X_N^{A^1_m}, \quad \text{and} \quad r\colon X^{A^1_m\overline{N_m}} \to X_N^{A^1_m}.$$

Suppose f in X is fixed under $A^1_m\overline{N_m}$. Then the function f_0 defined by:

$$g \mapsto (\operatorname{Vol} N_m)^{-1} \int_{N_m} (\rho(n)f)(g)\, Dn$$

belongs to $X^{K^1_m}$. Furthermore,

$$r(f_0)(a) = (\operatorname{Vol} N_m)^{-1} \int_{N_m} (\rho(n)f)(a)\, dn = f(a), \quad \text{for all } a \text{ in } A^1.$$

Thus the image of $X^{K^1_m}$ in X_N is identical to that of $X^{A^1_m\overline{N_m}}$.

Now let Z be any finite-dimensional subspace of $X_N^{A^1_m}$, and let Z' be any finite dimensional subspace of X mapping onto Z. Then we can find an open compact subgroup $\overline{O}$ in $\overline{N_m}$ such that $Z' \subset X^{A^1_m\overline{O}}$. Choose a_0 in

A^1 such that $a_0^{-1}\overline{N}_m a_0 \subseteq \overline{O}$. Then if f is in Z' and n is in $\overline{N}_m$, we have:

$$\rho(n)\rho(a_0)f = \rho(a_0)\rho(a_0^{-1}na_0)f = \rho(a_0)f.$$

Thus $\rho(a_0)Z' \subseteq X^{A_m\overline{N}_m}$. And $r(\rho(a_0)Z' = \rho_N(a_0)Z$ is contained in the image of $X^{K_m^1}$ (by the earlier remark). So $\dim Z = \dim \rho_N(a_0)Z \leq \dim X^{K_m^1}$. This shows that every finite-dimensional subspace of $X_N^{A_m^1}$ has dimension bounded by $\dim X^{K_m^1}$. Then $X_N^{A_m^1}$ itself is finite-dimensional. This proves the claim.

Now the claim implies in particular that $W^N|_{A^1}$ is A^1-finite. So there exist a finite number of characters μ_i of A^1 such that

$$W^N\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \sum_{i=1}^{N} \alpha_i \mu_i(a), \qquad N < \infty,\ \alpha_i \in \mathbf{C}.$$

But W^N: regular implies:

$$W^N\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = 0, \quad \text{if } |a| \ll 1.$$

This forces each α_i to be zero, and $W^N \equiv 0$. □

F: archimedean

When $F = \mathbf{C}$, there is nothing to prove due to the absence of discrete series. So assume $F = \mathbf{R}$. The regularity of Whittaker vectors W of square-integrable representations follow from **[4]**, **[5]**. It remains to show that $W^N \equiv 0$.

Let X denote the $(\mathrm{Lie}\, G^1, K^1)$-span of W^N. Then (ρ, X) is an admissible $(\mathrm{Lie}\, G^1, K^1)$-module. As in the p-adic case, consider the restriction map $r\colon f \to f|_{A^1}$ from X into the space of smooth functions on A^1. Let X_N denote $\mathrm{Im}(r)$. Then it is stable under the action of $(\mathrm{Lie}\, A^1, A_K^1)$. Note that

$$\rho(n)f(a) = f(an) = f(ana^{-1}a) = f(a),$$

for every a in A^1 and n in N. So $\rho(n)f - f$ is in $\mathrm{Ker}(r)$ for all f in X and for all n in N. On the other hand, if $n = \exp(T)$ with T in $\mathrm{Lie}(N)$, then

$$\rho(n)f = \rho(\exp T)f = \rho(1 + T)f = f + \rho(T)f.$$

Thus $\rho(n)f - f = \rho(T)f$.

Conversely, for any T in $\mathrm{Lie}\, N$, $\rho(T)f = r(T - 1)f - f$. Hence $\rho(\mathrm{Lie}\, N)X = \mathrm{Span}\{\rho(n)f - f\}$. By an earlier remark, we have $\rho(\mathrm{Lie}\, N)X \subset \mathrm{Ker}(r)$. So $\dim X_N \leq \dim(X/\rho(\mathrm{Lie}\, N)X)$.

Let $\mathfrak{U}(\mathrm{Lie}\, N)$ be the univerdal enveloping algebra of (the complexification of) Lie N. Then (cf. [**1**]) the $(\mathrm{Lie}\, G^1, K^1)$-module X is finitely generated as an $\mathfrak{U}(\mathrm{Lie}\, N)$-module. Since

$$\mathfrak{U}(\mathrm{Lie}\, N) = \mathbf{C} + \rho(\mathrm{Lie}\, N)\,\mathfrak{U}(\mathrm{Lie}\, N),$$

we get $\dim(X/\rho(\mathrm{Lie}\, N)X) < \infty$. Thus X_N is *finite-dimensional*, and hence an admissible $(\mathrm{Lie}\, A^1, A^1_K)$-module.

We can then find a finite set $\{\mu_K | k = 1, \ldots, N\}$ of characters of Lie A^1, and scalars $\{b_k\}$ such that

$$W^N\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \sum_{1 \le k \le N} b_k \mu_k \begin{pmatrix} \log a & 0 \\ 0 & -\log a \end{pmatrix}.$$

As the μ_k's are unitary, there exist real numbers $\{t_k\}$ such that

$$\mu_k \begin{pmatrix} \log a & 0 \\ 0 & -\log a \end{pmatrix} = a^{it_k}.$$

On the other hand, we know, since W is regular, that

$$W^N\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \le \text{const.}\ |a|, \quad \text{for } |a| \ll 1.$$

This yields

$$\left|\sum_k a^{it_k}\right| < \text{const.}\ |a|, \quad \text{for } |a| \ll 1.$$

Then for every $j \in \{1, 2, \ldots, N\}$

$$\left| b_j + \sum_{k \ne j} b_k a^{i(t_k - t_j)} \right| < \text{const.}\ |a|.$$

As we let $|a| \to 0$, this forces b_j to be zero. This is true for every j. Hence

$$W^N\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = 0, \quad \text{for all } a.$$

So W^N is identically zero, and we are done.

REMARK 1.41. A^1 acts on the characters of N via its conjugation action on N. Let S denote the set of equivalence classes of characters of N under this action. Then this set S is finite. Let $\{\eta_1, \eta_2, \ldots, \eta_r\}$ be a set of representatives for the elements of S in $\hat{N}$. Then it is a standard fact that *every* discrete series representation π of G^1 admits a Whittaker model relative to some η_j, $1 \le j \le r$. [**4**]

We conclude this section with a few remarks on the case of $G = \mathrm{GL}(2, F)$.

DEFINITION 1.42.

(a) For every Φ in $S(F^2)$ and ϕ in $S(F^*)$, set:

$$f(g)=f(\Phi,\phi)(g)=f_\Phi(g)\cdot\phi(\det g),\quad \text{for all } g \text{ in } G$$

(b) $S(N\setminus G)=\{f(\Phi,\phi)|\Phi\in S(F^2),\ \phi\in S(F^*)\}$

(c) $S_0(N\setminus G)=\{f(\Phi,\phi)|\Phi\in S_0(F^2),\ \phi\in S(F^*)\}$.

It may be verified that:

(1.43) For every f in $S_0(N\setminus G)$ and g in G,

$$\theta_f(g)=\int_N f(wng)\bar{\eta}(n)\,dn$$

is convergent and defines a K-finite function in $L^2(N\setminus G,\eta)$.

Clearly we have:

(1.44) $\theta_{f(\Phi,\phi)}(g)=\theta_\Phi(g)\cdot(\det g),$

for all $\Phi\in S_0(F^2)$, $\phi\in S(F^*)$ and $g\in G$.

The following result can be easily established using Th. 1.27:

PROPOSITION 1.45. *Suppose h is a K-finite function in $L^2(N\setminus G,\eta)$ which is orthogonal to $\{\theta_f|f\in S_0(N\setminus G)\}$. Then for every t in F^*, the function*:

$$g\mapsto h\left[\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}g\right],\qquad g\in G^1,$$

is in the discrete part of $L^2(N\setminus G^1,\eta_t)$, where η_t denotes the character: $\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\mapsto\eta(tx)$ *of N.*

2. The main duality. For any quasi-character $\mu=(\mu_1,\mu_2)$ of A, and for h in $L^2(N\setminus G)$, the formally define the μ-Mellin transform of h by:

$$L(h,\mu;g)=\int_A h(ag)\delta(a)^{-1/2}\mu(a)\,d^*a\qquad(\forall g\in G).\tag{2.0}$$

This expression can be seen to be convergent when μ is a unitary character and h is in $S(N\setminus G)$ or in $(\theta(S_0(N\setminus G)))^N$.

The result we are after is the following:

THEOREM 2.1. *For all f in $S_0(N\setminus G)$, μ in $\hat{A}$ and g in G,*

$$L(f,\mu;g)=\tfrac{1}{2}\left|\varepsilon'(\mu_1\mu_2^{-1})\right|^{-2}\cdot L\left(\theta_f^N,\mu;g\right),$$

where (for any quasi-character λ of F^*)

$$\varepsilon'(\lambda)=\varepsilon(\lambda,\psi,dx)\cdot\frac{L(\alpha\cdot\lambda^{-1})}{L(\lambda)}\quad \textit{with } \alpha(x)=|x|.$$

Here $\varepsilon(\lambda, \psi, dx)$ *and* $L(\lambda)$ *are respectively the* ε- *and* L-*factors attached to* λ (*and* ψ *and* dx) *by J. Tate* (*cf.* **[13]**, **[14]**.

COROLLARY 2.2 (*Preliminary scalar product formula*).

$$\langle \theta_f, \theta_h \rangle = \frac{1}{2} \int_K dk \int_{\hat{A}} L(\theta_f^N, \mu; k) \overline{L(\theta_h^N, \mu; k)} \cdot \left| \varepsilon'(\psi_1 \mu_2^{-1}) \right|^{-2} d\mu,$$

for all f, h *in* $S_0(N \setminus G)$.

To see that the theorem implies the corollary, note that if f_1, f_2 are in $L^2(N \setminus G)$, then by using of the Plancherel formula for A, we have:

$$\langle f_1, f_2 \rangle = \int_{K \times \hat{A}} L(f_1, \mu; k) \overline{L(f_2, \mu; k)} \, d\mu \, dk,$$

where $d\mu$ is the dual Haar measure on $\hat{A}$. Hence

$$\langle \theta_f, \theta_h \rangle = \langle f, \theta_h^N \rangle = \int_K dk \int_{\hat{A}} L(f, \mu; k) L(\theta_h^N, \mu; k) \, d\mu.$$

Now we will begin with the proof of the theorem.

Let $\mathbf{B}$ denote the skew-symmetric bilinear form on F^2 given by $\mathbf{B}(u, v) = ad - bc$, if $u = (a, b)$ and $v = (c, d)$. For Φ in $S(F^2)$, set:

$$\hat{\Phi}(v) = \int_{F^2} \Phi(u) \psi(\mathbf{B}(u, v)) \, du. \tag{2.3}$$

The map $\Phi \mapsto \hat{\Phi}$ can be seen to be an automorphism of $S(F^2)$ commuting with the right action of G^1. It also induces, for each σ in $\widehat{K^1}$, an automorphism of $S(F^2, \sigma)$ onto itself. Furthermore, if Φ is in $S(F^2)$, it can be shown for every L in $\mathscr{L}$ with Haar measure dz that $\int_L \Phi(z) \, dz = 0$ iff $\int_L \hat{\Phi}(z) \, dz = 0$. Hence this symplectic Fourier transform maps $S_0(F^2)$ (resp. $S_0(F^2, \sigma)$) onto itself.

DEFINITION 2.4. For Φ in $S_0(F^2)$ and y in F^*, set:

$$\Phi^*(y) = \int_F \Phi(y, x) \, dx.$$

Let F be p-adic. Then Φ^* clearly vanishes if $|y| \gg 1$. Also, since Φ is locally constant and since $\int_F \Phi(0, x) \, dx = 0$, we have:

$$\Phi^*(y) = 0 \quad \text{if } |y| \ll 1.$$

Now assume that F is archimedean. Then $\Phi^*(y)$ vanishes rapidly as $|y| \to \infty$. Moreover, proceeding along the lines of proof of Proposition 1.11, we can show that:

$$\Phi^*(y) \prec |y|_F^{c_F}, \quad \text{as } |y| \to 0,$$

where

$$c_{\mathbf{R}} = 1 \quad \text{and} \quad c_{\mathbf{C}} = \tfrac{1}{2}.$$

DEFINITION 2.5. For Φ in $S_0(F^2)$, set:

$$h_\Phi(y) = \begin{cases} |y|^{-1}\Phi^*(y), & \text{if } y \neq 0 \\ 0, & \text{if } y = 0. \end{cases}$$

From the above remarks about the asymptotics of Φ, it can be seen that h_ϕ is actually an integrable function on F.

For any open subset U of F, let $S(U)$ denote the space of functions on U which are restrictions to U of functions in $S(F)$. Furthermore, let U_ε denote, for every $\varepsilon > 0$, the open subset of F given by: $\{y \in F \,|\, |y| > \varepsilon\}$.

DEFINITION 2.6.

$$S_1(F) = \left\{ h \in L^1(F) \middle| h|_{U^\varepsilon} \in S(U_\varepsilon), \forall \varepsilon > 0 \right\}.$$

Then for every Φ in $S_0(F^2)$, h_Φ is in $S_1(F)$. Note that when F is p-adic, h_Φ is actually in $S(F)$.

For every h in $L^1(F)$, let $\hat{h}$ denote the usual Fourier transform relative to (ψ, dx).

PROPOSITION 2.7. *Let $f = f(\Phi, \phi)$ be in $S_0(N \setminus G)$ (see Definition 1.4.2) with $\Phi \in S_0(F^2)$ and $\phi \in S(F^*)$. Then*

$$\theta_f^N\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \phi(ab) \cdot |b|^{-1}\left(h_{\hat{\Phi}}(a^{-1}) + (h_{\Phi^\wedge})^\wedge(b)\right).$$

Given any $f = f(\Phi, \phi)$ in $S_0(N \setminus G)$, let $f^\sim$ denote the function $f(\hat{\Phi}, \Phi^\vee)$, where $\phi^\vee(t) = |t| \cdot \phi(t^{-1})$ for t in F^*.

COROLLARY 2.8. *For every f in $S_0(N \setminus G)$ and g in G, we have:*

$$\theta_f^N(g) = \theta_{f^\sim}^N(g \det \cdot g^{-1}).$$

The corollary easily follows from the proposition when g is a diagonal matrix. We may write any element of G as nhk with n in N, h in A and k in K^1. So it suffices to show that the truth of the corollary for g implies it for gk as well for k in K^1. We may take Φ to be in $S_0(F^2, \sigma)$ for some σ in $\widehat{K^1}$. Then, noting that $\theta_f^N(gk)$ equals $\sigma(k^{-1})\theta_f^N(g)$, we obtain the desired result.

We also have the following

COROLLARY 2.9. *$f(g) = f^{\sim}(g \det g^{-1})$, for all f in $S_0(N \setminus G)$ and g in G.*

Indeed we have

$$\left\langle f, \theta^N_{f^{\sim}} \right\rangle = \left\langle \theta_f, \theta_{f^{\sim}} \right\rangle = \left\langle \theta^N_f, f^{\sim} \right\rangle.$$

This, together with Corollary 2.8, yields the formula.

Proof of Proposition 2.7. We have to show that

$$\theta^N_f \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

equals:

$$\phi(ab) \cdot \left[|b|^{-1} \int_F \psi(a^{-1}y) |y|^{-1} \, dy \int_F \Phi(y, x) \, dx \right.$$

$$\left. + |b|^{-1} \int_F \psi(by) |y|^{-1} \, dy \int_F \hat{\Phi}(y, x) \, dx \right].$$

From the definition of θ_f, we get

$$\theta^N_f \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \phi(ab) \int_F dx \int_F \Phi(ay, b + bxy) \psi(y) \, dy.$$

Since Φ is in $S(F^2)$ we can find a function Φ_1 which is integrable on F^2 such that

$$\Phi(u, v) = \Phi(0, v) + u\Phi_1(u, v) \quad \text{for all } (u, v) \text{ in } F^2.$$

Thus

$$\theta^N_f \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \phi(ab) \cdot (I + J),$$

where

$$I = \int dx \int \Phi(0, b + bxy) \psi(y) \, dy,$$

and

$$J = \int dx \int ay\Phi_1(ay, b + bxy) \psi(y) \, dy.$$

We write $\int$ for $\int_F$. Then

$$I = |b|^{-1} \int \frac{dx}{|x|} \int \Phi(0, b+y)\psi(b^{-1}x^{-1}y)\,dy, \quad \text{as } y \to b^{-1}x^{-1}y$$

$$= |b|^{-1} \int \frac{dx}{|x|} \int \Phi(0, b+y)\psi(xy)\,dy, \quad \text{as } x \to b^{-1}x^{-1}$$

$$= |b|^{-1} \int \psi(-bx)\frac{dx}{|x|} \int \phi(0, y)\psi(x, y)\,dy, \quad \text{as } y \to y - b$$

$$= |b|^{-1} \int \psi(bx)\frac{dx}{|x|} \int \hat{\Phi}(x, y)\,dy$$

$$= |b|^{-1} \int \psi(by)\frac{dy}{|y|} \int \hat{\Phi}(y, x)\,dx,$$

by simply interchanging the labels x and y.

Next we claim that J is absolutely convergent. For this it is enough to show that

$$J^* \stackrel{\text{def}}{=} \int ay\psi(y)\,dy \int \Phi_1(ay, b + bxy)\,dx$$

is absolutely convergent. This is indeed so because

$$J^* = |b|^{-1} \int ay\psi(y)\frac{dy}{|y|} \int \Phi_1(ay, b+x)\,dx, \quad \text{as } x \to b^{-1}y^{-1}x$$

$$= |b|^{-1} \int \psi(a^{-1}y)(y/|y|)\,dy \int \Phi_1(y, x)\,dx,$$

which is absolutely convergent. So by Fubini's theorem $J = J^*$, and we obtain:

$$J = |b|^{-1} \int \psi(a^{-1}y)\frac{dy}{|y|} \int y\Phi_1(y, x)\,dx$$

$$= |b|^{-1} \int \psi(a^{-1}y)\frac{dy}{|y|} \int [\Phi(y, x) - \Phi(0, x)]\,dx$$

$$= |b|^{-1} \int \psi(a^{-1}y) \int \Phi(y, x)\,dx, \quad \text{since } \int \Phi(0, x)\,dx = 0. \qquad \square$$

For every character μ of A, let μ^w denote the character

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \mu\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}.$$

PROPOSITION 2.10. *For every f in $S_0(N \setminus G)$ and $\mu = (\mu_1, \mu_2)$ in $\hat{A}$,*

$$L\left(\theta_f^N, \mu; g\right) = \left|\varepsilon'\left(\mu_1\mu_2^{-1}\right)\right|^2\left[L(f, \mu; g) + L(f^{\sim}, \mu^w; g \cdot \det g^{-1})\right].$$

Theorem 2.1 can be easily seen to follow from this proposition and Corollary 2.9.

Replacing f by $\rho(g)f$, we see that we will be done once we show that the proposition holds when $g = e$.

For h in $S_1(F)$ and ν a quasi-character of F^*, set:

$$Z(h, \nu) = \int_{F^*} h(x)\nu(x)\, d^*x. \tag{2.11}$$

This is a local zeta integral as defined by J. Tate ([**13**]). We have the following functional equation:

$$Z(\hat{h}, \nu^{-1} \cdot \alpha) = \varepsilon'(\nu) Z(h, \nu), \tag{2.12}$$

where $\alpha(x) = |x|$ for x in F^*.

A simple calculation, using Proposition 2.7, yields:

$$\begin{aligned} L\left(\theta_f^N, \mu; e\right) = {} & Z\left(\phi, \mu_2 \cdot \alpha^{-1/2}\right) Z\left(\hat{h}_\Phi, \mu_2\mu_1^{-1}\right) \\ & + Z\left(\phi, \mu_1 \cdot \alpha^{-1/2}\right) Z\left(\hat{g}_{\hat{\Phi}}, \mu_2\mu_1^{-1}\right). \end{aligned} \tag{2.13}$$

By using the functional equation (2.12) and the definition of h_Φ we see that

$$Z\left(\hat{h}_\Phi, \mu_2\mu_1^{-1}\right) = \varepsilon'\left(\mu_2^{-1}\mu_1\alpha\right) \int_{F^* \times F} \Phi(a, x)\mu_2^{-1}\mu^1(a)\, d^*a\, dx,$$

which equals

$$\left[\frac{\varepsilon'\left(\mu_1\mu_2^{-1}\alpha\right)}{\varepsilon'\left(\mu_1\mu_2^{-1}\right)}\right] \int_{F^*} \hat{\Phi}(0, b)\mu_1\mu_2^{-1}(b)|b|\, d^*b.$$

From this it can be deduced that

$$L(f, \mu; e) = \left[\frac{\varepsilon'\left(\mu\mu_2^{-1}\alpha\right)}{\varepsilon'\left(\mu_1\mu_2^{-1}\right)}\right] \cdot Z\left(\phi, \mu_1\alpha^{-1/2}\right) \cdot Z\left(\hat{h}_\Phi, \mu_2\mu_1^{-1}\right). \tag{2.14}$$

Similarly, we get

$$L(f^{\sim}, \mu^w; e) = \left[\frac{\varepsilon'\left(\mu_1\mu_2^{-1}\alpha\right)}{\varepsilon'\left(\mu_1\mu_2^{-1}\right)}\right] \cdot Z\left(\phi, \mu_2\alpha^{-1/2}\right) \cdot Z\left(\hat{h}_{\hat{\Phi}}, \mu_2\mu_1^{-1}\right). \tag{2.15}$$

The proposition, and hence the theorem, follow from (2.13), (2.14), and (2.15), once we note (cf. [13]) that when ν is a character (of module one) of F^*,

$$[\varepsilon'(\nu)/\varepsilon'(\nu\alpha)] = |\varepsilon'(\nu)|^2. \qquad \square$$

3. The scalar product formula. For any Φ in $S(F^2)$, not necessarily in $S_0(F^2)$, and quasicharacter $\mu = (\mu_1, \mu_2)$ of A, set (as in [7]):

(3.0) $b_\Phi(\mu; g)$

$$= \mu_1(\det g)|\det g| \int_{F^*} \Phi[(0,t)g]\mu_1\mu_2^{-1}(t)|t|\, d^*t, \qquad \forall g \in G.$$

Then $b_\Phi(\mu)$ belongs to the space of the smoothly induced representation $\operatorname{Ind}(G, B; \mu)$, and every function in this space is of this form. The associated Whittaker vector is:

$$W_\Phi(\mu; g) = \int_N b_\Phi(\mu; wng)\overline{\eta(n)}\, dn \qquad (g \in G). \tag{3.1}$$

For any ξ in $L^2(N \setminus G, \eta)$, formally set:

$$\Psi(\xi, W_\Phi(\mu)) = \int_{N\setminus G} \xi(g) W_\Phi(\mu; \delta g)\, dg, \tag{3.2}$$

where $\delta = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

It may be seen that, when $\xi = \theta_f$ for some f in $S(N \setminus G)$ and when $W_\Phi(\mu)$ belongs to a unitary principal series representation, this expression makes sense. Furthermore it equals, in the sense of analytic continuation:

$$\mu(w\delta w^{-1}) \int_K b_\Phi(\mu; k) L(\xi^N, \mu; k)\, dk. \tag{3.3}$$

DEFINITION 3.4. For every σ in $\hat{K}$, let I_σ denote the set of pairs (i, j) of integers such that $1 \le i, j \le \dim\sigma$, and write $\Lambda = \{(\sigma, I_\sigma) | \sigma \in \hat{K}\}$.

THEOREM 3.5. *There exists a family* $\{\Phi_l \in S(F^2) | l \in \Lambda\}$ *such that, for all* f, h *in* $S_0(N \setminus G)$,

$$\langle \theta_f, \theta_h \rangle = \frac{1}{2} \sum_{l \in \Lambda} \int_{\hat{A}} \Psi(\theta_f, W_{\Phi_l}(\mu)) \cdot \overline{\Psi(\theta_h, W_{\Phi_l}(\mu))} \cdot |\varepsilon'(\mu_1\mu_2^{-1})|^{-2}\, d\mu.$$

Proof. Let (σ, H_σ) be a unitary irreducible of K^1, and let μ be a character of A. Then, for β in $L^2(N \setminus G)$, we can formally define an operator:

$$L(\beta,\mu;\sigma)=\iint_{A\,K}\sigma(k)\beta(ak)\mu(a)\delta(a)^{-1/2}\,d^*a\,dk. \tag{3.6}$$

This expression converges whenever β is in $S(N\setminus G)$ or in $(\theta(S_0(N\setminus G)))^N$.

Let $L(\beta,\mu;\sigma)^*$ be the ajoint of $L(\beta,\mu;\sigma)$. Then, for all f, h in $S_0(N\setminus G)$, the preliminary scalar product formula (Corollary 2.2) yields:

$$\langle\theta_f,\theta_h\rangle=\frac{1}{2}\sum_{\hat{K}}\int_{\hat{A}}\operatorname{sp}\left[L(\theta_h^N,\mu;\sigma)^*L(\theta_f^N,\mu;\sigma)\right]\left|\varepsilon'(\mu_1^{-1}\mu_2)\right|^{-2}d\mu, \tag{3.7}$$

where sp denotes $\operatorname{tr}/\deg\sigma$.

For each character μ of A, let μ_0 denote the restriction to A_K. Recall that the eigenspace $H_\sigma(\mu_0)$ is at most one-dimensional (cf. [**3**]). Let $P(\sigma,\mu_0)$ be the corresponding projector. Then

$$P(\sigma,\mu_0)L(\beta,\mu;\sigma)=L(\beta,\mu;\sigma). \tag{3.8}$$

When F is p-adic, let χ denote the characteristic function of the unit group U_F divided by the volume of U_F. For t in F^*, set:

$$\chi_\mu(t)=\begin{cases}\mu(t)\cdot\chi(t), & \text{if } F\text{: } p\text{-adic}\\ \mu(t)\cdot\exp(-\pi t^2), & \text{if } F=\mathbf{R}\\ \mu(t)\cdot\exp(-2\pi t\bar{t}), & \text{if } F=\mathbf{C}.\end{cases} \tag{3.9}$$

By construction $\int_{F^*}\chi_\mu(t)\mu^{-1}(t)|t|\,d^*t=1$. Set

$$\Phi_\sigma[(0,t)k]=\chi_\mu(t)\sigma(k^{-1})P(\sigma,\mu_0). \tag{3.10}$$

This defines an operator-valued function Φ_σ in $S(F^2)$. Furthermore,

$$b_{\Phi_\sigma}(\bar{\mu};\delta)=\int_{F^*}\Phi_\sigma(0,t)\mu^{-1}(t)\,d^*t=P(\sigma,\mu_0).$$

Thus we get, using (3.8)

$$L(\beta,\mu;\sigma)=\int_{N\setminus G}\beta(g)b_{\Phi_\sigma}(\bar{\mu}^w;g)\,dg. \tag{3.11}$$

From (3.7) and (3.11) we get:

$$\langle\phi_f,\theta_h\rangle=\frac{1}{2}\sum_{\sigma\in\hat{K}'}\int_{\hat{A}}d\mu\iint_{(N\setminus G)^2}\left(\theta_f^N(g)\overline{\theta_h^N(g')}\cdot\left|\varepsilon'(\mu_1^{-1}\mu_2)\right|^{-2}\right) \tag{3.12}$$
$$\cdot\operatorname{sp}\left[b_{\Phi_\sigma}(\bar{\mu}^w;g)^*\cdot b_{\Phi_\sigma}(\bar{\mu}^w;g')\right]dg\,dg'.$$

Finally, choosing a basis $\{e_j\}$ of each H_σ, set

$$\Phi_{i,j} = (\dim H_\sigma)^{-1} \cdot \langle \Phi e_i, \Phi e_j \rangle. \tag{3.13}$$

Then

$$\operatorname{sp}\left[b_{\Phi_\sigma}(\mu^w; g)^* b_{\Phi_\sigma}(\mu^w; g)^* b_{\Phi_\sigma}(\mu^w; g')\right] = \sum_{1 \le i,\, j \le \dim H_\sigma} b_{\Phi_{ij}}(\mu^w s; g') b_{\Phi_{i,j}}(\mu^w; g). \tag{3.14}$$

Now the theorem follows from (3.2), (3.7), (3.12) and (3.14). □

Note added in proof. The main results of this paper admit an extension to GL_n by an analogous but not so explicit a method ([**12**]). By employing different (and much more powerful) principles involving the weak inequality and Maass-Selberg relations, Professor Harish Chandra obtained a complete spectral decomposition in 1982 valid for all real reductive groups. A sketch of proof is in his unpublished paper, "On the theory of the Whittaker integral."

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