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AN EVALUATION OF THE CONDITIONAL YEH-WIENER INTEGRAL

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Yeh obtained the conditional Wiener integral of

$$\exp\{-\textstyle\int_0^t V[x(u)]\,du\}$$

given $x(t)$ where x is in Wiener space $C[0,t]$ and V is a function on $\mathbf{R}^1$ satisfying certain conditions. In this paper we extend Yeh's result to the conditional Yeh-Wiener integral of $\exp\{-\int_0^t\int_0^s V[x(u,v)]\,du\,dv\}$ given $x(s,t)$ where x is in Yeh-Wiener space $C_2(Q)$ and V is a nonnegative continuous function on $\mathbf{R}^1$ satisfying the condition

$$\int_{\mathbf{R}^1} V(w)\cdot\exp\left\{-\frac{w^2}{2st}\right\}dm_L(w)<\infty.$$

1. Introduction. Yeh recently derived inversion formulae for conditional expectations [**5**] and for conditional Wiener integrals [**6**]. He also evaluated some conditional Wiener integrals using these inversion formulae. In [**2**] and [**3**], they introduced the conditional Yeh-Wiener integral and extended some of Yeh's results for the conditional Wiener integrals to the conditional Yeh-Wiener integrals.

Here the probability space is the Yeh-Wiener measure space on the Yeh-Wiener space $C_2(Q)$ of the real valued continuous functions x defined on $Q=[0,s]\times[0,t]$ for some fixed positive real numbers s and t such that $x(0,v)=x(u,0)=0$ for all $0\le u\le s$ and $0\le v\le t$. In this paper we shall always denote Q as a fixed above rectangle. Let $(C_2(Q),\mathscr{Y},m_y)$ be the Yeh-Wiener measure space. For a complete discussion of Yeh-Wiener measure space, see [**7**].

A real valued functional F on $C_2(Q)$ is said to be Yeh-Wiener measurable if it is $\mathscr{Y}$-measurable. Its integral with respect to m_y if it exists, is called its Yeh-Wiener integral which is denoted by $E^y(F)$. In this case we write

$$E^y(F)=\int_{C_2(Q)}F(x)\,dm_y(x). \tag{1.1}$$

We say that F is Yeh-Wiener integrable or m_y-integrable when the Yeh-Wiener integral of F, $E^y(F)$, exists and is finite. The Yeh-Wiener measurability and Yeh-Wiener integrability of a complex valued functional on $C_2(Q)$ are defined in terms of its real and imaginary parts.

Let X and Y be the $\mathbf{R}^n$-valued and real valued Yeh-Wiener measurable functions on $C_2(Q)$, respectively, with $E^{y}(|Y|) < \infty$. Let P_X be the probability distribution determined by X. By the conditional Yeh-Wiener integral of Y given X we mean the conditional expectation $E^{y}(Y \mid X)$ which is given as a function on the value space of X. Throughout this paper we shall be exclusively concerned with X and Y given by $X(x) = x(s,t)$ and $Y(x) = \exp\{-\int_Q V[x(u,v)]\,du\,dv\}$ for $x \in C_2(Q)$ in which V is a nonnegative continuous function on $\mathbf{R}^1$ satisfying the condition

$$\int_{\mathbf{R}^1} V(w) \cdot \exp\left\{-\frac{w^2}{2st}\right\} dm_L(w) < \infty \tag{1.2}$$

where m_L is the Lebesgue measure on $\mathbf{R}^1$.

The techniques of this paper are closely related to those of paper [**6**] of Yeh, but for the evaluation of the conditional Yeh-Wiener integral we use slightly different techniques. In Theorem 2.1 we evaluate the conditional Yeh-Wiener integral of $Y(x)$ given $X(x)$ which is the extension of Yeh's result [**6**; Theorem 5]. The proof of Theorem 2.1 is simpler than that of Yeh. To do this we will use the following Proposition which comes from [**3**; Theorem 3.5].

PROPOSITION 1.1. *Let X and Y be measurable transformations of $(C_2(Q), \mathscr{Y})$ into $(\mathbf{R}^1, \mathscr{B}(\mathbf{R}^1))$ with $E^{y}(|Y|) < \infty$. Assume that P_X is absolutely continuous with respect to m_L on $(\mathbf{R}^1, \mathscr{B}(\mathbf{R}^1))$ and $E^{y}(e^{iuX}Y)$ is a m_L-integrable function of u on $\mathbf{R}^1$. Then there exists a version of* $E^{y}[Y \mid X](dP_X/dm_L)$ such that for $\xi \in \mathbf{R}^1$,

$$E^{y}[Y \mid X](\xi)\frac{dP_X}{dm_L}(\xi) = \frac{1}{2\pi}\int_{\mathbf{R}^1} e^{-iu\xi}E^{y}(e^{iuX}Y)\,dm_L(u). \tag{1.3}$$

2. The conditional Yeh-Wiener integral of

$$\exp\left\{-\int_0^t \int_0^s V[x(u,v)]\,du\,dv\right\}$$

given $x(s,t)$.

THEOREM 2.1. *For some fixed positive real numbers s and t, let*

$$X_{(s,t)}(x) = x(s,t) \quad \textit{and} \tag{2.1}$$

$$Y_{(s,t)}(x) = \exp\left\{-\int_0^t \int_0^s V[x(u,v)]\,du\,dv\right\}$$

for $x \in C_2(Q)$ where V is a nonnegative continuous function on $\mathbf{R}^1$ satisfying the condition

$$\int_{\mathbf{R}^1} V(w)\exp\left\{-\frac{w^2}{2st}\right\} dm_L(w) < \infty \tag{2.2}$$

for every s and t in $(0, \infty)$. *Then the conditional Yeh-Wiener integral of* $Y_{(s,t)}$ *given by* $X_{(s,t)}$ *is*

$$(2.3)\quad E^y\left[Y_{(s,t)} \mid X_{(s,t)}\right](\zeta)$$

$$= 1 - \int_Q \Bigg[\int_{\mathbf{R}^3} \sqrt{\frac{st}{(s-u)(t-v)}}$$

$$\times \left\{ \exp\left(- \frac{(\zeta - u_{21} - u_{12} + u_{11})^2}{2(s-u)(t-v)} + \frac{\zeta^2}{2st} \right) \right\}$$

$$\times \left\{ V(u_{11}) E^y\left[Y_{(u,v)} \mid X\right](u_{11}, u_{12}, u_{21}) \right.$$

$$\left. - E^y\left[Z_{(u,v)} Y_{(u,v)} \mid X\right](u_{11}, u_{12}, u_{21}) \right\} dP_X(u_{11}, u_{12}, u_{21}) \Bigg] dm_L(u, v)$$

where $x(s,t) = \zeta \in \mathbf{R}^1$,

$$(2.4)\qquad Z_{(u,v)}(x) = \int_0^v V[x(u,r)]\, dr \int_0^u V[x(q,v)]\, dq,$$

$$(2.5)\qquad X(x) = \left(X_{(u,v)}(x), X_{(u,t)}(x), X_{(s,v)}(x) \right)$$

for $x \in C_2(Q)$, *and*

$$(2.6)\quad dP_X(u_{11}, u_{12}, u_{21}) = \left\{ (2\pi)^3 u^2 v_1^2 (s-u)(t-v) \right\}^{-1/2}$$

$$\times \exp\left\{ - \frac{u_{11}^2}{2uv} - \frac{(u_{12} - u_{11})^2}{2u(t-v)} - \frac{(u_{21} - u_{11})^2}{2(s-u)v} \right\} dm_L(u_{11}, u_{12}, u_{21}).$$

REMARK. The existence of V on (2.2) follows if V satisfies the order of growth condition

$$V(w) = O\left(\exp\{w^{2-\delta}\}\right) \quad \text{as } w \to \pm\infty$$

for some $\delta \in (0, 2)$. Under (2.2), if we define

$$(2.7)\qquad \phi((s,t)) = \frac{1}{\sqrt{2\pi st}} \int_{\mathbf{R}^1} V(w) \exp\left\{ - \frac{w^2}{2st} \right\} dm_L(w)$$

for $(s,t) \in (0,\infty)^2$, then ϕ is a nonnegative continuous function on $(0,\infty)^2$ and furthermore

$$\lim_{(s,t) \to (\sigma,\tau)} \phi(s,t) = V(o) \quad \text{for } (\sigma,\tau) \in [0,\infty)^2 - (0,\infty)^2.$$

Let us define

$$\phi(\sigma, \tau) = \lim_{(s,t)\to(\sigma,\tau)} \phi(s,t) \tag{2.8}$$

for $(\sigma, \tau) \in [0, \infty)^2 - (0, \infty)^2$ so that ϕ is continuous on $[0, \infty)^2$.

LEMMA 2.1. *For $0 < u \leq s$ and $0 < v \leq t$,*

$$E^y\left[e^{iw(x(s,t)-x(s,v)-x(u,t)+x(u,v))}\right] = \exp\left\{-\frac{(s-u)(t-v)}{2}w^2\right\} \tag{2.9}$$

for $x \in C_2(Q)$ and $w \in \mathbf{R}^1$.

The lemma can be followed from the fact that the left-hand side of (2.9) is the characteristic function of the random variable $x(s,t) - x(s,v) - x(u,t) + x(u,v)$ whose probability distribution is the normal distribution with mean 0 and variance $(s-u)(t-v)$.

Proof of Theorem 2.1. We can easily obtain that $X_{(s,t)}$ and $Y_{(s,t)}$ are measurable transformations of $(C_2(Q), \mathscr{Y})$ into $(\mathbf{R}^1, \mathscr{B}(\mathbf{R}^1))$, with $E^y(|Y|) < \infty$. Now

$$\begin{aligned}\frac{\partial^2}{\partial u \partial v}&\left[\exp\left\{-\int_0^v \int_0^u V[x(q,r)]\,dq\,dr\right\}\right] \\ &= \exp\left\{-\int_0^v \int_0^u V[x(q,r)]\,dq\,dr\right\} \\ &\quad\times\left\{\int_0^v V[x(u,r)]\,dr \cdot \int_0^u V[x(q,v)]\,dq - V[x(u,v)]\right\},\end{aligned}$$

thus we have by (2.1)

$$\begin{aligned}Y_{(s,t)}(x) = 1 &+ \int_Q \left\{\exp\left(-\int_0^v \int_0^u V[x(q,r)]\,dq\,dr\right)\right\} \\ &\times\left\{\int_0^v V[x(u,r)]\,dr \int_0^u V[x(q,v)]\,dq - V[x(u,v)]\right\} dm_L(u,v).\end{aligned} \tag{2.11}$$

To show that $E^y[e^{iwX_{(s,t)}}Y_{(s,t)}]$ is a m_L-integrable function of w in $\mathbf{R}^1$, let

$$E^y\left[e^{iwX_{(s,t)}}Y_{(s,t)}\right] = E^y\left[e^{iwx(s,t)}\right] - J_1(s,t) + J_2(s,t) \tag{2.12}$$

where

$$\begin{aligned}J_1(s,t) &= E^y\left[e^{iwx(s,t)}\int_Q V[x(u,v)]\right. \\ &\qquad\left.\times\exp\left\{-\int_0^v \int_0^u V[x(q,r)]\,dq\,dr\right\} dm_L(u,v)\right]\end{aligned} \tag{2.13}$$

and

$$(2.14)\quad J_2(s,t) = E^y\left[e^{iwx(s,t)}\int_Q\left\{\int_0^v V[x(u,r)]\,dr\int_0^u V[x(q,v)]\,dq\right\}\right.$$
$$\left.\times\exp\left\{-\int_0^v\int_0^u V[x(q,r)]\,dq\,dr\right\}dm_L(u,v)\right].$$

Then we can have

$$(2.15)\quad E^y\left[e^{iwx(s,t)}\right] = \frac{1}{\sqrt{2\pi st}}\int_{\mathbf{R}^1} e^{iw\xi}e^{-\xi^2/2st}\,dm_L(\xi) = e^{-(st/2)w^2}$$

by the basic Yeh-Wiener integration formula and the formula

$$(2.16)\qquad \int_{\mathbf{R}^1}\exp\{-(a\xi^2+b\xi)\}\,dm_L(\xi) = \sqrt{\frac{\pi}{a}}\exp\left\{\frac{b^2}{4a}\right\}$$

for $a > 0$ and real or imaginary b. Thus

$$(2.17)\qquad \int_{\mathbf{R}^1}\left|E^y\left[e^{iwx(s,t)}\right]\right|dm_L(w) = \sqrt{\frac{2\pi}{st}} < \infty.$$

To interchange the order of the Yeh-Wiener integral and the integral with respect to $dm_L(u,v)$ on Q in (2.13), note that

$$\left|e^{iwx(s,t)}V[x(u,v)]\exp\left\{-\int_0^v\int_0^u V[x(q,r)]\,dq\,dr\right\}\right| \le V[x(u,v)]$$

for $((u,v),x) \in Q\times C_2(Q)$ and note that by (2.7) and the continuity of ϕ on $[0,\infty)^2$,

$$\int_Q E^y[V(x(u,v))]\,dm_L(u,v) = \int_Q \phi(u,v)\,dm_L(u,v) < \infty.$$

By the Fubini Theorem we have

$$(2.18)\quad J_1(s,t) = \int_Q E^y\left[e^{iwx(s,t)}V[x(u,v)]\right.$$
$$\left.\times\exp\left\{-\int_0^v\int_0^u V[x(q,r)]\,dq\,dr\right\}\right]dm_L(u,v).$$

Since

$$\{x(s,t)-x(s,v)-x(u,t)+x(u,v),\ \{x(s,v),x(u,t),x(q,r)\}\}$$

is an independent system of random variables on $(C_2(Q), \mathscr{Y}, m_y)$ for every $(q, r) \in [0, u] \times [0, v]$, we can have by Lemma 2.1,

$$J_1(s,t) = \int_Q \exp\left\{-\frac{(s-u)(t-v)}{2}w^2\right\} \times E^y\Big[e^{iw(x(s,v)+x(u,t)-x(u,v))}V[x(u,v)] \times \exp\left\{-\int_0^v\int_0^u V[x(q,r)]\,dq\,dr\right\}\Big]\,dm_L(u,v). \tag{2.19}$$

Let X be a three dimensional random vector on $(C_2(Q), \mathscr{Y}, m_y)$ given by

$$X(x) = (X_1(x), X_2(x), X_3(x)) \tag{2.20}$$

where $X_1 \equiv X_{(u,v)}$, $X_2 \equiv X_{(u,t)}$, and $X_3 \equiv X_{(s,v)}$. Let $Y_1 \equiv Y_{(u,v)}$. Then the regular conditional distribution of Y_1 given X, $P(Y_1 \mid X)$, exists since Y_1 is a real valued random variable. With fixed $w \in \mathbf{R}^1$ consider a complex valued function f on $\mathbf{R}^3 \times \mathbf{R}^1$ defined by

$$f((\xi_1, \xi_2, \xi_3), \eta) = e^{iw(\xi_3+\xi_2-\xi_1)}V(\xi_1)\eta \tag{2.21}$$

for $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3$ and $\eta \in \mathbf{R}^1$. By Proposition 2 and Proposition 1 in [5], we have

$$\begin{aligned} E^y\Big[&e^{iw(x(s,v)+x(u,t)-x(u,v))}V[x(u,v)] \times \exp\left\{-\int_0^v\int_0^u V[x(q,r)]\,dq\,dr\right\}\Big] \\ &= \int_{\mathbf{R}^3}\left\{\int_{\mathbf{R}^1} e^{iw(\xi_3+\xi_2-\xi_1)}V(\xi_1)\eta P(Y_1 \mid X)(d\eta, \xi)\right\} dP_X(\xi) \\ &= \int_{\mathbf{R}^3} e^{iw(\xi_3+\xi_2-\xi_1)}V(\xi_1)E^y(Y_1 \mid X)(\xi)\,dP_x(\xi) \\ &= \int_{\mathbf{R}^3} e^{iw(u_{21}+u_{12}-u_{11})}V(u_{11})E^y(Y_1 \mid X)(u_{11}, u_{12}, u_{21}) \\ &\qquad dP_X(u_{11}, u_{12}, u_{21}) \end{aligned} \tag{2.22}$$

where

$$dP_X(u_{11}, u_{12}, u_{21}) = \left\{(2\pi)^3u^2v^2(s-u)(t-v)\right\}^{-1/2} \times \exp\left\{-\frac{u_{11}^2}{2uv} - \frac{(u_{12}-u_{11})^2}{2u(t-v)} - \frac{(u_{21}-u_{11})^2}{2(s-u)v}\right\} dm_L(u_{11}, u_{12}, u_{21}). \tag{2.23}$$

By (2.19) and (2.22) we can obtain

$$(2.24)\quad J_1(s,t) = \int_Q \exp\left\{-\frac{(s-u)(t-v)}{2}w^2\right\} \times \left[\int_{\mathbf{R}^3} e^{iw(u_{21}+u_{12}-u_{11})}V(u_{11}) \times E^y(Y_1 \mid X)(u_{11},u_{12},u_{21})\,dP_X(u_{11},u_{12},u_{21})\right] dm_L(u,v).$$

To show that $J_1(s,t)$ is integrable, observe that

$$\left|V(u_{11})E^y(Y_1 \mid X)(u_{11},u_{12},u_{21})e^{iw(u_{21}+u_{12}-u_{11})}e^{-((s-u)(t-v)/2)w^2}\right| \le V(u_{11})E^y(Y_1 \mid X)(u_{11},u_{12},u_{21})e^{-((s-u)(t-v)/2)w^2}$$

for $((u_{11},u_{12},u_{21}),(u,v)) \in \mathbf{R}^3 \times Q$. Let π_1 be a function from $\mathbf{R}^3$ to $\mathbf{R}^1$ defined by $\pi_1(u_{11},u_{12},u_{21}) = u_{11}$. Then $X_1 = \pi_1 \circ X$. Thus by Proposition 3 in [5] and (2.7) we have

$$(2.25)\quad \int_{\mathbf{R}^3} V(u_{11})E^y(Y_1 \mid X)(u_{11},u_{12},u_{21})\,dP_X(u_{11},u_{12},u_{21}) = E^y[(V\circ X_1)Y_1] \le E^y[V(x(u,v))] = \phi(u,v).$$

Now

$$\int_Q \phi(u,v)e^{-((s-u)(t-v)/2)w^2}\,dm_L(u,v) = Ke^{-(st/2)w^2}$$

where

$$K \equiv \int_Q \phi(u,v)e^{((sv+ut-uv)/2)w^2}\,dm_L(u,v) < \infty$$

by the continuity of ϕ on $[0,\infty)^2$. Thus $J_1(s,t)$ is integrable since

$$(2.26)\quad \int_{\mathbf{R}^1} |J_1(s,t)|\,dm_L(w) \le K\int_{\mathbf{R}^1} e^{-(st/2)w^2}\,dm_L(w) < \infty.$$

To interchange the order of the integrals in (2.14) note that

$$\left|e^{iwx(s,t)}\left\{\int_0^v V[x(u,r)]\,dr\int_0^u V[x(q,v)]\,dq\right\} \times \exp\left\{-\int_0^v\int_0^u V[x(q,r)]\,dq\,dr\right\}\right| \le \int_0^v V[x(u,r)]\,dr\cdot\int_0^u V[x(q,v)]\,dq$$

for $x \in C_2(Q)$ and

$$\int_Q E^y\left[\int_0^v V[x(u,r)]\,dr\cdot\int_0^u V[x(q,v)]\,dq\right] dm_L(u,v) \le Mst < \infty$$

for some $M > 0$ since $V \circ x$ is a continuous function. Thus from (2.14) we obtain

$$(2.27)\quad J_2(s,t) = \int_Q \exp\left\{-\frac{(s-u)(t-v)}{2}w^2\right\} \times E^y\Bigg[e^{iw(x(s,v)+x(u,t)-x(u,v))} \times\left\{\int_0^v V[x(u,r)]\,dr\int_0^u V[x(q,v)]\,dq\right\} \times\exp\left\{-\int_0^v\int_0^u V[x(q,r)]\,dq\,dr\right\}\Bigg]\,dm_L(u,v)$$

by the Fubini Theorem and the same way as in (2.19). Let X be given as in (2.20) and let $Y_1 = Y_{(u,v)}$. Note that $E^y(|Y_1|) < \infty$. Let

$$Z_1(x) = Z_{(u,v)}(x) = \int_0^v V[x(u,r)]\,dr\int_0^u V[x(q,v)]\,dq$$

for $x \in C_2(Q)$. Then it is obvious that Z_1 is Yeh-Wiener measurable and Yeh-Wiener integrable on $C_2(Q)$. Put $F_1 \equiv Z_1Y_1$. Then F_1 is a real valued random variable on $(C_2(Q), \mathscr{Y}, m_y)$ with $E^y(|F_1|) < \infty$. For fixed $w \in \mathbf{R}^1$ consider a complex valued function g on $\mathbf{R}^3 \times \mathbf{R}^1$ defined by

$$g((\xi_1,\xi_2,\xi_3),\eta) = e^{iw(\xi_3+\xi_2-\xi_1)}\eta$$

for $\xi = (\xi_1,\xi_2,\xi_3) \in \mathbf{R}^3$ and $\eta \in \mathbf{R}^1$. Applying Proposition 2 and Proposition 1 in [**5**] to the real and imaginary parts of g we have by (2.1),

$$(2.28)\quad E^y\Bigg[e^{iw(x(s,v)+x(u,t)-x(u,v))}\left\{\int_0^v V[x(u,r)]\,dr\int_0^u V[x(q,v)]\,dq\right\} \times\exp\left\{-\int_0^v\int_0^u V[x(q,r)]\,dq\,dr\right\}\Bigg]$$

$$= \int_{\mathbf{R}^3}\left\{\int_{\mathbf{R}^1} e^{iw(\xi_3+\xi_2-\xi_1)}\eta P(F_1\,|\,X)(d\eta,\xi)\right\}dP_X(\xi)$$

$$= \int_{\mathbf{R}^3} e^{iw(u_{21}+u_{12}-u_{11})}E^y(F_1\,|\,X)(u_{11},u_{12},u_{21})\,dP_X(u_{11},u_{12},u_{21})$$

where $dP_X(u_{11},u_{12},u_{21})$ is given as in (2.23). By (2.27) and (2.28) we have

$$(2.29)\quad J_2(s,t) = \int_Q\left\{\exp\left(-\frac{(s-u)(t-v)}{2}w^2\right)\right\} \times\left\{\int_{\mathbf{R}^3} e^{iw(u_{21}+u_{12}-u_{11})}E^y(F_1\,|\,X)(u_{11},u_{12},u_{21})\,dP_X(u_{11},u_{12},u_{21})\right\} dm_L(u,v).$$

To show that $J_2(s, t)$ is integrable, observe that

$$\begin{aligned}&\left| E^y(F_1 \mid X)(u_{11}, u_{12}, u_{21}) e^{iw(u_{21}+u_{12}-u_{11})} e^{-((s-u)(t-v)/2)w^2} \right| \\ &\quad \leq E^y(F_1 \mid X)(u_{11}, u_{12}, u_{21}) e^{-((s-u)(t-v)/2)w^2}\end{aligned}$$

for $((u_{11}, u_{12}, u_{21}), (u, v)) \in \mathbf{R}^3 \times Q$. Since the conditional Yeh-Wiener integral is P_X-integrable, we have

$$N \equiv \int_{\mathbf{R}^3} E^y(F_1 \mid X)(u_{11}, u_{12}, u_{21})\, dP_X(u_{11}, u_{12}, u_{21}) < \infty.$$

Thus

$$\int_Q N e^{-((s-u)(t-v)/2)w^2}\, dm_L(u, v) = NLe^{-(st/2)w^2}$$

where

$$L \equiv \int_Q e^{(sv+ut-uv/2)w^2}\, dm_L(u, v) < \infty.$$

Hence $J_2(s, t)$ is integrable since

$$\text{(2.30)} \quad \int_{\mathbf{R}^1} |J_2(s, t)|\, dm_L(w) \leq NL \int_{\mathbf{R}^1} e^{-(st/2)w^2}\, dm_L(w) < \infty.$$

Therefore by (2.12), (2.17), (2.26) and (2.30), we have that $E^y[e^{iwX_{(s,t)}} Y_{(s,t)}]$ is a m_L-integrable function of w on $\mathbf{R}^1$ and thus, by Proposition 1.1, there exists a version of

$$E^y\left[Y_{(s,t)} \middle| X_{(s,t)}\right] \frac{dP_{X_{(s,t)}}}{dm_L}$$

such that

$$\begin{aligned}\text{(2.31)} \quad & E^y\left[Y_{(s,t)} \middle| X_{(s,t)}\right](\zeta) \frac{dP_{X_{(s,t)}}}{dm_L}(\zeta) \\ &= \frac{1}{2\pi} \int_{\mathbf{R}^1} e^{-iw\zeta} E^y\left[e^{iwX_{(s,t)}} Y_{(s,t)}\right] dm_L(w) \\ &= \frac{1}{2\pi} \int_{\mathbf{R}^1} e^{-iw\zeta} \left\{ E^y\left[e^{iwx(s,t)}\right] - J_1(s, t) + J_2(s, t) \right\} dm_L(w)\end{aligned}$$

by (2.12). To evaluate the above integral first note that

$$\text{(2.32)} \quad \frac{1}{2\pi} \int_{\mathbf{R}^1} e^{-iw\zeta} E^y\left[e^{iwx(s,t)}\right] dm_L(w) = \frac{1}{\sqrt{2\pi st}} \exp\left\{ -\frac{\zeta^2}{2st} \right\}$$

by (2.15) and (2.16). And also note that

$$(2.33)\quad \frac{1}{2\pi}\int_{\mathbf{R}^1} e^{-iw\zeta}J_1(s,t)\ dm_L(w)$$

$$= \frac{1}{2\pi}\int_Q \left\{\int_{\mathbf{R}^3} V(u_{11})E^y(Y_1\,|\,X)(u_{11},u_{12},u_{21}) \times\left[\int_{R^1} e^{-iw(\zeta-u_{21}-u_{12}+u_{11})} \times e^{-((s-u)(t-v)/2)w^2}\,dm_L(w)\right] dP_X(u_{11},u_{12},u_{21})\right\} dm_L(u,v)$$

by (2.24) and the Fubini Theorem. By (2.16) we can have

$$(2.34)\quad \frac{1}{2\pi}\int_{\mathbf{R}^1} e^{-iw(\zeta-u_{21}-u_{12}+u_{11})-((s-u)(t-v)/2)w^2}\,dm_L(w)$$

$$= \frac{1}{\sqrt{2\pi(s-u)(t-v)}}\exp\left\{-\frac{(\zeta-u_{21}-u_{12}+u_{11})^2}{2(s-u)(t-v)}\right\}.$$

Substituting (2.34) in (2.33) we obtain

$$(2.35)\quad \frac{1}{2\pi}\int_{\mathbf{R}^1} e^{-iw\zeta}J_1(s,t)\,dm_L(w)$$

$$= \int_Q \left\{\int_{\mathbf{R}^3} V(u_{11})E^y(Y_1\,|\,X)(u_{11},u_{12},u_{21})\frac{1}{\sqrt{2\pi(s-u)(t-v)}} \times\exp\left\{-\frac{(\zeta-u_{21}-u_{12}+u_{11})^2}{2(s-u)(t-v)}\right\} dP_X(u_{11},u_{12},u_{21})\right\} dm_L(u,v).$$

Finally we can note that

$$(2.36)\quad \frac{1}{2\pi}\int_{\mathbf{R}^1} e^{-iw\zeta}J_2(s,t)\,dm_L(w)$$

$$= \int_Q \left\{\int_{\mathbf{R}^3} E^y[F_1\,|\,X](u_{11},u_{12},u_{21})\frac{1}{\sqrt{2\pi(s-u)(t-v)}} \times\exp\left\{-\frac{(\zeta-u_{21}-u_{12}+u_{11})^2}{2(s-u)(t-v)}\right\} dP_X(u_{11},u_{12},u_{21})\right\} dm_L(u,v)$$

by (2.29), the Fubini Theorem and (2.34). Hence by (2.31), (2.32), (2.35) and (2.36) we have

$$(2.37)\quad E^y\left[Y_{(s,t)}\middle| X_{(s,t)}\right](\zeta)\frac{dP_{X_{(s,t)}}}{dm_L}(\zeta)$$

$$= \frac{1}{\sqrt{2\pi st}}\exp\left\{-\frac{\zeta^2}{2st}\right\}$$

$$-\int_Q\left[\int_{\mathbf{R}^3}\{V(u_{11})E^y[Y_1 \mid X](u_{11},u_{12},u_{21})\right.$$

$$-E^y[F_1 \mid X](u_{11},u_{12},u_{21})\}\frac{1}{\sqrt{2\pi(s-u)(t-v)}}$$

$$\left.\times\exp\left\{-\frac{(\zeta-u_{21}-u_{12}+u_{11})^2}{2(s-u)(t-v)}\right\}dP_X(u_{11},u_{12},u_{21})\right]dm_L(u,v).$$

Since

$$\frac{dP_{X_{(s,t)}}}{dm_L}(\zeta) = \frac{1}{\sqrt{2\pi st}}\exp\left\{-\frac{\zeta^2}{2st}\right\},$$

we can have the desired result (2.3) from (2.37).

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