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NOTE ON THE PWB-METHOD IN THE NON-LINEAR CASE

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The Perron-Wiener-Brelot (PWB) method is applied to an important nonlinear situation. Unbounded subsolutions, their approximation and a counterpart of the harmonic measure are considered.

Introduction. The Perron-Wiener-Brelot (PWB-) method as introduced by O. Perron [**P**] and refined by several mathematicians is well-known in Potential Theory and it is mainly used in the theory of harmonic functions although it has a wider scope of applications [**CC**]. The PWB-method was generalized by E. Beckenbach and L. Jackson [**BJ, J**] to the non-linear situation. Their approach used the strong maximum principle for the difference of two solutions [**BJ**, Postulate 2]. The purpose of this note is to show that the PWB-method can be employed without this assumption in certain important non-linear cases. We are also able to deal with unbounded subsolutions.

We consider weak solutions, called F-extremals, of an Euler equation

$$\nabla \cdot \nabla_h F(x, \nabla u) = 0, \tag{1.1}$$

where the variational kernel F: $G \times R^n \to R$ satisfies the assumptions:

(a) For each $\varepsilon > 0$ there is a closed set C in the domain $G \subset R^n$ such that $m(G \setminus C) < \varepsilon$ and $F|C \times R^n$ is continuous.

(b) For a.e. $x \in G$ the function $h \mapsto F(x, h)$ is strictly convex and differentiable in R^n.

(c) There are $0 < \alpha \leq \beta < \infty$ such that for a.e. $x \in G$

$$\alpha|h|^n \leq F(x, h) \leq \beta|h|^n,$$

$h \in R^n$.

(d) For a.e. $x \in G$

$$F(x, \lambda h) = |\lambda|^n F(x, h),$$

$\lambda \in R$, $h \in R^n$.

For a thorough analysis of the above assumptions see [**GLM1**]. Some of the assumptions are not necessary for the constructions. The exponent n in (c) is essential for applications in conformal geometry, cf. [**GLM1-2**].

A function $u \in C(G) \cap \operatorname{loc} W_n^1(G)$, i.e. u is ACL^n, is called an F-extremal in G if for all domains $D \subset\subset G$

$$I_F(u, D) = \inf_{v \in \mathscr{F}_u} I_F(v, D),$$

where

$$I_F(v, D) = \int_D F(x, \nabla v(x))\, dm(x)$$

is the variational integral with the kernel F and

$$\mathscr{F}_u = \{ v \in C(\overline{D}) \cap W_n^1(D) \,|\, v = u \text{ in } \partial D \}.$$

A function u is an F-extremal if and only if $u \in C(G) \cap \operatorname{loc} W_n^1(G)$ is a solution of (1.1) in the weak sense.

An upper semi-continuous function u: $G \to R \cup \{-\infty\}$ is called a sub-F-extremal if u satisfies the F-comparison principle in G, i.e. if $D \subset\subset G$ is a domain and $h \in C(\overline{D})$ is an F-extremal in D, then $h \geq u$ in ∂D implies $h \geq u$ in D. A function u: $G \to R \cup \{\infty\}$ is a super-F-extremal if $-u$ is a sub-F-extremal. For the basic properties of F-extremals and sub-F-extremals we refer to **[GLM1]**.

Let $G \subset R^n$ be a domain and let f: $\partial G \to R \cup \{\pm\infty\}$ be any function. The fundamental concepts in the PWB-method are the upper and lower classes $\mathscr{U}_f$ and $\mathscr{L}_f$ determined by f. Our first theorem, Theorem 2.2, states in the complete analogy with the PWB-method that the function

$$\underline{H}_f(x) = \sup\{ u(x) \,|\, u \in \mathscr{L}_f \}$$

is either $+\infty$, $-\infty$ or an F-extremal in G. The proof differs in several aspects from the classical proof, cf. e.g. [H]. First, a method like the Poisson modification of a sub-F-extremal is needed and since no Poisson formula is available in the non-linear case, our modification is based on approximation and on the solvability of the Dirichlet's problem in balls. The crucial step in the proof is to show that the function $\underline{H}_f$ is continuous if it is not $+\infty$ or $-\infty$. The proof is based on a uniform Hölder-estimate, see [**GLM1**, Theorem 4.7], which is quite similar to Harnack's inequality. Moreover, the proof for Theorem 2.2 uses a uniform approximation argument, Lemma 2.14, for the function $\underline{H}_f$ and Harnack's principle several times.

The rest of the paper is devoted to applications of the PWB-method and to byproducts of the method. In Chapter 3 we develop the barrier method for the non-linear case. Here the method works as in the linear case. Moreover, this method gives a necessary and sufficient condition for the solvability of the Dirichlet problem with continuous boundary values. In the non-linear case the best condition for solvability has been the celebrated Wiener criterion, see [**Maz**], [**GZ**]. In the harmonic case these conditions are equivalent but this is not known in the non-linear situation.

Approximation of sub-F-extremals by means of regular sub-F-extremals is studied in Chapter 4. These results which are usually proved using a simple convolution argument, cf. [**R**], are more difficult to obtain in the non-linear case and we need a solution to an obstacle problem in the calculus of variations, cf. [**GLM1**, Theorem 5.15]. As a consequence we especially show that a bounded subharmonic function in a plane domain belongs to the Sobolev space $\operatorname{loc} W_2^1$. These results are needed in the variational interpretation of subharmonicity and, more generally, sub-F-extremality.

Chapter 5 is devoted to the construction of the F-harmonic measure in general domains. This concept has turned out useful in studying the boundary behavior of F-extremals, see [**GLM2**], [**GLM3**].

Our notation is standard and generally as in [**GLM1**].

2. The Perron-Wiener-Brelot method. Suppose that $G \subset R^n$ is a domain and that $f\colon \partial G \to R \cup \{\pm\infty\}$ is a function.

2.1. DEFINITION. The lower class $\mathscr{L}_f$ consists of the functions u: $G \to R \cup \{-\infty\}$ for which

(a) u is a sub-F-extremal in G,

(b) u is bounded above,

(c) $\overline{\lim}_{x\to y} u(x) \le f(y)$ for all $y \in \partial G$,

(d) there is a compact set $K_u \subset R^n$ such that $u \le 0$ in $G \setminus K_u$.

The upper class $\mathscr{U}_f$ is defined analogously via super-F-extremals.

Let $\overline{H}_f = \inf\{u \mid u \in \mathscr{U}_f\}$ and $\underline{H}_f = \sup\{u \mid u \in \mathscr{L}_f\}$. The next theorem is fundamental for the PWB-method.

2.2. THEOREM. *The function $\underline{H}_f$ satisfies one of the following conditions*:

(i) *$\underline{H}_f$ is an F-extremal in G,*

(ii) *$\underline{H}_f = \infty$ in G,*

(iii) *$\underline{H}_f = -\infty$ in G.*

The same is true for the funciton $\overline{H}_f$.

Some auxiliary results are needed in the proof of the above theorem. The so-called F-comparison principle is a basic tool.

2.3. LEMMA. *Let $G \subset R^n$ be a bounded open set, u a sub-F-extremal and v a super-F-extremal in G. Suppose*

$$\overline{\lim_{x\to y}}\, u(x) \le \underline{\lim_{x\to y}}\, v(x) \tag{2.4}$$

for all $y \in \partial G$. *If the left and right-hand sides are neither* ∞ *nor* $-\infty$ *at the same time, then* $u \leq v$ *in* G.

Proof. Fix any $x \in G$. We will show that $u(x) \leq v(x)$. Let $\varepsilon > 0$ and consider the open set $H = \{y \in G | u(y) < v(y) + \varepsilon\}$. There exists a regular domain D_ε, $\overline{D}_\varepsilon \subset G$, such that $x \in D_\varepsilon$ and $\partial D_\varepsilon \subset H$. Choose a decreasing sequence $\varphi_i \in C^\infty(G)$ and an increasing sequence $\psi_i \in C^\infty(G)$ such that $\varphi_i \to u$ and $\psi_i \to v + \varepsilon$. Since ∂D_ε is compact we have $\varphi_i < \psi_i$ on ∂D_ε for some $i \in N$. Let h_i^1 and h_i^2 be F-extremals such that $h_i^1 | \partial D_\varepsilon = \varphi_i | \partial D$, $h_i^2 | \partial D_\varepsilon = \psi_i | \partial D$. It follows from **[GLM1**, Definition 5.1] that

$$u \leq h_i^1 \leq h_i^2 \leq v + \varepsilon \quad \text{in } D_\varepsilon.$$

Since $x \in D_\varepsilon$ and $\varepsilon > 0$ was arbitrary, we obtain the desired inequality $u(x) \leq v(x)$.

The F-comparison principle yields:

2.5. LEMMA. $\underline{H}_f \leq \overline{H}_f$.

The Poisson modification of a subharmonic function so as to be harmonic over part of its domain is a basic operation in the classical potential theory. In the proof of Theorem 2.2 we employ a similar modification method for sub-F-extremals, cf. [**R**].

2.6. *Modification of sub-F-extremals*. Suppose that $G \subset R^n$ is a domain and that u: $G \to R \cup \{-\infty\}$ is a sub-F-extremal. Let $\overline{B} \subset G$ be a ball. We modify the sub-F-extremal by an approximation argument. Since u is upper semicontinuous in G, there exists a sequence $\varphi_i \in C^\infty(G)$ such that $\varphi_1 \geq \varphi_2 \geq \cdots \geq u$ in $\overline{B}$ and $\lim_{i \to \infty} \varphi_i = u$ in $\overline{B}$.

Choose F-extremals h_i in $\overline{B}$ such that $h_i | \partial B = \varphi_i | \partial B$ and $h_i \in C(\overline{B}) \cap W_n^1(B)$. By Lemma 2.3, $h_1 \geq h_2 \geq \cdots \geq u$ in $\overline{B}$. The function $h = \lim_{i \to \infty} h_i$ is an F-extremal or identically $-\infty$ in B, see **[GLM1**, Theorem 4.22]. For any $\zeta \in \partial B$

$$\varlimsup_{\substack{x \to \zeta \\ x \in B}} h(x) \leq \varlimsup_{\substack{x \to \zeta \\ x \in B}} h_i(x) = \varphi_i(\zeta), \qquad i = 1, 2, 3, \ldots,$$

and thus

$$\varlimsup_{\substack{x \to \zeta \\ x \in B}} h(x) \leq u(\zeta). \tag{2.7}$$

Write

$$P(u, B)(x) = \begin{cases} u(x), & x \in G \setminus B \\ h(x), & x \in B. \end{cases}$$

It is easy to see that $P(u, B)$ is independent of the sequence φ_i, although this fact is not needed in the sequel.

Now $P(u, B) \geq u$ in G and we shall prove that the function $P(u, B)$ is a sub-F-extremal. For that purpose an auxiliary result is needed.

2.8. LEMMA. *A sub-F-extremal u is identically $-\infty$ if and only if it is $-\infty$ in some nonempty open subset of G.*

PROOF. Write $H = \inf\{x \in G | u(x) = -\infty\}$ and suppose $H \neq G$. Let $x_0 \in \partial H \cap G$ and choose $\delta > 0$ such that $\overline{B}^n(x_0, \delta) \subset G$ and $S^{n-1}(x_0, \delta) \cap H \neq \varnothing$. Pick a closed cap $K \subset S^{n-1}(x_0, \delta) \cap H$ and denote the F-harmonic measure $\omega(K, B^n(x_0, \delta); F)$ by h. It follows from [**GLM2**, Theorem 4.10] that h is not identically zero. Let $\alpha > 0$ and consider the F-extremal $v = M - \alpha h$ where $M = \sup_{B^n(x_0, \delta)} u$. Now

$$\varliminf_{x \to y} v(x) \geq \varlimsup_{x \to y} u(x)$$

for all $y \in S^{n-1}(x_0, \delta)$ and the left-hand side is finite. Hence by Lemma 2.3, $v \geq u$ in $B^n(x_0, \delta)$. Letting $\alpha \to \infty$ we obtain $u(x) = -\infty$ for all $x \in B^n(x_0, \delta)$, a contradiction since $B^n(x_0, \delta)$ contains points not in H. The lemma follows.

We are ready to prove

2.9. LEMMA. *The function $\mathscr{U} = P(u, B)$ is a sub-F-extremal in G.*

Proof. If h is identically $-\infty$, so is u by Lemma 2.8 and there is nothing to prove. Otherwise, h is an F-extremal and we first show that $\mathscr{U}$ is upper semicontinuous. We need only consider points $\zeta \in \partial B$. Now

$$\varlimsup_{\substack{x \to \zeta \\ x \in G \setminus B}} \mathscr{U}(x) = \varlimsup_{\substack{x \to \zeta \\ x \in G \setminus B}} u(x) \leq u(\zeta) = \mathscr{U}(\zeta).$$

By combining (2.7) and the inequality above it readily follows that $\mathscr{U}$ is upper semicontinuous.

Next we prove that $\mathscr{U}$ satisfies the F-comparison principle in G. Suppose that $D \subset\subset G$ is a domain and that $H \in C(\overline{D})$ is an F-extremal in D with $H|\partial D \geq \mathscr{U}|\partial D$. We will show that $H \geq \mathscr{U}$ in D. Now $H \geq u$ in D, since $\mathscr{U} \geq u$ in G and u is a sub-F-extremal. Let $\zeta \in \partial(D \cap B)$. Now

$$H(\zeta) \geq u(\zeta) \geq \varlimsup_{\substack{x \to \zeta \\ x \in D \cap B}} h(x),$$

and hence

$$\varliminf_{\substack{x \to \zeta \\ x \in D \cap B}} H(x) \geq \varlimsup_{\substack{x \to \zeta \\ x \in D \cap B}} h(x).$$

Lemma 2.3 implies $H \geq h = \mathscr{U}$ in $D \cap B$. Consequently $H \geq \mathscr{U}$ in the whole D as desired.

2.10 *Three lemmas for $\underline{H}_f$.* In what follows we shall only consider the function $\underline{H}_f$. The lemmas and proofs for $\overline{H}_f$ are similar. First we prove that it is possible to replace $\mathscr{L}_f$ by a subfamily, which is bounded from below in compact subsets of G. This new family gives the same $\underline{H}_f$.

2.11. LEMMA. *Suppose $K \subset G$ is compact and $\underline{H}_f$ is not identically $-\infty$. Then there exists $\mathscr{L}_K \subset \mathscr{L}_f$ such that $\mathscr{L}_K$ is bounded from below in K and $\underline{H}_f = \sup \mathscr{L}_K$.*

Proof. There exists $u_0 \in \mathscr{L}_f$ such that u_0 is not identically $-\infty$. Choose a finite cover $\{B^n(x_1, R_{x_1}), \ldots, B^n(x_k, R_{x_k})\}$ of K such that $\overline{B}^n(x_i, 2R_{x_i}) \subset G$ for $i = 1, \ldots, k$, and let $\mathscr{U}_{x_i} = P(u_0, B^n(x_i, 2R_{x_i}))$, $i = 1, \ldots, k$. Lemma 2.9 yields $\mathscr{U}_{x_i} \in \mathscr{L}_f$ and Lemma 2.8 shows that $\mathscr{U}_{x_i} > -\infty$ in $B^n(x_i, 2R_{x_i})$, $i = 1, \ldots, k$. Since $\mathscr{U}_{x_i}$ is an F-extremal in $B^n(x_i, 2R_{x_i})$ the continuity of $\mathscr{U}_{x_i}$ gives $M_{x_i} < \infty$ such that $\mathscr{U}_{x_i} > -M_{x_i}$ in $B^n(x_i, R_{x_i})$. Choose $\mathscr{L}_K = \{\max\{u, \mathscr{U}_{x_1}, \ldots, \mathscr{U}_{x_k}\} \mid u \in \mathscr{L}_f\}$. Observe that for all $u \in \mathscr{L}_K$ we have $u \geq \min\{-M_{x_1}, \ldots, -M_{x_k}\}$ in K. Let $u \in \mathscr{L}_f$. Then there exists $u^* \in \mathscr{L}_K$ such that $u^* \geq u$ in G. It follows that $\sup \mathscr{L}_K = \underline{H}_f$.

The next lemma is the basic step for the proof of Theorem 2.2.

2.12. LEMMA. *If $\underline{H}_f$ is locally bounded above, then $\underline{H}_f$ is continuous or identically $-\infty$.*

Proof. Let $\varepsilon > 0$ and assume that $\underline{H}_f$ is not identically $-\infty$. We will show that there is $r > 0$ such that

$$|\underline{H}_f(x_1) - \underline{H}_f(x_2)| < \varepsilon \quad \text{for } x_1, x_2 \in B^n(x_0, r). \tag{2.13}$$

Fix a ball $\overline{B}^n(x_0, R) \subset G$ and let $K = \overline{B}^n(x_0, R)$. By Lemma 2.11 we can restrict to the class $\mathscr{L}_K$ and there is a constant $0 < M_K < \infty$ such that $|u(x)| < M_K$, $x \in K$, for all $u \in \mathscr{L}_K$. Suppose $x_1, x_2 \in B^n(x_0, r) \subset B^n(x_0, R)$, where $r > 0$ will be fixed later. Assume, for example, that $\underline{H}_f(x_2) \geq \underline{H}_f(x_1)$. We can choose a sequence of functions $u_i \in \mathscr{L}_K$

such that $\lim_{i\to\infty} u_i(x_2) = \underline{H}_f(x_2)$. Consider the functions $\mathscr{U}_i = P(u_i, B^n(x_0, R))$. Again we have

$$\lim_{i\to\infty} \mathscr{U}_i(x_2) = \lim_{i\to\infty} P(u_i, B^n(x_0, R))(x_2).$$

Choose $i_0 \in N$ so large that $\underline{H}_f(x_2) - \mathscr{U}_i(x_2) \leq \varepsilon/2$ for $i > i_0$. Now

$$\begin{aligned} 0 \leq \underline{H}_f(x_2) - \underline{H}_f(x_1) &< \mathscr{U}_i(x_2) + \frac{\varepsilon}{2} - \mathscr{U}_i(x_1) \\ &\leq \operatorname{osc}(\mathscr{U}_i, B^n(x_0, r)) + \frac{\varepsilon}{2}. \end{aligned}$$

It is possible to choose r independent of i such that $\operatorname{osc}(\mathscr{U}_i, B^n(x_0, r)) < \varepsilon/2$. Since $\mathscr{U}_i$ is an F-extremal in $B^n(x_0, R)$, this follows from the Hölder-estimate of F-extremals

$$\operatorname{osc}(\mathscr{U}_i, B^n(x_0, r)) \leq c\left(\frac{r}{R}\right)^{\kappa} \operatorname{osc}(\mathscr{U}_i, B^n(x_0, R)) \leq 2c\left(\frac{r}{R}\right)^{\kappa} M_K,$$

cf. **[GLM1**, Theorem 4.7]. In the same way (2.13) can be proved if $\underline{H}_f(x_2) \leq \underline{H}_f(x_1)$.

2.14. LEMMA. *Suppose that $C \subset G$ is compact and that $\underline{H}_f$ is locally bounded from above in G. Then for arbitrary $\varepsilon > 0$ there exists $v_\varepsilon \in \mathscr{L}_f$ such that*

$$\underline{H}_f(x) \leq v_\varepsilon(x) + \varepsilon \quad \textit{for } x \in C. \tag{2.15}$$

Proof. By Lemma 2.12 there are two possibilities: either $\underline{H}_f$ is identically $-\infty$ or continuous. In the first case choose $v_\varepsilon \equiv -\infty$. In the second case for each $x \in C$ choose $\overline{B}^n(x, 2R_x) \subset G$. Let

$$K = \overline{\bigcup_{x\in C} B^n(x, R_x)},$$

and replace $\mathscr{L}_f$ by the class $\mathscr{L}_K$. Now $\mathscr{L}_K$ is uniformly bounded on the compact subset K of G.

For all $x \in C$ there exists $u_x \in \mathscr{L}_K$ such that $u_x(x) \geq \underline{H}_f(x) - \varepsilon/3$. Consider $\mathscr{U}_x = P(u_x, B^n(x, R_x))$. There exists $M_K > 0$ such that $\operatorname{osc}(\mathscr{U}_x, B^n(x, R_x)) \leq M_K$. Let $x \in K$. Since $\underline{H}_f$ is continuous and the Hölder-estimate **[GLM1**, Theorem 4.7] is valid for $\mathscr{U}_x$ in $B^n(x, R_x)$ we can choose a ball $B^n(x, r_x)$, $0 < r_x < R_x$, such that

$$\operatorname{osc}(\mathscr{U}_x, B^n(x, r_x)) < \frac{\varepsilon}{3}, \qquad \operatorname{osc}\big(\underline{H}_f, B^n(x, r_x)\big) < \frac{\varepsilon}{3}.$$

Now $\{B^n(x, r_x) | x \in C\}$ is an open cover for C and there is a finite subcover $\{B^n(x_1, r_{x_1}), \ldots, B^n(x_k, r_{x_k})\}$ of C. For $y \in B^n(x_i, r_{x_i})$, $i \in$

$\{1,\ldots,k\}$, we have

$$\begin{aligned}\underline{H}_f(y)-\mathscr{U}_{x_i}(y)&=\left(\underline{H}_f(y)-\underline{H}_f(x_i)\right)+\left(\underline{H}_f(x_i)-\mathscr{U}_{x_i}(x_i)\right)\\&\quad+\left(\mathscr{U}_{x_i}(x_i)-\mathscr{U}_{x_i}(y)\right)\\&<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.\end{aligned}$$

The function $v_\varepsilon=\max\{\mathscr{U}_{x_1},\ldots,\mathscr{U}_{x_k}\}$ is a sub-F-extremal [**GLM1**, Lemma 5.2] and has the desired property.

Proof of Theorem 2.2. Assume first that $\underline{H}_f$ is not locally bounded from above. Then there are a sequence of functions $u_i\in\mathscr{L}_f$ and a sequence of points $x_i\in G$ such that $\lim_{i\to\infty}u_i(x_i)=\infty$, $\lim_{i\to\infty}x_i=x_0\in G$.

Suppose that $y\in G$. We will prove that $\underline{H}_f(y)=\infty$. There is a domain D such that $\overline{D}$ is compact in G, $y\in D$ and $x_i,x_0\in D$. By Lemma 2.11 we can restrict to a sublcass $\mathscr{L}_{\overline{D}}\subset\mathscr{L}_f$, which is uniformly bounded from below in $\overline{D}$. For that reason we may assume that the functions in $\mathscr{L}_{\overline{D}}$ are non-negative in D. Choose balls $\overline{B}^n(z_j,2r_j)\subset D$, $j=1,\ldots,k$, with the following properties.

(i) $x_i,x_0\in B^n(z_1,r_1)$, for $i>i_0$,

(ii) $y\in B^n(z_k,r_k)$,

(iii) $B^n(z_j,r_j)\cap B^n(z_{j+1},r_{j+1})\neq\varnothing$, $j=1,\ldots,r-1$.

Let $i>i_0$ and define the functions $\mathscr{U}_i^j$ as follows: $\mathscr{U}_i^1=u_i$, $\mathscr{U}_i^{j+1}=P(\mathscr{U}_i^j,\ B^n(z_j,2r_j))$, $j=1,\ldots,k-1$. By iterated use of Harnacks's inequality for the functions $\mathscr{U}_i^j$, $j=1,\ldots,k$, it is easy to see that there is a constant $c>0$ independent of i such that

$$u_i(x_i)\leq\frac{1}{c}\mathscr{U}_i^k(y)\leq\frac{1}{c}\underline{H}_f(y).$$

By letting $i\to\infty$ we obtain $\underline{H}_f(y)=\infty$.

Next assume that $\underline{H}_f$ is locally bounded from above. According to Lemma 2.12 either $\underline{H}_f$ is continuous or identically $-\infty$. In the latter case the proof is complete. Suppose that $\underline{H}_f$ is continuous. Let $\overline{B}^n(x_0,r)\subset G$ and choose $C=\overline{B}^n(x_0,r)$ in Lemma 2.14. Lemma 2.14 shows that there is a sequence $v_i\in\mathscr{L}_{\overline{D}}$ such that $v_i>\underline{H}_f-1/i$ in $\overline{B}^n(x_0,r)$. Consider the functions $V_i=P(v_i,B^n(x_0,r))$. Again we have $\lim_{i\to\infty}V_i=\underline{H}_f$ uniformly in $\overline{B}^n(x_0,r)$. It follows from Harnack's principle [**GLM1**, Theorem 4.21], that $\underline{H}_f$ is an F-extremal in $B^n(x_0,r)$ and thus in G.

3. Regular boundary points. As in the classical harmonic case it is possible to define a barrier function for the boundary value problem of F-extremals. In this chapter we show that it gives a necessary and sufficient condition for the regularity of boundary points. The proof for necessity differs considerably from the linear situation. Our variational principle also gives a new proof for Bouligand's theorem [**H**, p. 169].

Let $G \subset R^n$ be a bounded domain. A point $x_0 \in \partial G$ has an F-barrier if there exists a sub-F-extremal w: $G \to R$ such that

(a) $\overline{\lim}_{x \to y} w(x) < 0$ for all $y \in \partial G$, $y \neq x_0$,

(b) $\lim_{x \to x_0} w(x) = 0$.

3.1. THEOREM. *Suppose that f: $\partial G \to R$ is bounded and continuous at $x_0 \in \partial G$. If x_0 has an F-barrier, then*

$$\lim_{x \to x_0} \underline{H}_f(x) = f(x_0).$$

Proof. The proof is completely analogous to the classical proof. Let $\varepsilon > 0$ and $M = \sup|f|$. By virtue of the assumptions there are constants $\delta > 0$ and $k < 0$ such that $|f(x) - f(x_0)| < \varepsilon$ if $|x - x_0| < \delta$ and $kw(x) \geq 2M$ if $|x - x_0| \geq \delta$. Note that the functions $f(x_0) + \varepsilon + kw$ and $f(x_0) - \varepsilon - kw$ belong to the classes $\mathscr{U}_f$ and $\mathscr{L}_f$ respectively. Observe that kw is a super-F-extremal and $-kw$ is a sub-F-extremal. Then

$$f(x_0) - \varepsilon - kw(x) \leq \underline{H}_f(x) \leq \overline{H}_f(x) \leq f(x_0) + \varepsilon + kw(x)$$

or

$$\left|\underline{H}_f(x) - f(x_0)\right| \leq \varepsilon + kw(x).$$

Since $w(x) \to 0$ as $x \to x_0$ we obtain $\underline{H}_f(x) \to f(x_0)$ as $x \to x_0$.

3.2. DEFINITION. A bounded domain $G \subset R^n$ is called F-regular, if for all continuous f: $\partial G \to R$ there is an F-extremal $u \in C(\overline{G})$ with $u|\partial G = f$.

3.3. LEMMA. *A domain $G \subset R^n$ is F-regular if and only if $\lim_{x \to y} \underline{H}_f(x) = f(y)$ for all $y \in \partial G$.*

Proof. Suppose that G is F-regular. Then for $f \in C(\partial G)$ there is u as in Definition 3.2. Since $u \in \mathscr{L}_f$ it follows that $\underline{H}_f \geq u$ in G. Let $v \in \mathscr{L}_f$ and $y \in \partial G$. Now

$$\overline{\lim_{x \to y}}\, v(x) \leq f(y) = \lim_{x \to y} u(x)$$

and the F-comparison principle implies $v \leq u$ in G. Then $\underline{H}_f = \sup\{v | v \in \mathscr{L}_f\} \leq u$ in G. We have proved that $\underline{H}_f = u$ and thus $\lim_{x \to y} \underline{H}_f(x) = \lim_{x \to y} u(x) = f(y)$ for all $y \in \partial G$. The converse is trivial.

3.4. DEFINITION. A boundary point x_0 of a bounded domain $G \subset R^n$ is called F-regular, if for all continuous $f\colon \partial G \to R$

$$\lim_{x \to x_0} \underline{H}_f(x) = f(x_0).$$

Lemma 3.3 implies

3.5. COROLLARY. *A bounded domain G is F-regular if and only if each boundary point of G is F-regular.*

Theorem 3.1 gives

3.6. COROLLARY. *A point $x_0 \in \partial G$ is F-regular if it has an F-barrier.*

The converse of Corollary 3.6 is also true.

3.7. THEOREM. *A point $x_0 \in \partial G$ is F-regular if and only if x_0 has an F-barrier.*

Proof. Suppose that $x_0 \in \partial G$ has an F-barrier. It follows from Corollary 3.6 that x_0 is F-regular. To show the converse assume that $x_0 \in \partial G$ is F-regular. Let $\overline{G} \subset B^n(x_0, R)$. We shall construct a barrier function w at x_0. For this purpose we need a continuous sub-F-extremal u in $B^n(x_0, R)$ such that $u(x_0) = 0$ and $u(x) > 0$ for $x \in B^n(x_0, R)$. The function u is constructed as a solution of an obstacle problem.

We will use the function $\varphi = |x - x_0|$ as an obstacle. Let $B = B^n(x_0, R)$ and

$$\mathcal{F}(\varphi) = \left\{ v \in C(\overline{B}) \cap W_n^1(B) | v \le \varphi \text{ in } B,\, v = \varphi \text{ in } \partial B \right\}.$$

There exists $u \in \mathcal{F}(\varphi)$ such that $I_F(u, B) = \inf\{ I_F(v, B) | v \in \mathcal{F}(\varphi)\}$, see **[GLM1**, Theorem 5.15]. Now **[GLM1**, Theorem 5.17(ii)] implies that the function u is a sub-F-extremal.

In what follows we will show that $u(x_0) = 0$ and $u(x) > 0$ for $x \in \overline{B}$, $x \neq x_0$. The function $h = \max\{u, 0\}$ belongs to the class $\mathcal{F}(\varphi)$, hence $u(x_0) = 0$ and $u \ge 0$ in $B^n(x_0, R)$. Suppose that there is $x_1 \in B^n(x_0, R)$ such that $u(x_1) = 0$ and $x_1 \neq x_0$. Now $x_1 \in \tilde{A}$, where $\tilde{A}$ is a component of the open set $\{x \in B^n(x_0, R) | u(x) < \varphi(x)\}$. Observe that u is an F-extremal in the set $\tilde{A}$ **[GLM1**, pp. 39–40]. Harnack's inequality implies that $u(x) = 0$ for $x \in \tilde{A}$. This is a contradiction. Hence $u(x) > 0$ for $x \in B^n(x_0, R) \setminus \{x_0\}$.

We are ready to construct the barrier at $x_0 \in \partial G$. Consider the function $\underline{H}_u$. Now $u \in \mathscr{L}_u$ and hence $\underline{H}_u \geq u$ in G. This yields

$$\varliminf_{x \to y \in \partial G} \underline{H}_u(x) \geq \varliminf_{x \to y \in \partial G} u(x) = u(y) > 0 \quad \text{for } y \neq x_0.$$

Since x_0 is F-regular it follows from Definition 3.4 that $\lim_{x \to x_0} \underline{H}_u(x) = u(x_0) = 0$.

For the barrier we choose the function $-\underline{H}_u$.

3.8. REMARK. The function $\underline{H}_u$ is the barrier sought in Bouligand's theorem.

4. Approximation of sub-F-extremals. In the classical potential theory it is well-known that subharmonic functions can be approximated by regular subharmonic functions. The following theorem gives a corresponding approximation result for sub-F-extremals. In particular, it follows that a general sub-F-extremal which is locally bounded from below is in the Sobolev-space $\operatorname{loc} W_n^1(G)$.

4.1. THEOREM. *Suppose u: $G \to R \cup \{-\infty\}$ is a sub-F-extremal and $D \subset\subset G$ a domain. Then there exists a decreasing sequence of sub-F-extremals $u_i \in C(\overline{D}) \cap W_n^1(D)$ such that* $\lim_{i \to \infty} u_i = u$ *in D. If u is locally bounded from below then u is in* $\operatorname{loc} W_n^1(G)$ *and*

$$\int_{\operatorname{spt} \eta} F(x, \nabla u)\, dm \leq \int_{\operatorname{spt} \eta} F(x, \nabla(u - \eta))\, dm, \tag{4.2}$$

for all non-negative $\eta \in C_0^\infty(G)$.

Proof. Since u is upper semicontinuous there exists a decreasing sequence $\varphi_i \in C^\infty(D) \cap C(\overline{D})$ such that $\lim_{i \to \infty} \varphi_i = u$ in $\overline{D}$. We may assume that the domain D is regular. We shall again employ the solutions of an obstacle problem. Choose functions u_i which minimize the integral

$$I_F(u, D) = \int_D F(x, \nabla u)\, dm \tag{4.3}$$

in the class $\mathscr{F}(\varphi_i) = \{u \in C(\overline{D}) \cap W_n^1(D) | u \leq \varphi_i \text{ in } D,\ u = \varphi_i \text{ in } \partial D\}$, see [**GLM1**, Theorem 5.15]. The functions u_i are sub-F-extremals.

Next we show that $u \leq u_i \leq \varphi_i$ in $\overline{D}$. Consider the set $A_i = \{x \in D | u_i(x) < \varphi_i(x)\}$. Let $\tilde{A}_i$ be a component of A_i. Then u_i is an F-extremal in $\tilde{A}_i$, see [**GLM1**, the proof of Theorem 5.17], and $u_i | \partial \tilde{A}_i = \varphi_i | \partial \tilde{A}_i \geq u | \partial \tilde{A}_i$. By the F-comparison principle $u_i \geq u$ in $\tilde{A}_i$ and clearly in the whole $\overline{D}$. Thus $u = \lim_{i \to \infty} \varphi_i \geq \lim_{i \to \infty} u_i \geq u$ in $\overline{D}$.

Next we prove that the sequence u_i is decreasing. Assume the contrary. Then the open set $A = \{x \in D | u_{i+1}(x) > u_i(x)\}$ is non-empty for some i. The function $\min(u_i, u_{i+1})$ belongs to the class $\mathscr{F}(\varphi_{i+1})$. Now

$$\begin{aligned} I_F(u_{i+1}, D) &= I_F(u_{i+1}, A) + I_F(u_{i+1}, D \setminus A) \\ &\le I_F(u_i, A) + I_F(u_{i+1}, D \setminus A) \end{aligned}$$

and hence $I_F(u_{i+1}, A) \le I_F(u_i, A)$. In the same way we obtain

$$\begin{aligned} I_F(u_i, D) &= I_F(u_i, A) + I_F(u_i, D \setminus A) \\ &\le I_F(\max(u_i, u_{i+1}), D) \\ &= I_F(u_{i+1}, A) + I(u_i, D \setminus A), \end{aligned}$$

i.e. $I_F(u_i, A) \le I_F(u_{i+1}, A)$. Thus $I_F(u_i, A) = I_F(u_{i+1}, A)$ and it follows from the strict convexity of the kernel F that the set A is empty and $u_{i+1} \le u_i$ in D.

Suppose u is locally bounded from below. We prove that u is in $\operatorname{loc} W_n^1(D)$. Since $u_i \in C(\overline{D}) \cap W_n^1(D)$, [**GLM1**, Theorem 5.17] implies that

$$\int_{\operatorname{spt}\eta} F(x, \nabla u_i)\, dm \le \int_{\operatorname{spt}\eta} F(x, \nabla(u_i - \eta))\, dm$$

for all non-negative $\eta \in C_0^\infty(D)$. Since u is locally bounded from below we may assume that it is non-negative in D. Then also the functions u_i are non-negative. Let $\overline{B}^n(x_0, r) \subset D$ and consider the condenser $(D, \overline{B}^n(x_0, r))$. Analogously to the proof of [**GLM1**, Lemma 4.2] it can be shown that

$$\begin{aligned} \int_{B^n(x_0, r)} |\nabla u_i|^n\, dm &\le c \operatorname{osc}(u_i, D)^n \operatorname{cap}_n(D, B^n(x_0, r)) \\ &\le L \operatorname{cap}_n(D, B^n(x_0, r)), \end{aligned} \tag{4.4}$$

where the constant L does not depend on i. This shows that the L^n-norms of ∇u_i are uniformly bounded. Hence there is a subsequence of ∇u_i converging weakly in $L^n(B^n(x_0, r))$ to the generalized gradient ∇u of u, which is in $L^n(B^n(x_0, r))$. Since the ball $B^n(x_0, r)$ was arbitrary, u belongs to $\operatorname{loc} W_n^1(D)$.

In order to prove the inequality (4.2) we show that there is a subsequence of ∇u_i such that $\nabla u_i \to \nabla u$ a.e. in compact subsets of D. The expression

$$(\nabla_h F(x, h_1) - \nabla_h F(x, h_2)) \cdot (h_1 - h_2), \tag{4.5}$$

is strictly positive for a.e. $x \in G$, and all $h_1, h_2 \in R^n$, $h_1 \neq h_2$. Since the functions u_i are sub-F-extremals in D and belong to $C(\overline{D}) \cap W_n^1(D)$, they satisfy the inequality

$$\int_{\operatorname{spt}\eta} \nabla_h F(x, \nabla u_i) \cdot \nabla \eta \, dm \leq 0, \tag{4.6}$$

for all non-negative $\eta \in C_0^\infty(D)$.

Let $\overline{B} = \overline{B}^n(x_0, r) \subset D$, $0 < r' < r$, $\zeta \in C_0^\infty(B)$, $0 \leq \zeta \leq 1$, and $\zeta(x) = 1$ for $x \in B^n(x_0, r')$. Put $\eta = \zeta(u_i - u)$ and use (4.6) to obtain

$$\begin{aligned}
&\int_{\operatorname{spt}\zeta} (\nabla_h F(x, \nabla u_i) - \nabla_h F(x, \nabla u)) \cdot \nabla(\zeta(u_i - u)) \, dm \\
&\quad = \int_{\operatorname{spt}\zeta} \zeta(\nabla_h F(x, \nabla u_i) - \nabla_h F(x, \nabla u)) \cdot (\nabla u_i - \nabla u) \, dm \\
&\quad + \int_{\operatorname{spt}\zeta} (u_i - u)(\nabla_h F(x, \nabla u_i) - \nabla_h F(x, \nabla u)) \cdot \nabla \zeta \, dm \\
&\quad = I_i^1 + I_i^2 \leq -\int_{\operatorname{spt}\zeta} \nabla_h F(x, \nabla u) \cdot \nabla(\zeta(u_i - u)) \, dm \\
&\quad = -\int_{\operatorname{spt}\zeta} (u_i - u) \nabla_h F(x, \nabla u) \cdot \nabla \zeta \, dm \\
&\quad - \int_{\operatorname{spt}\zeta} \zeta \nabla_h F(x, \nabla u) \cdot (\nabla u_i - \nabla u) \, dm.
\end{aligned}$$

Because of the inequality (4.4) we can choose a subsequence of u_i such that $u_i \to u$ in $L^n(B^n(x_0, r))$ and $\nabla u_i \to \nabla u$ weakly in $L^n(B^n(x_0, r))$, see [**M**, p. 75, Theorem 3.4.4]. Then the last two integrals and the integral I_i^2 tend to zero for $i \to \infty$. Now (4.5) yields $I_i^1 \geq 0$ and hence $\lim_{i \to \infty} I_i^1 = 0$. Then we employ the condition (4.5) to show that there is a subsequence of ∇u_i such that $\nabla u_i \to \nabla u$ for a.e. $x \in B^n(x_0, r')$. Write

$$g_i(x) = (\nabla_h F(x, \nabla u_i(x)) - \nabla_h F(x, \nabla u(x))) \cdot (\nabla u_i(x) - \nabla u(x)).$$

Then $g_i \to 0$ in $L^1(B^n(x_0, r'))$ and hence there is a subsequence such that $g_i(x) \to 0$ for a.e. $x \in B^n(x_0, r')$. It follows from (4.5) that $\nabla u_i(x) \to \nabla u(x)$ for a.e. $x \in B^n(x_0, r')$.

Finally choose a non-negative $\eta \in C_0^\infty(B^n(x_0, r'))$ in (4.6). Since the integrals

$$\int_{B^n(x_0, r')} |\nabla_h F(x, \nabla u_i)|^{n/(n-1)} \, dm$$

are uniformly bounded and $\nabla_h F(x, \nabla u_i(x)) \to \nabla_h F(x, \nabla u(x))$ a.e. in $B^n(x_0, r')$, the inequality (4.6) yields via weak convergence

$$\int_{B^n(x_0, r')} \nabla_h F(x, \nabla u) \cdot \nabla \eta \, dm \leq 0.$$

Thus the above inequality holds in D and (4.2) follows from [**GLM1**, Theorem 5.17].

5. F-harmonic measure. The PWB-method can be used in the definition of the F-harmonic measure. In [**GLM2**] the F-harmonic measure was constructed via generating sequences. This method can only be used in regular domains.

Suppose $G \subset R^n$ is a bounded open set. Let $C \subset \partial G$ be a closed set and let $f\colon \partial G \to R$ be the characteristic function of C. The function $\overline{H}_f$, which is an F-extremal, is the F-harmonic measure of C with respect to G. The next theorem shows that in regular domains this concept gives the same F-harmonic measure.

5.1. THEOREM. *Suppose that $G \subset R^n$ is a regular domain, and that $C \subset \partial G$ is a closed set. If f is the characteristic function of C, then $\overline{H}_f = \omega(C, G; F)$, where $\omega(C, G; F)$ is the F-harmonic measure as in* [**GLM2**, *Definition* 2.16].

Proof. Let φ_i be a (C, G)-boundary sequence, see [**GLM2**, pp. 235–236]. Consider the F-extremals $u_i \in C(\overline{G}) \cap W_n^1(G)$ with $u_i|\partial G = \varphi_i|\partial G$. It was shown in [**GLM2**, pp. 3–4] that $\lim_{i\to\infty} u_i = \omega(C, G; F)$ locally uniformly in G. Now $u_i \in \mathscr{U}_f$ and hence $u_i \geq \overline{H}_f$. Thus $\omega(C, G; F) = \lim_{i\to\infty} u_i \geq \overline{H}_f$. On the other hand, for $u \in \mathscr{U}_f$, $\underline{\lim}_{x\to y\in\partial G}\, u(x) \geq f(y) \geq \overline{\lim}_{x\to y\in\partial G}\, \omega(C, G; F)$, see [**GLM2**, Remark 2.20]. Lemma 2.3 implies that $u \geq \omega(C, G; F)$, in G and thus $\overline{H}_f \geq \omega(C, G; F)$, which together with the previous inequality completes the proof.

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