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ON FUNCTIONS AND STRATIFIABLE μ-SPACES

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It is shown that a space X is a stratifiable μ-space if and only if X has a topology induced by the collection $\bigcup_{n=1}^{\infty} \Phi_n$ of $[0,1]$-valued continuous functions of X such that each Φ_n satisfies the conditions (α), (β) and (γ) stated below.

1. Introduction. Throughout, all spaces are assumed to be regular Hausdorff. N always denotes the positive integers. For a space X, $C(X, I)$ denotes the collection of all continuous functions $f\colon X \to I = [0,1]$. For $f \in C(X, I)$ we denote by $\operatorname{coz} f$ the cozero set of f in X. We are assumed to be familiar with the class of stratifiable spaces in the sense of [**1**]. For a stratifiable space X, every closed subset F of X has a stratification $\{O_n(F) : n \in N\}$ in X. As is well-known, every stratifiable space X is monotonically normal, that is, X has a monotonically normal operator $D(M, N)$ for each disjoint pair (M, N) of closed subsets of X.

J. Guthrie and M. Henry characterized metrizable spaces X in terms of collections of continuous functions with continuous sup and inf as follows: A space X is metrizable if and only if X has the weak topology induced by a σ-relatively complete collection $\Phi \subset C(X, I)$, that is, $\Phi = \bigcup_{n=1}^{\infty} \Phi_n$, where for each n, each subcollection of Φ_n has both continuous sup and inf, [**3**]. On the other hand, C. R. Borges and G. Gruenhage obtained the characterization of stratifiable spaces as follows: A space X is stratifiable if and only if for each open set U of X there exists $f_U \in C(X, I)$ such that $\operatorname{coz} f_U = U$ and such that for each family $\mathscr{U}$ of open subsets of X, $\sup\{f_U : U \in \mathscr{U}\} \in C(X, I)$, [**2**, Theorem 2.1]. In the discussion below, we also give a characterization of the class of stratifiable μ-spaces in terms of collections of continuous functions with continuous sups with an additional condition. This is the main purpose of this paper.

In an earlier paper [**6**], the author introduced the notion of M-structures and studied the class $\mathscr{M}$ of all stratifiable spaces having an M-structure. This class $\mathscr{M}$ is shown to coincide with that of stratifiable μ-spaces, [**5**]. The kernel of M-structures is the term "$\mathscr{H}$-preserving in both sides". Therefore, first we state the definition. For the definition of M-structures, we refer the reader to [**6**].

Let $\mathscr{U}$, $\mathscr{H}$ be families of subsets of a space X. Then we call that $\mathscr{U}$ is *inside $\mathscr{H}$-preserving* at a point $p \in X$ if for each $\mathscr{U}_0 \subset \mathscr{U}$, $p \in \bigcap \mathscr{U}_0$

implies $p \in H \subset \bigcap \mathscr{U}_0$ for some $H \in \mathscr{H}$. We call that $\mathscr{U}$ is *outside* $\mathscr{H}$-preserving at p if for each $\mathscr{U}_0 \subset \mathscr{U}$, $p \in X - \bigcup \mathscr{U}_0$ implies $p \in H \subset X - \bigcup \mathscr{U}_0$ for some $H \in \mathscr{H}$. If $\mathscr{U}$ is both inside and outside $\mathscr{H}$-preserving at p, then $\mathscr{U}$ is called $\mathscr{H}$-preserving *in both sides at* p.

2. Continuous functions and stratifiable μ-spaces.

LEMMA 2.1. *For a stratifiable space X, the following are equivalent*:

(1) $X \in \mathscr{M}$.

(2) *Every closed subset F of X has an open neighborhood base $\mathscr{U}$ in X such that $\mathscr{U}$ is $\mathscr{H}$-preserving in both sides at each point of X for some σ-discrete family $\mathscr{H}$ of closed subsets of X.*

(3) *Every closed subset F of X has an open neighborhood base $\mathscr{U}$ in X such that $\mathscr{U}$ is inside $\mathscr{H}$-preserving at each point of X for some σ-discrete family $\mathscr{H}$ of closed subsets of X.*

(4) *Every closed subset F of X has an open neighborhood base $\mathscr{U}$ in X such that for each $U \in \mathscr{U}$ there exists a sequence $\{F_n(U) : n \in N\}$ of closed subsets of X satisfying the following*:

(a) $U = \bigcup_{n=1}^{\infty} F_n(U)$ *for each* $U \in \mathscr{U}$.

(b) *For each* n, $\{F_n(U) \colon U \in \mathscr{U}\}$ *is a closure-preserving family in* X.

(c) *For each* $\mathscr{U}_0 \subset \mathscr{U}$, *if* $p \in \bigcap \mathscr{U}_0$, *then* $p \in \bigcap \{F_n(U) : U \in \mathscr{U}_0\}$ *for some* n.

Proof. (1) $\rightleftarrows$ (2) is given in [**6**]. (2) $\to$ (3) is trivial. (3) $\to$ (4): Let F be a closed subset of X and $\mathscr{U}$, $\mathscr{H}$ be families given by (3). Write $\mathscr{H} = \bigcup_{n=1}^{\infty} \mathscr{H}_n$, where each $\mathscr{H}_n$ is a discrete family of closed subsets of X. For each $U \in \mathscr{U}$ and each n, set

$$F_n(U) = \bigcup \left\{ H \in \bigcup_{t=1}^{n} \mathscr{H}_t : H \subset U \right\}.$$

Then it is easy to see that $\{F_n(U) : n \in N\}$, $U \in \mathscr{U}$, satisfy the required conditons. (4) $\to$ (1): Let F be a closed subset of X and let $\mathscr{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an open neighborhood base of F in X such that for each $\lambda \in \Lambda$, there exists a sequence $\{F_{\lambda n} : n \in N\}$ of closed subsets of X satisfying the conditions (a), (b) and (c) with $F_n(U) = F_{\lambda n}$ and $U = U_\lambda$ for each $\lambda \in \Lambda$. Define an equivalence relation R on X as follows: For $x, y \in X$, xRy if and only if $\Lambda(x) = \Lambda(y)$, where $\Lambda(x) = \{\lambda \in \Lambda : x \in U_\lambda\}$. Let $\mathscr{P}$ be the disjoint partition of X with respect to R. $\mathscr{P}$ is written as follows: $\mathscr{P} = \{P(\delta) : \delta \in \Delta\}$, where for each $\delta \in \Delta \subset 2^\Lambda$

$$P(\delta) = \bigcap \{U_\lambda : \lambda \in \delta\} - \bigcup \{U_\lambda : \lambda \in \Lambda - \delta\}.$$

For each n, $k \in N$ and $\delta \in \Delta$, set

$$F(n,k,\delta) = \left[\bigcap\{F_{\lambda n} : \lambda \in \delta\} - O_k\left(\bigcup\{F_{\lambda n} : \lambda \in \Lambda - \delta\}\right)\right] \cap \left[X - \bigcup\{U_\lambda : \lambda \in \Lambda - \delta\}\right].$$

Then we can show that

$$\mathscr{F}(n,k) = \{F(n,k,\delta) : \delta \in \Delta\}$$

is a discrete family of closed subsets of X. To see it, let p be an arbitrary point and let $\delta_0 = \{\lambda \in \Lambda : p \in F_{\lambda n}\}$. Then, we easily see that if we define

$$N(p) = \left(X - \bigcup\{F_{\lambda n} : \lambda \in \Lambda - \delta_0\}\right) \cap O_k\left(\bigcap\{F_{\lambda n} : \lambda \in \delta_0\}\right)$$

when $\delta_0 \neq \varnothing$ and

$$N(p) = X - \bigcup\{F_{\lambda n} : \lambda \in \Lambda\}$$

when $\delta_0 = \varnothing$, then $N(p)$ is an open neighborhood of p in X such that $N(p) \cap F(n,k,\delta) = \varnothing$ for each $\delta \in \Delta - \{\delta_0\}$. This shows that $\mathscr{F}(n,k)$ is a discrete family in X. It is easily seen that each $F(n,k,\delta)$ is closed in X. Let

$$\mathscr{H} = \bigcup\{\mathscr{F}(n,k) : n, k \in N\}.$$

To see that $\mathscr{U}$ is $\mathscr{H}$-preserving in both sides at each point of X, it suffices to see that if $p \in P(\delta)$, then there exists $F(n,k,\delta) \in \mathscr{H}$ such that $p \in F(n,k,\delta) \subset P(\delta)$. But this is obvious from the construction of $\mathscr{H}$. This completes the proof.

LEMMA 2.2. *For a stratifiable space X, the following are equivalent*:

(1) $X \in \mathscr{M}$.

(2) *X has a base $\mathscr{U}$ such that $\mathscr{U}$ is σ-$\mathscr{H}$-preserving in both sides at each point of X for some σ-discrete family $\mathscr{H}$ of closed subsets of X.*

(3) *X has a base $\mathscr{U}$ such that $\mathscr{U}$ is σ-inside $\mathscr{H}$-preserving at each point of X for some σ-discrete family $\mathscr{H}$ of closed subsets of X.*

Proof. (1) $\rightarrow$ (2): Let $\mathscr{H} = \bigcup_{n=1}^{\infty} \mathscr{H}_n$ be a network of X, where each $\mathscr{H}_n$ is a discrete family of closed subsets of X. For each n, let $\{U_H : H \in \mathscr{H}_n\}$ be a family of open subsets of X such that $H \subset U_H$ for each $H \in \mathscr{H}_n$ and $\{\overline{U_H} : H \in \mathscr{H}_n\}$ is discrete in X. For each $H \in \mathscr{H}_n$, $n \in N$, by [**6**, Lemma 3.3] there exists an open neighborhood base $\mathscr{U}(H)$ of H

such that $\mathscr{U}(H)$ is $\mathscr{F}(H)$-preserving in both sides at each point of X for some σ-discrete family $\mathscr{F}(H)$ of closed subsets of X and $H \subset U \subset U_H$ for each $U \in \mathscr{U}(H)$. Set $\mathscr{U}_n = \bigcup\{\mathscr{U}(H): H \in \mathscr{H}_n\}$ for each n. Then $\mathscr{U} = \bigcup_{n=1}^{\infty} \mathscr{U}_n$ is a base for X and each $\mathscr{U}_n$ is $\mathscr{F}$-preserving in both sides at each point of X, where $\mathscr{F} = \bigcup_{n=1}^{\infty} \mathscr{F}_n \cup \mathscr{H}$ and

$$\mathscr{F}_n = \bigcup\{\mathscr{F}(H)/\overline{U_H}: H \in \mathscr{H}_n\}$$

for each n. Since $\mathscr{F}_n$ is a σ-discrete family of closed subsets of X, $\mathscr{F}$ is also a σ-discrete family of closed subsets of X. This completes the proof of (1) $\to$ (2). (2) $\to$ (3) is trivial. (3) $\to$ (1): By a routine check, we can show that every closed subset F of X has an open neighborhood base which is inside $\mathscr{H}$-preserving at each point of X for some σ-discrete family $\mathscr{H}$ of closed subsets of X. Then by Lemma 2.1(3), $X \in \mathscr{M}$. This completes the proof.

LEMMA 2.3. *Let $\mathscr{H}$ be a σ-discrete family of closed subsets of a stratifiable space X and $\mathscr{U} = \{U_\alpha : \alpha \in A\}$ a family of open subsets of X which is $\mathscr{H}$-preserving in both sides at each point of X. Then there exists a collection $\Phi = \{\phi_\alpha : \alpha \in A\} \subset C(X, I)$ satisfying the following conditions*:

(α) *For each $A_0 \subset A$, $\sup\{\phi_\alpha : \alpha \in A_0\} \in C(X, I)$.*

(β) $U_\alpha = \operatorname{coz} \phi_\alpha$ *for each $\alpha \in A$.*

(γ) *For each point $p \in X$, $\{\phi_\alpha(p) : \alpha \in A\}$ is a finite set.*

Proof. Write $\mathscr{H} = \bigcup_{n=1}^{\infty} \mathscr{H}_n$, where each $\mathscr{H}_n$ is a discrete family of closed subsets of X. Let Q_0 be the set of all rational numbers of $(0, 1]$. For each $\alpha \in A$, set

$$\mathscr{H}(\alpha) = \{H \in \mathscr{H}: H \subset X - U_\alpha\}.$$

Then obviously, $\bigcup\mathscr{H}(\alpha) = X - U_\alpha$. For each n, there exists a discrete family $\{\mathscr{U}_H : H \in \mathscr{H}_n\}$ of open subsets of X such that $H \subset U_H$ for each $H \in \mathscr{H}_n$. Since X is a monotonically normal space, X has the operator $D(M, N)$. For each $H \in \mathscr{H}_n$, $n \in N$, we choose a regular open set V_H of X such that

$$H \subset V_H \subset \overline{V_H} \subset U_H \cap D\left(H, \bigcup\left\{H' \in \bigcup_{t=1}^{n} \mathscr{H}_t : H' \cap H = \varnothing\right\}\right).$$

As a preliminary for the discussion below, we observe the following (1) by the same argument as in the proof of [7, Theorem 2, (1) $\to$ (2)].

(1) If for each $H \in \mathscr{H}$, G_H is a regular open set of X such that $H \subset G_H \subset \overline{G_H} \subset V_H$, then the families

$$\{X - \bigcup\{G_H : H \in \mathscr{H}(\alpha)\} : \alpha \in A\}$$

and

$$\left\{X - \bigcup\{\overline{G_H}: H \in \mathscr{H}(\alpha)\}: \alpha \in A\right\}$$

are closure-preserving families of closed and open subsets of X, respectively.

For each $H \in \mathscr{H}$, there exists a function $f_H \in C(X, I)$ such that $f_H^{-1}(0) = H$ and $f_H^{-1}(1) = X - V_H$. We write Q_0 as $Q_0 = \{q_1 = 1, q_2, \dots\}$. By induction on n, we shall construct families $\{V(H, q_n): H \in \mathscr{H}\}$ and $\mathscr{B}(q_n)$, $n \in N$, of subsets of X. For $n = 1$, let $V(H, q_1) = V_H$ for each $H \in H$, and let

$$B(\alpha, q_1) = X - \bigcup\{V(H, q_1): H \in \mathscr{H}(\alpha)\}$$

for each $\alpha \in A$. Then by (1), $\mathscr{B}(q_1) = \{B(\alpha, q_1): \alpha \in A\}$ is a closure-preserving family of closed subsets of X. Let $n \in N$ and assume that for each $k \leq n$, we have constructed families $\mathscr{B}(q_k) = \{B(\alpha, q_k): \alpha \in A\}$ and $\{V(H, q_k): H \in \mathscr{H}\}$ satisfying the following:

$(2)_n$ $\bigcup_{k=1}^n \mathscr{B}(q_k)$ is a closure-preserving families of closed subsets of X and each $B(\alpha, q_k) \in \mathscr{B}(q_k)$ is defined by

$$B(\alpha, q_k) = X - \bigcup\{V(h, q_k): H \in \mathscr{H}(\alpha)\}.$$

$(3)_n$ If $q_k < q_{k'}$ with $k, k' \leq n$, then $\overline{V(H, q_k)} \subset V(H, q_{k'})$ and $B(\alpha, q_{k'}) \subset \operatorname{Int} B(\alpha, q_k)$ for each $H \in \mathscr{H}$ and $\alpha \in A$.

$(4)_n$ If $q_t = \min\{q_1, \dots, q_n\}$, then $V(H, q_t) \subset f_H^{-1}[0, q_t)$.

To obtain $\mathscr{B}(q_{n+1})$, we define $V(H, q_{n+1})$ and $B(\alpha, q_{n+1})$ as follows:

(1) If $q_{n+1} < q_k$ for each $k \leq n$, then we choose a regular open set $(V(H, q_{n+1})$ by

$$H \subset B(H, q_{n+1}) \subset \overline{V(H, q_{n+1})} \subset f_H^{-1}[0, q_{n+1}) \cap \bigcap_{k=1}^{n} V(H, q_k).$$

(2) Otherwise, we choose a regular open set $V(H, q_{n+1})$ by

$$\bigcup\{\overline{V(H, q_t)}: t \leq n \text{ and } q_t < q_{n+1}\} \subset V(H, q_{n+1}) \subset \overline{V(H, q_{n+1})}$$
$$\subset \bigcap\{V(H, q_t): t \leq n \text{ and } q_t > q_{n+1}\}.$$

For each $\alpha \in A$, we define

$$B(\alpha, q_{n+1}) = X - \bigcup\{V(H, q_{n+1}): H \in \mathscr{H}(\alpha)\}$$

and also define the family $\mathscr{B}(q_{n+1}) = \{B(\alpha, q_{n+1}): \alpha \in A\}$. By (1), $\mathscr{B}(q_{n+1})$ is a closure-preserving family of closed subsets of X. Therefore, $(2)_{n+1}$ is satisfied. $(4)_{n+1}$ is trivial by the definition of $V(H, q_{n+1})$ in (1).

To see $(3)_{n+1}$, let $q_t < q_{n+1}$ for some t with $t \leq n$. Then by (2) we easily see

$$B(\alpha, q_{n+1}) \subset X - \bigcup\{\overline{V(H, q_t)} : H \in \mathscr{H}(\alpha)\}$$
$$\subset X - \bigcup\{V(H, q_t) : H \in \mathscr{H}(\alpha)\} = B(\alpha, q_t).$$

Since $\overline{V(H, q_t)} \subset V_H$ in (2), by (1) the second set is open in X. This implies $B(\alpha, q_{n+1}) \subset \operatorname{Int} B(\alpha, q_t)$. If $q_t > q_{n+1}$ with $t \leq n$, then by (2) we have $\overline{V(H, q_{n+1})} \cap B(\alpha, q_t) = \varnothing$. This implies

$$B(\alpha, q_t) \subset X - \bigcup\{\overline{V(H, q_{n+1})} : H \in \mathscr{H}(\alpha)\} \subset B(\alpha, q_{n+1}).$$

Again, the second set is open in X by (1). Hence we have $B(\alpha, q_t) \subset \operatorname{Int} B(\alpha, q_{n+1})$. In this manner, we repeat the construction of a sequence $\{\mathscr{B}(q) : q \in Q_0\}$ of families of subsets of X. Then, by induction the following are obvious:

(5) For each $q \in Q_0$, $\mathscr{B}(q) = \{B(\alpha, q) : \alpha \in A\}$ is a closure-preserving family of closed subsets of X.

(6) If $q, q' \in Q_0$ with $q < q'$, then for each $\alpha \in A$ $B(\alpha, q') \subset \operatorname{Int} B(\alpha, q)$.

Since $\mathscr{U}$ is inside $\mathscr{H}$-preserving at each point and $\bigcap\{V(h, q) : q \in Q_0\} = H$ for each $H \in \mathscr{H}$, by the method of the construction of V_H we get that

(7) For each $\alpha \in A$, $U_\alpha = \bigcup\{B(\alpha, q) : q \in Q_0\}$.

Also, from the fact that $\mathscr{U}$ is inside $\mathscr{H}$-preserving at each point, we get that

(8) For $A_0 \subset A$, if $p \in \bigcap\{U_\alpha : \alpha \in A_0\}$, then there exist $n \in N$ and $H \in \mathscr{H}_n$ such that $p \in H \subset \bigcap\{U_\alpha : \alpha \in A_0\}$ and $H \cap V(H', q) = \varnothing$ for each $q \in Q_0$ and each $H' \in (\bigcup_{t=n}^{\infty} \mathscr{H}_t) \cap (\bigcup\{\mathscr{H}(\alpha) : \alpha \in A_0\})$.

Now, for each $\alpha \in A$ we define $\phi_\alpha : X \to I$ by

$$\phi_\alpha(x) = \begin{cases} 1 & \text{if } x \in B(\alpha, 1), \\ \inf\{q \in Q_0 : x \notin B(\alpha, q)\}. \end{cases} \tag{9}$$

Then, as shown in the proof of [**2**, Theorem 2], $\phi_\alpha \in C(X, I)$ and $\operatorname{coz} \phi_\alpha = U_\alpha$ for each $\alpha \in A$, and (α) is satisfied. The condition (γ) is easily obtained by (8). This completes the proof.

COROLLARY 2.4. *Under the hypothesis for Lemma* 2.3, *there exist a collection* $\Phi \subset C(X, I)$ *and a* σ-*discrete family* $\mathscr{H}$ *of closed subsets of* X *such that* (α), (β) *and the following are satisfied*:

$(\gamma)'$ *For each* $H \in \mathscr{H}$ *and* $A_0 \subset A$, $\inf\{\phi_\alpha / H : \alpha \in A_0\} \in C(H, I)$.

Proof. In the proof above, without loss of generality we can assume $H \cap U_\alpha \neq \varnothing$ if and only if $H \subset U_\alpha$ for each $H \in \mathscr{H}$ and $\alpha \in A$. By the same method, we can construct $\mathscr{B}(q) = \{B(\alpha, q) : \alpha \in A\}$, $q \in Q_0$, satisfying (5), (6), (7) and (8) above. If we define $\Phi = \{\phi_\alpha : \alpha \in A\}$ by (9) above, then Φ is shown to be the desired collection. In fact (α) and (β) are obvious. By the similar argument to that of the proof of [7, Theorem 2, (1) $\to$ (2)], we can observe that for each $H \in \mathscr{H}$ and each $q \in Q_0$, $\{B(\alpha, q) : \alpha \in A\}/H$ is interior-preserving in the subspace H.

Now, we establish the following general assertion, from which $(\gamma)'$ follows directly:

Assertion. Let $\{B(\alpha, q) : \alpha \in A\}$ and $\Phi = \{\phi_\alpha : \alpha \in A\}$ be the same as in the proof of Lemma 2.3. If for each $q \in Q_0$, $\{B(\alpha, q) : \alpha \in A\}$ is interior-preserving in X, then for each $A_0 \subset A$, $\inf\{\phi_\alpha : \alpha \in A_0\} \in C(X, I)$.

Proof of the assertion. Let t be an arbitrary number of $[0, 1)$. Since

$$(\inf\{\phi_\alpha : \alpha \in A_0\})^{-1}[t, 1] = \bigcap\{\phi_\alpha^{-1}[t, 1] : \alpha \in A_0\}$$

is closed in X, it suffices to show that $S = (\inf\{\phi_\alpha : \alpha \in A_0\})^{-1}(t, 1]$ is open in X. Let p be an arbitrary point of S. Then

$$t < \inf\{\phi_\alpha(p) : \alpha \in A_0\} = \delta \leq 1.$$

Take r and $s \in Q_0$ such that $t < r < s < \delta$. Since for each $\alpha \in A_0$, $s < \delta \leq \phi_\alpha(p)$, $p \in B(\alpha, s)$. By (6) above, $p \in \operatorname{Int} B(\alpha, r)$ for each $\alpha \in A_0$. Therefore

$$N(p) = \bigcap\{\operatorname{Int} B(\alpha, r) : \alpha \in A_0\}$$

is an open neighborhood of p in X because $\{B(\alpha, r) : \alpha \in A\}$ is interior-preserving in X. Since $N(p) \subset S$, S is open in X. This completes the proof.

REMARK 2.5. If we slightly modify the argument above, then we can establish the following: Let $\mathscr{H}$ be a σ-discrete family of closed subsets of a stratifiable space X and $\mathscr{U} = \{U_\alpha : \alpha \in A\}$ a family of open subsets of X which is $\mathscr{H}$-preserving in both sides at each point of X. Then there exist a contraction ρ: $X \to \hat{X}$ with $\hat{X}$ metrizable and a collection $\{f_\alpha : \hat{X} \to I : \alpha \in A\}$ of correspondences satisfying the following:

(1) For each $\alpha \in A$, $\phi_\alpha = f_\alpha \rho \in C(X, I)$ and $\operatorname{coz} \phi_\alpha = U_\alpha$.

(2) $\rho(\mathscr{H})$ is a σ-discrete family of closed subsets of X.

(3) For each $H \in \mathscr{H}$ and each $\alpha \in A$,

$$f_\alpha/\rho(H) \in C(\rho(H), I).$$

In fact, let $\mathscr{H} = \bigcup_{i=1}^{\infty} \mathscr{H}_i$, where each $\mathscr{H}_i$ is discrete in X and for each $H \in \mathscr{H}$ and each $\alpha \in A$, $H \cap U_\alpha \neq \varnothing$ if and only if $H \subset U_\alpha$. By the same argument as in the proof of Lemma 2.3, we can construct families $\{V(H,q): q \in Q_0, H \in \mathscr{H}\}$ and $\{\mathscr{B}(q): q \in Q_0\}$ of subsets of X. Let ρ be a contraction of X onto a metrizable space $\hat{X}$ satisfying the following:

(1) $\rho(\mathscr{H})$ is a σ-discrete family of closed subsets of $\hat{X}$.

(2) For each $q \in Q_0$ and each i, $\{\rho(V(H,q)): H \in \mathscr{H}_i\}$ and $\{\rho(\overline{V(H,q)}): H \in \mathscr{H}_i\}$ are discrete families of open and closed subsets of $\hat{X}$, respectively.

(3) For each $q \in Q_0$, $\rho(\mathscr{B}(q))$ is a closure-preserving family of closed subsets of $\hat{X}$.

For each $\alpha \in A$, we define a correspondence $f_\alpha: \hat{X} \to I$ as follows:

$$f_\alpha(x) = \begin{cases} 1 & \text{if } x \in \rho(B(\alpha,1)), \\ \inf\{q \in Q_0 : x \notin \rho(B(\alpha,q))\}. \end{cases}$$

Then it is easy to see that $\{f_\alpha: \alpha \in A\}$ and $\rho: X \to \hat{X}$ satisfy the required conditions.

If we apply the essential argument of [**4**, Theorem 2.1] to this case, we can construct a one-to-one continuous mapping g: $X \to Y$ with Y a stratifiable σ-metric space such that $g(U_\alpha)$ is open in Y for each $\alpha \in A$. As a consequence, we reach to the coincidence theorem of the class $\mathscr{M}$ with stratifiable μ-spaces of [**5**].

LEMMA 2.6. *Let X be a stratifiable space and $\Phi = \{\phi_\alpha: \alpha \in A\} \subset C(X, I)$ satisfy the conditions* (α), (β) *and* (γ) *above. Then there exists a σ-discrete family $\mathscr{H}$ of closed subsets of X such that $\{\operatorname{coz}\phi_\alpha: \alpha \in A\}$ is $\mathscr{H}$-preserving in both sides at each point of X.*

Proof. For each $\alpha \in A$ and each n, set $F_{\alpha n} = \phi_\alpha^{-1}[1/n, 1]$. Then obviously each $F_{\alpha n}$ is closed in X and $\operatorname{coz}\phi_\alpha = \bigcup_{n=1}^{\infty} F_{\alpha n}$. Moreover, for each n $\mathscr{F}_n = \{F_{\alpha n}: \alpha \in A\}$ is closure-preserving in X. To see it, let $p \in X - \bigcup\{F_{\alpha n}: \alpha \in A_0\}$ for $A_0 \subset A$. This implies $0 \leq \phi_\alpha(p) < 1/n$ for each $\alpha \in A_0$. By (γ) $\sup\{\phi_\alpha(p): \alpha \in A_0\} < 1/n$. Since $\sup\{\phi_\alpha: \alpha \in A_0\}$ is continuous at p,

$$N(p) = (\sup\{\phi_\alpha : \alpha \in A_0\})^{-1}[0, 1/n)$$

is an open neighborhood of p such that $N(p) \cap F_{\lambda n} = \varnothing$ for each $\alpha \in A_0$. Hence $\mathscr{F}_n$ is closure-preserving in X. Assume

$$p \in \bigcap\{\operatorname{coz}\phi_\alpha : \alpha \in A_0\} \quad \text{for } A_0 \subset A.$$

By (γ), there exists $n \in N$ such that $1/n \leq \inf\{\phi_\alpha(p): \alpha \in A_0\}$. This implies $p \in \bigcap\{F_{\alpha n}: \alpha \in A_0\}$. By the same argument as in the proof of

(4) $\to$ (1) in Lemma 2.1, we have a σ-discrete family $\mathscr{H}$ of closed subsets of X such that $\{\operatorname{coz}\phi_\alpha : \alpha \in A\}$ is $\mathscr{H}$-preserving in both sides at each point in X. This completes the proof.

We state the main result.

THEOREM 2.7. *For a space X, the following are equivalent*:

(1) $X \in \mathscr{M}$, *that is*, X *is a stratifiable* μ-*space*.

(2) X *has a topology induced by the collection* $\Phi = \bigcup_{n=1}^{\infty} \Phi_n \subset C(X, I)$ *such that each* Φ_n *satisfies* (α), (β) *and* (γ) *of Lemma* 2.3.

Proof. (1) $\to$ (2): Let $X \in \mathscr{M}$. By Lemma 2.2, X has a σ-discrete family $\mathscr{H}$ of closed subsets of X and a base $\bigcup_{n=1}^{\infty} \mathscr{U}_n$, where each $\mathscr{U}_n$ is $\mathscr{H}$-preserving in both sides at each point of X. By Lemma 2.3, for each n there exists a collection $\Phi_n \subset C(X, I)$ satisfying (α), (β) and (γ). Then $\Phi = \bigcup_{n=1}^{\infty} \Phi_n$ is the desired collection. (2) $\to$ (1): By the argument of [**2**, Theorem 2.1] and by (α), X is a stratifiable space. By Lemma 2.5, for each n there exists a σ-discrete family $\mathscr{H}_n$ of closed subsets of X such that $\mathscr{U}_n = \{\operatorname{coz}\phi : \phi \in \Phi_n\}$ is $\mathscr{H}_n$-preserving in both sides at each point of X. Then it is easy to see that each $\mathscr{U}_n$ is $\mathscr{H}$-preserving in both sides at each point of X, where $\mathscr{H} = \bigcup_{n=1}^{\infty} \mathscr{H}_n$ is also a σ-discrete family of closed subsets of X. This completes the proof.

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