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EXTENDED ADAMS-HILTON'S CONSTRUCTION

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Let $F \xrightarrow{J} E \xrightarrow{p} B$ be a Hurewicz fibration. The homotopy lifting property defines (up to homotopy) an action of the H-space ΩB on the fibre F which makes $H_*(F)$ into a $H_*(\Omega B)$-module. Suppose B is connected. We prove that if $E \xrightarrow{p} B$ is the cofibre of a map g: $W \to E$ where W is a wedge of spheres, then the reduced homology of F, $\tilde{H}_*(F)$ is a free $H_*(\Omega B)$-module generated by $\tilde{H}_*(W)$. This result implies in particular a characterization of aspherical groups.

The key point in the proof of this theorem is the following generalization of the Adams-Hilton construction. In their famous paper, Adams and Hilton construct for every simply connected C.W. complex B a graded differential algebra whose homology computes the algebra $H_*(\Omega B)$. Extending their construction to any fibration p we construct a differential graded module $C(F)$ whose homology computes the $H_*(\Omega B)$-module $H_*(F)$. We suppose E is a subcomplex of B, then $C(F)$ is a free $H_*(\Omega B)$-module generated by the cells of E. The differential is defined inductively on generators in accordance with the way the cells of E are attached.

Our construction has many applications. For instance, let $\tilde{K} \xrightarrow{p} K$ be a normal covering of a finite C.W. complex. $\tilde{K}$ is the homotopy fibre of some classifying map $K \to K(G,1)$. As $H_*(\Omega K(G,1))$ is isomorphic to $\mathbf{Z}[G]$, our construction yields an explicit chain complex whose homology computes the homology of $\tilde{K}$ as a $\mathbf{Z}[G]$-module. In particular, we establish some properties of infinite cyclic coverings in low dimensions.

1. The algebra structure of $H_*(\Omega X;\ R)$. Let X be an arcwise connected space with x_0 as base point. For sake of simplicity, we denote by G the fundamental group $\pi_1(X, x_0)$. Then

$$\Omega X = \coprod_{g \in G} (\Omega X)_g$$

where $(\Omega X)_g$ denotes the arcwise connected component of ΩX whose elements are the based loops γ belonging to the homotopy class g.

We denote by e the homotopy class of the constant loop at x_0. For each $\gamma \in g$, the homotopy equivalence

$$L_\gamma\colon (\Omega X)_e \to (\Omega X)_g$$

defined by $L_\gamma(\omega) = \gamma * \omega$, induces for each ring R a unique R-module isomorphism $(L_g)_*$: $H_*((\Omega X)_e;\ R) \to H_*((\Omega X)_g;\ R)$. Let $R[G]$ be the group ring of G. If $g = \Sigma_i \lambda_i g_i$ belongs to $R[G]$ and f belongs to $H_*((\Omega X)_e;\ R)$, the map

$$\Phi\colon H_*((\Omega X)_e;\ R) \otimes R[G] \to H_*(\Omega X;\ R)$$

defined by

$$\Phi(f, g) = \sum_i \lambda_i (L_{g_i})_*(f)$$

is an isomorphism of R-module.

Moreover, Φ is an algebra isomorphism when $H_*(\Omega X;\ R)$ is equipped with the canonical Pontryagin algebra structure and if the product in $H_*(\Omega X_e;\ R) \otimes R[G]$ is given by the formula

$$(f_1, g_1)(f_2, g_2) = f_1 f_2^{g_1} \otimes g_1 g_2,$$

where $f^g \in H_*((\Omega X)_e;\ R)$ denotes the image of f by the unique homomorphism $H_*((\Omega X)_e;\ R) \to H_*((\Omega X)_e;\ R)$ induced by the conjugation map $\omega \mapsto \gamma\omega\gamma^{-1}$ with $\gamma \in g$.

REMARKS. (1) Suppose that X admits a universal covering $p\colon \tilde{X} \to X$, then $\Omega p\colon \Omega\tilde{X} \to (\Omega X)_e$ is an isomorphism of topological monoids.

(2) By the natural inclusion $(\Omega X)_e \to \Omega X$, $H_*((\Omega X)_e;\ R)$ is a subalgebra of $H(\Omega X;\ R)$, and so $H_*(\Omega X;\ R)$ is a free left module on the ring $H(\Omega\tilde{X};\ R)$.

(3) The conjugation map $\omega \to \gamma\omega\gamma^{-1}$ in $(\Omega X)_e$ corresponds via Ωp to the map in $\Omega\tilde{X}$ defining the operation of $\pi_1(X, x_0) = G$ on $\pi_n(X, x_0)$.

(4) If R is a field of characteristic zero, then by the Milnor-Moore theorem **[10]** the Hopf algebra $H(\Omega\tilde{X};\ R)$ is isomorphic to the enveloping algebra $U(\pi(\Omega\tilde{X}) \otimes R)$. In this case Φ induces a Hopf algebra isomorphism

$$H_*(\Omega X;\ R) \cong U(\pi_{\geq 1}(\Omega X) \otimes R) \otimes R[G]$$

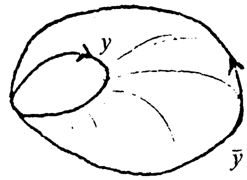
where the operation of $R[G]$ on $U(\pi_{\geq 1}(\Omega X) \otimes R)$ is induced by the natural operation of $\pi_1(X, x_0)$ on $\pi_{\geq 2}(X, x_0)$.

2. Adams-Hilton construction in the non-simply connected case. Recall the Baues' construction **[2]**.

Let K be a 0-reduced CW complex. There exists a 0-reduced CW complex $\overline{K}$ together with a homotopy equivalence

$$q\colon \overline{K} \to K$$

such that the attaching map of a 2-cell of $\overline{K}$ belongs to the free monoid generated by the 1-cells of $\overline{K}$. In order to do that, replace each 1-sphere in K^1 by the 2-dimensional complex

$\overline{S^1}$ = 

with one 2-cell and two 1-cells y and $\bar{y}$. The attaching maps of the 2-cell is $y\bar{y}$. The attaching maps of the n-cells of K define attaching map of $\overline{K}$ and for the cellular chains complex of $\overline{K}$ we have the relations:

$$\tilde{C}_n(\overline{K}) = \begin{cases} C_1(K) \oplus C_1(K), & n = 1, \\ C_2(K) \oplus sC_1(K), & n = 2, \\ C_n(K), & n \geq 3. \end{cases}$$

THEOREM 1 [**2**, D3.7 and 3.16]. *Let K be a* 0-*reduced CW-complex. There is a differential d on $T(s^{-1}\tilde{C}_*(\overline{K}))$ together with a weak equivalence of chain algebras*

$$v\colon A(\overline{K}) = T\big(s^{-1}\tilde{C}_*(\overline{K})\big) \to C_*(\Omega\overline{K}).$$

Moreover, the construction of d and v is inductive. Assume constructed v_n: $A(K^n) \to C_*(\Omega K^n)$ then for each $(n+1)$-cell e, with attaching map f: $S^n \to X$, put $ds^{-1}e = z$ where $(v_n)_*[z] = (\Omega f)_*(\xi)$ with ξ a generator of $H_{n-1}(\Omega S^{n-1})$.

Each 1-cell y of $\overline{K}$ yields a loop $y \in \Omega K \subset C_0(\Omega K)$. Then $v(s^{-1}y) = y$.

For a 2-cell e in $\overline{K}$, $ds^{-1}e = \alpha - 1$, where α is an element of the free monoid generated by the 1-cells of K, representing the attaching map of e.

REMARK. These formulas differ slightly from the Baues' ones. (Simply, subtitute formally y by $y + 1$).

Now, consider the canonical fibration

$$\Omega\overline{K} \xrightarrow{j} P\overline{K} \xrightarrow{p} \overline{K}.$$

Let denote by $S_*(\Omega\overline{K})$ (resp. $S_*(\overline{K})$, $S_*(P\overline{K})$). The singular chain group generated by non-degenerated cubes (resp. whose vertices are at the base point, in $\Omega(\overline{K})$). Following, the original Adams-Hilton construction it is easy now to obtain.

THEOREM 2. *If K is a 0-reduced CW-complex there is a commutative diagram of augmented chain complexes*

$$\begin{array}{ccc}
\left(A(\overline{K}), d\right) & \xrightarrow{v} & S_*\left(\Omega(\overline{K})\right) \\
r\downarrow & & \downarrow j \\
\left(B(\overline{K}) \otimes A(\overline{K}), d\right) & \xrightarrow{\theta_1} & S_*\left(P(\overline{K})\right) \\
\pi\downarrow & & \downarrow p \\
\left(B(\overline{K}), \bar{d}\right) & \xrightarrow{\theta} & S_*(\overline{K})
\end{array}$$

with $B(\overline{K}) = \mathbf{Z} \oplus \tilde{C}(\overline{K})$, such that

1. *v is a homomorphism of $\mathbf{Z}$-algebras;*
2. *θ_1 is a homomorphism of differential modules;*
3. *The induced maps v_*, $(\theta_1)_*$, θ_* are isomorphisms.*

REMARKS. (a) Denote by Λ_n the set of n-dimensional cells. Then $\langle t_\alpha, \alpha \in \Lambda_1;\ r_\beta, \beta \in \Lambda_2 \rangle$ is a presentation of the fundamental group G of K. This defines a group extension:

$$1 \to H \to F \to G \to 1$$

where F denotes the free group $\langle t_\alpha, \alpha \in \Lambda_1 \rangle$ and H the normal subgroup of F generated by the elements r_β, $\beta \in \Lambda_2$.

The group ring $\mathbf{Z}[F]$ is an augmented $\mathbf{Z}$-algebra concentrated in degree zero. We denote by

$$\hat{A}(K) = \mathbf{Z}[F] * T\left(s^{-1}C_{\leq 2}(K)\right)$$

the free product of the two associative $\mathbf{Z}$-algebras. As $A(\overline{K}) = T(s^{-1}C_1(K) \oplus s^{-1}C_1(K) \oplus C_1(K) \oplus s^{-1}C_{\geq 2}(K))$, the homomorphism ρ: $A(\overline{K}) \to \hat{A}(K)$ defined by

$$\rho(t_\alpha) = t_\alpha, \quad \rho(\bar{t}_\alpha) = t_\alpha^{-1}, \quad \rho(C_1(K)) = 0, \quad \rho|_{s^{-1}C_{\geq 2}} = \mathrm{id}$$

induces an isomorphism in homology. If K is countable, Milnor constructs a topological group $G(K)$ which has the homotopy type of $\Omega(K)$. In this case it is possible to construct directly an equivalence of chain algebras, between $(\hat{A}(K), D)$ and $S_*(G(K))$.

(b) As in the classical construction, we define on the chain complex $B(\overline{K}) \otimes A(\overline{K})$ (resp. $B(\overline{K}) \otimes \hat{A}(K)$) an ε-derivation s such that

$$sd + ds = 1 - \varepsilon$$

where ε denotes the augmentation of the complex.

In partiuclar, using Fox calculus we obtain in $B(\overline{K}) \otimes \hat{A}(K)$ the following relations, in low degrees;

$$\begin{aligned} dt_i &= 0, && i \in \Lambda_1, \\ d1 \otimes v_j^1 &= 1 \otimes r_j - 1 \otimes 1, && j \in \Lambda_2, \\ db_i^1 \otimes 1 &= 1 \otimes t_i - 1 \otimes 1, && i \in \Lambda_1, \\ db_j^2 \otimes 1 &= 1 \otimes v_j^1 - \sum b_i^1 \otimes \frac{\partial r_j}{\partial t_i}, && j \in \Lambda_2, \end{aligned}$$

where

$$\begin{aligned} \hat{A}(K) &= \mathbf{Z}[t_i, t_i^{-1}] * \langle v_j^l \rangle, && i \in \Lambda_1,\ j \in \Lambda_l,\ l \geq 2, \\ B(\overline{K}) &= (1, b_j^k), && j \in \Lambda_k,\ k \geq 1. \end{aligned}$$

NOTATIONS. $\langle v_\alpha \rangle$, $\alpha \in \Lambda$ denotes the free group (resp. the free association algebra) generated by the v_α's when the degree of the v_α's is zero (resp. is positive) (b_α), $\alpha \in \Lambda$ denotes the abelian group freely generated by the b_α's.

EXAMPLES.

EXAMPLE 1. $K = P^4(\mathbf{R})$,

$$\hat{A}(K) = \left(\mathbf{Z}[t, t^{-1}] * \langle v_1, v_2, v_3 \rangle, d\right),$$

$$dt = 0,\ dv_1 = t^2 - 1,\ dv_2 = tv_1t^{-1} - v_1,$$

$$dv_3 = tv_2t^{-1} + v_2 - v_1^2t^{-2}.$$

EXAMPLE 2. $K = S^1 \times S^2$,

$$\hat{A}(K) = \left(\mathbf{Z}[t, t^{-1}] * \langle v_1, v_2 \rangle, d\right),$$

$$dv_1 = 0,\ dv_2 = tv_1t^{-1} - v_1.$$

Therefore the natural projection $\hat{A}(K) \twoheadrightarrow (Z[t, t^{-1}] \otimes \langle v_1 \rangle,\ 0)$ is a quasi-isomorphism.

3. Adams-Hilton construction for homotopy fiber and applications.

3.1. Let f: $K \to L$ be a cellular map between 0-reduced C.W. complexes. Denote by g: $F \to K$ the homotopy fibre of f and by δ the connecting homomorphism in the Puppe sequence.

THEOREM 2. *With the notations introduced in* §2, *there is a commutative diagram of augmented chain complexes*

$$\begin{array}{ccc}
(A(\bar{L}), d) & \xrightarrow{v_L} & S_*(\Omega\bar{L}) \\
\downarrow r & & \downarrow \delta \\
(B(\bar{K}) \otimes A(\bar{L}), d) & \xrightarrow{\Psi} & S_*(F) \\
\downarrow 1 \otimes \varepsilon & & \downarrow f \\
(B(\bar{K}), \bar{d}) & \xrightarrow{\theta_K} & S_*(\bar{K})
\end{array}$$

such that Ψ *is a homomorphism of differential modules and* Ψ_* *is an isomorphism.*

Proof. Clearly we may suppose that f is an inclusion. We have only to define d and Ψ on $B(\bar{K}) \otimes A(\bar{L})$. d is defined as the restriction of the differential d of $B(\bar{L}) \otimes A(\bar{L})$ to $B(\bar{K}) \otimes A(\bar{L})$. This is possible since f is an inclusion. The cellular construction of Theorem 2.2 shows that the restriction of $\theta_1(L)$ to $B(\bar{K}) \otimes A(\bar{L})$ factors into a homomorphism of differential modules Ψ, making commutative the above diagram.

(i) Suppose that $K = V_\alpha S^1_\alpha$ and denote by $\Omega\bar{L} \to F' \to K$ the induced fibration by the inclusion $K \to \bar{L}$. Then we obtain a commutative diagram

$$\begin{array}{ccc}
B(K) \otimes A(\bar{L}) & \xrightarrow{\Psi_K} & S_*(F') \\
j \downarrow & & \downarrow j' \\
B(\bar{K}) \otimes A(\bar{L}) & \underset{\Psi_{\bar{K}}}{\rightarrow} & S_*(F).
\end{array}$$

As j_* and j'_* are isomorphism, it suffices to prove that $(\Psi_K)_*$ is an isomorphism.

The Leray-Serre spectral sequence of the fibration $\Omega\bar{L} \to F' \to K$ on one hand and the spectral sequence obtained using the filtration $B_{\leq p}(K) \otimes A(\bar{L})$ on the other hand, yield the commutative diagram

$$\begin{array}{ccccccccc}
\to & H_{q+1}(B(K) \otimes A(\bar{L})) & \to & B_1(K) \otimes H_q(A(\bar{L})) & \xrightarrow{d_1} & H_q(A(\bar{L})) & \to & H_q(B(K) \otimes A(L)) & \\
& \downarrow (\Psi_K)_* & & \cong \downarrow \theta_1 \otimes (v_L)_* & & \cong \downarrow (v_L)_* & & \downarrow (\Psi_K)_* & \\
\to & H_{q+1}(F) & \to & C_1(K) \otimes H_q(\Omega\bar{L}) & \to & H_q(\Omega L) & \to & H_q(F) & \to
\end{array}$$

So, from the five lemma we deduce that Ψ_* is an isomorphism.

(ii) Suppose we have proved Theorem 3 for C.W. complexes of dimension less or equal to n and let $K = K^{n+1}$. The following diagram defines then F' as the total space of a pull-back fibration

$$\begin{array}{ccccc} \Omega(\bar{L}) & = & \Omega(\bar{L}) & = & \Omega(\bar{L}) \\ \downarrow & & \downarrow & & \downarrow \\ F' & \to & F & \to & P(\bar{L}) \\ \downarrow & & \downarrow & & \downarrow p \\ K^n & \to & \bar{K} & \xrightarrow{f} & \bar{L} \end{array}$$

So obtain the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & S_*(F') & \to & S_*(F) & \to & S_*(F)/S_*(F') & \to & 0 \\ & & \uparrow \Psi' & & \uparrow \Psi & & \uparrow \bar{\Psi} & & \\ 0 & \to & B(\overline{K^n}) \otimes A(\bar{L}) & \to & B(\bar{K}) \otimes A(\bar{L}) & \to & B(\bar{K}) \otimes A(\bar{L})/B(\overline{K^n}) \otimes A(\bar{L}) & \to & 0 \end{array}$$

From the inductive assumption and the five lemma it suffices to prove that $(\bar{\Psi})_*$ is an isomorphism.

We denote by $\chi\colon (E^{n+1}, S^n) \to (\bar{K}, \overline{K^n})$ the characteristic map of the cell e, and suppose that $\bar{K} = \overline{K^n} \cup e$.

Now from the commutativity of the diagram

$$\begin{array}{ccc} \left(B(\bar{K})_{/B(\overline{K^n})}\right) \otimes A(\bar{L}) & \xrightarrow{\sim} & \left(B(\overline{L^n} \cup e)_{/B(\overline{L^n})}\right) \otimes A(\bar{L}) \\ \bar{\Psi} \downarrow & & \bar{\theta}_1 \downarrow \\ S_*(F)/S_*(F') & \xrightarrow{\sim} & S_*\left(p^{-1}(\overline{L^n} \cup e),\, p^{-1}(\overline{L^n})\right) \end{array}$$

where the two horizontal maps are quasi-isomorphisms, we might as well suppose that

$$\overline{K^n} = \overline{L^n} \quad \text{and} \quad \bar{\Psi} = \bar{\theta}_1 \qquad (\theta_1, \text{as in Th. 2}).$$

Now, let us recall the construction of $\theta_1\colon B(\bar{L}) \otimes A(\bar{L}) \to S_*(P\bar{L})$. We denote by ζ a cycle of $S_*(\Omega S^n)$ corresponding by homology suspension to a generator of $H_n(S^n)$. Let $\xi \in S_n(PS^n)$ and $\eta \in S_n(\Omega E^{n+1})$ such that $d\xi = \zeta$ and $d\eta = \zeta$ when ζ is considered as an element of $S_*(PS^n)$ or of $S_*(\Omega E^{n+1})$. Considering now, all these chains in $S_*(PE^{n+1})$ we obtain the relation $d\kappa = \xi - \eta$ for some $\kappa \in S_{n+1}(PE^{n+1})$. Now θ_1 is defined such that

$$\theta_1(e \otimes 1) + P\chi(\kappa) \in S_{n+1}(P\bar{L}^n) \subset S_{n+1}(p^{-1}\bar{L}^n)$$

with $P\chi$: the canonical map $PE^{n+1} \to PK \hookrightarrow PL$.

From this formula we deduce the following commutative diagram,

$$
\begin{tikzcd}
B(\overline{L^n} \cup e) \otimes A(\overline{L}) \arrow[r, "\theta_1"] \arrow[d] & S_*\left(p^{-1}(\overline{L^n} \cup e)\right) \arrow[d] \\
\left(B(\overline{L^n} \cup e)/_{B(\overline{L^n})}\right) \otimes A(L) \arrow[r, "\overline{\theta}_1"] \arrow[d, "\alpha"'] & S_*\left(p^{-1}(\overline{L^n} \cup e),\, p^{-1}(\overline{L^n})\right) \\
S_*\left(PE^{n+1}, \Omega E^{n+1} \bigcup_{\Omega S^n} PS^n\right) \otimes S_*(\Omega\overline{L}) \arrow[d, "\mu"'] & \\
S_*\left(\chi^{-1}(p),\, \chi^{-1}(p)|_{S^n}\right) \arrow[ruu, bend right, "\chi'"'] &
\end{tikzcd}
$$

where,

(i) $\alpha = \gamma \otimes v_{\overline{L}}$ with $\gamma(e) = -\rho(\kappa)$ and ρ is the canonical map $S_*(PE^{n+1}) \to S_*(PE^{n+1}, \Omega E^{n+1} \cup_{\Omega S^n} PS^n)$.

(ii) χ' is defined by the following diagram

$$
\begin{tikzcd}
\left(\chi^{-1}(p), \chi^{-1}(p)|_{S^n}\right) \arrow[r, "\chi'"] \arrow[d, "p'"] & \left(p^{-1}(\overline{L^n} \cup e),\, p^{-1}(\overline{L^n})\right) \arrow[d, "p"] \\
(E^{n+1}, S^n) \arrow[r, "\chi"] & (\overline{L^n} \cup e, \overline{L^n})
\end{tikzcd}
$$

(iii) μ is induced by the homotopy equivalence

$$\left(PE^{n+1}, \Omega E^{n+1} \bigcup_{\Omega S^n} PS^n\right) \times \Omega\overline{L} \xrightarrow{\mu} \left(\chi^{-1}(p), \chi^{-1}(p)|_{S^n}\right)$$

with $\mu(c) = (P\chi(c), c(1))$ if $c \in PE^{n+1}$ and extended using the operation of $\Omega\overline{L}$ on $\chi^{-1}(p)$.

By excision χ'_* is an isomorphism and since α_* and μ_* are also isomorphisms, so is $(\overline{\theta}_1)_*$.

3.2. *Fibre of a cofibre.*

PROPOSITION. *Let K and L be connected C.W. complexes. If $f\colon K \to L$ is the cofibre of a map $\bigvee_\alpha S^{n_\alpha} \to K$ and F the homotopy fibre of f, then $H_+(F)$ is a free $H_*(\Omega L)$-module generated by $H_+(\bigvee_\alpha S^{n_\alpha})$.*

Proof. The 1-connected version of this theorem soon appears in **[6]**. Nevertheless, for the convenience of the reader, we sketch the proof again. By 3.1,

$$H_*(F) \cong H_*(B(K) \otimes A(L)).$$

Consider then the exact sequence of differential chain complexes

$$(*)\quad 0 \to (B(K) \otimes A(L), d) \to (B(L) \otimes A(L), d) \to (B(L)/B(K) \otimes A(L), \bar{d}) \to 0$$

The inductive property of the Adams-Hilton construction shows that:

$$H_*(B(L)/B(K) \otimes A(L), \bar{d}) \cong B(L)/B(K) \otimes H_*(A(L))$$

The long exact sequence induced by $(*)$ is an exact sequence of $H_*(A(L))$-modules. So on we obtain an isomorphism of $H_*(A(L))$-modules

$$B(L)/B(K) \otimes H_*(A(L)) \to H_*(B(K) \otimes A(L)). \qquad \square$$

3.3. *Coverings*. Let K be a connected finite C.W. complex and $H \to \pi_1(K) = G$ a normal subgroup with quotient group $N = G/H$. Denote by θ_2: $\hat{A}(K) \to C_*(G(K))$ an Adams-Hilton model of K, by $\tilde{K} \to K$ a covering corresponding to H and by $\tau\nu_K$: $\hat{A}(K) \to \mathbf{Z}[\pi_1(K)] \to \mathbf{Z}[N]$ the composite of the canonical projections. The following proposition results then directly from Theorem 3.

PROPOSITION.

$$B(K) \otimes \mathbf{Z}[N] \underset{\text{def}}{=} (B(K) \otimes \hat{A}(K)) \otimes_{\hat{A}(K)} \mathbf{Z}[N]$$

is a chain complex whose homology is isomorphic to $H_*(\tilde{K};\, \mathbf{Z})$ *as* $\mathbf{Z}[N]$-*module*.

Proof. The homotopy fibre of the inclusion

$$K \hookrightarrow L = K(N,1)$$

has the homotopy type of $\tilde{K}$. From Theorem 3 and the definition of ρ: $A \to \mathrm{i}\hat{\mathrm{A}}$ we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathbf{Z}[N] \arrow[d] & A(L) \arrow[l, "\nu_L"', "\sim"] \arrow[d] & (A(\bar{L}), d) \arrow[l, "\rho"', "\sim"] \arrow[r, "\nu_L", "\sim"'] \arrow[d] & S_*(\Omega \bar{L}) \arrow[d] \\
B(\bar{K}) \otimes \mathbf{Z}[N] \arrow[drr] & B(\bar{K}) \otimes \hat{A}(L) \arrow[l, "1 \otimes \nu_L"] \arrow[dr] & (B(\bar{K}) \otimes A(\bar{L}), d) \arrow[l, "1 \otimes \rho"'] \arrow[r, "\Psi", "\sim"'] \arrow[d] & S_*(\tilde{K}) \arrow[d] \\
 & & (B(\bar{K}), \bar{d}) \arrow[r, "\sim"'] & S_*(K)
\end{tikzcd}$$

It is easy, then to prove that $1 \otimes \nu_L$ and $1 \otimes \rho$ induce isomorphisms at the homological level.

If we choose α, $\hat{A}(K) \to \hat{A}(\bar{L})$ such that $\tau\nu_K = \alpha\nu_L$, and if we define a $\hat{A}(K)$-module structure on $\mathbf{Z}[N]$ with $\tau\nu_K$, we obtain a commutative diagram

$$\begin{array}{ccc} B(\bar{K}) \otimes \hat{A}(\bar{K}) \otimes_{\hat{A}(\bar{K})} \hat{A}(K) & \underset{\mu}{\to} & B(\bar{K}) \otimes \hat{A}(L) \\ \downarrow 1 \otimes \nu_L & & \downarrow 1 \otimes \nu_L \\ B(\bar{K}) \otimes \hat{A}(\bar{K}) \otimes_{\hat{A}(\bar{K})} \mathbf{Z}[N] & \underset{\mu'}{\to} & B(\bar{K}) \otimes \mathbf{Z}[N] \end{array}$$

where the canonical isomorphisms μ and μ' commute with differentials, and so induce isomorphisms between homologies. □

With the notations of remark (b) below Theorem 2, the differential d of the complex $B(K) \otimes \mathbf{Z}[N]$ is defined in low degrees as follow:

$$d\left(b_i^1 \otimes 1\right) = 1 \otimes [t_1] - 1 \otimes 1,$$

$$d\left(b_j^2 \otimes 1\right) = -\sum b_i^1 \otimes \left[\frac{\partial r_j}{\partial t_i}\right],$$

where $[\alpha]$ denotes the image of α by the projection $\mathbf{Z}[t_i, t_i^{-1}] \to \mathbf{Z}[N]$. So we recover the classical formulaes of [**5**].

3.4. *Infinite cyclic coverings in low dimensions.* Let $\tilde{K} \to K$ be an infinite cyclic covering of a 0-reduced finite C.W. complex K. Denote by $\mathscr{A}$ the matrix $([\partial r_j/\partial t_i])$ defined in 3.3 and by rank $\mathscr{A}$ the maximal r such that there exists in $\mathscr{A}$ a non-zero $r \times r$ minor. Then

PROPOSITION. *If $\tilde{K} \to K$ is a connected infinite cyclic covering of a 0-reduced finite C.W. complex, then $H_1(\tilde{K}; \mathbf{Q})$ is finite dimensional if and only if* rank $\mathscr{A} = n - 1$, *where n is the number of 1-cells in K.*

Proof. $H_1(\tilde{K})$ is a finitely generated $\mathbf{Z}[t, t^{-1}]$-module. If we write,

$$H_1(\tilde{K}) = \frac{\mathbf{Z}[t, t^{-1}]}{(\alpha_1)} \oplus \cdots \oplus \frac{\mathbf{Z}[t, t^{-1}]}{(\alpha_r)}$$

$H_1(\tilde{K}; \mathbf{Q})$ will be finite dimensional if and only if all $\alpha_i \neq 0$, and so if and only if

$$H_1(\tilde{K}) \otimes_{\mathbf{Z}[t, t^{-1}]} \mathbf{Q}(t) = 0.$$

Tensoring the complex $C_*(\tilde{K})$ by the field $\mathbf{Q}(t)$ over $\mathbf{Z}[t, t^{-1}]$, we obtain a chain complex of $\mathbf{Q}$-vector spaces

$$(*) \qquad 0 \leftarrow C_0(\tilde{K}) \otimes_{\mathbf{Z}[t, t^{-1}]} \mathbf{Q}(t) \overset{\partial_1}{\leftarrow} C_1(\tilde{K}) \otimes_{\mathbf{Z}[t, t^{-1}]} \mathbf{Q}(t)$$

$$\overset{\partial_2}{\leftarrow} C_2(\tilde{K}) \otimes_{\mathbf{Z}[t, t^{-1}]} \mathbf{Q}(t) \leftarrow$$

(whose Euler characteristic coincide with $\chi(K)$). As $H_0(\tilde{K}) = \mathbf{Z}$, $H_0(\tilde{K}) \otimes_{\mathbf{Z}[t,t^{-1}]} \mathbf{Q}(t) = 0$ and $\dim \operatorname{Im} \partial_1 = 1$. So rank $\mathscr{A} = \dim \operatorname{Im} \partial_2 = n - 1$ if and only if

$$H_1\big(C_*(\tilde{K}) \otimes_{\mathbf{Z}[t,t^{-1}]} \mathbf{Q}(t)\big) = H_1(\tilde{K}) \otimes_{\mathbf{Z}[t,t^{-1}]} \mathbf{Q}(t) + 0.$$

COROLLARY 1. *Let K be a 0-reduced finite 2-dimensional C.W. complex whose Euler characteristic is zero and satisfying rank $\mathscr{A} = n - 1$ (n = number of 1-cells). If $\tilde{K} \to K$ is a connected infinite cyclic covering, then $H_i(\tilde{K}; \mathbf{Q})$ is finite dimensional for each i.*

Proof. In the chain complex $(*)$ as $\chi(K) = 0$, ∂_2 becomes injective, so $\dim H_2(\tilde{K}; \mathbf{Q})$ and $\dim H_*(\tilde{K}; \mathbf{Q})$ are finite.

COROLLARY 2. *Let K be a 0-reduced finite 3-dimensional C.W. complex satisfying*

(i) *K satisfies Poincaré Duality with rational coefficients*

(ii) *rank $\mathscr{A} + 1$ = number of 1-cells.*

Then each connected infinite cyclic covering $\tilde{K}$ has the rational homotopy type of a compact manifold.

Proof. In this proof we assume a lot of material and notation from S. Halperin's paper [8]. Consider the K.S. model [9, 20–2] of the classifying map $\varphi\colon K \to S^1$ of the covering $\tilde{K}$:

$$(\Lambda t, 0) \to (\Lambda t \otimes \Lambda V, D) \to (\Lambda V, \overline{D})$$

In [7] we show that $\dim_{\mathbf{Q}} H^i(\Lambda V; \overline{D}) < \infty$ if and only if $\dim H_i(\tilde{K}; \mathbf{Q}) < \infty$. From the duality assumption we deduce a surjective quasi-isomorphism

$$(\Lambda t \otimes \Lambda V, D) \xrightarrow{\theta} (A, D)$$

such that $A^{>3} = 0$ and $A^3 = \mathbf{Q}U$. Moreover, since K is arcwise connected, $H_1(\varphi) \neq 0$ and there exist a cocycle $v \in \Lambda V$ such that $\theta(tv) = U$. Consider now the c.d.g.a. $(\Lambda t \otimes \Lambda V \otimes \Lambda \bar{t}, D')$ with $D'(\bar{t}) = t$, $D'|_{\Lambda t \otimes \Lambda V} = D$, $\deg(\bar{t}) = 0$. Denote now by $(A \otimes \Lambda \bar{t}, D)$ the tensor product of the two commutative differential graded algebras

$$(A, D) \otimes_{(\Lambda t \otimes \Lambda V)} (\Lambda t \otimes \Lambda V \otimes \Lambda \bar{t}, D').$$

Clearly, $(A \otimes \Lambda \bar{t}, D)$ is quasi-isomorphic to $(\Lambda V, \overline{D})$. Now $(A \otimes \Lambda \bar{t})^3 = \mathbf{Q}U \otimes \Lambda \bar{t}$. As $U \otimes \bar{t}^n = D(\theta(v)\bar{t}^{n-1}/n)$ for $n \geq 1$, $H^3(\Lambda V, \overline{D}) = \mathbf{Q}$ and thus $H_3(\tilde{K}; \mathbf{Q})$ is finite dimensional.

On the other hand, the above proposition shows that $H_1(\tilde{K}; \mathbf{Q})$ and $H_0(\tilde{K}; \mathbf{Q})$ are finite dimensional. As $\chi(K) = 0$, in the chain complex $(*)$ we obtain $H_2(\tilde{K}) \otimes_{\mathbf{Z}[t,t^{-1}]} \mathbf{Q}(t) = 0$, so $H_2(\tilde{K})$ is also finite dimensional.

The corollary results then of the Milnor theorem ([**11**]).

3.5. *Aspherical groups*. Let (W, w_0) be a wedge of S^1's and let X be obtained by attaching 2-cells to W:

$$X = W \cup \left(\bigcup_{i \in I} e_i^2 \right).$$

For each, $k \in I$, φ_k: $S^1 \to W$ denotes the attaching map of the 2-cell e_k^2.

Let N_X be the normal subgroup of $\pi_1(W, *)$ generated by the homotopy classes $[\varphi_k]$, $k \in I$.

Note that the group extension

$$1 \to N_X \to \pi_1(W, w_0) \xrightarrow{(i_{WX})_\#} \pi_1(X, w_0) \to 1$$

induces on the abelianized group $(N_X)_{\mathrm{ab}}$ a canonical structure of $\mathbf{Z}[\pi_1(X)]$-module. Denote by ϕ_i the image of $[\varphi_i]$ in $(N_X)_{\mathrm{ab}}$.

PROPOSITION. $(i_{WX})_\#$: $\pi_1(W, w_0) \to \pi_1(X, w_0)$ *is surjective iff* $(N_X)_{\mathrm{ab}}$ *is freely generated by the* ϕ_i*'s as* $\mathbf{Z}[\pi_1(X)]$*-module.*

Proof. We denote by j: $F_X \to W$ the homotopy fibre of i_{WX}. Then each φ_i, $i \in I$, factorises into $\overline{\varphi}_i$: $S^1 \to F_X$ and so induces $\overline{\Phi}_i$ belonging to $H_1(F_X)$. From 3.2, the reduced homology $H_+(F_X)$ is freely generated as $H_*(\Omega X)$-module by the $\overline{\Phi}_i$'s. An argument of degree shows that $H_1(F_X)$ is isomorphic to $\oplus_{i \in I} \mathbf{Z}[\pi_1(X)]\overline{\Phi}_i$, since $H_0(\Omega X) = \mathbf{Z}[\pi_1(X)]$.

(a) If $(i_{WX})_\#$ is surjective, then F_X has the homotopy type of a wedge of S^1's and so

$$(N_X)_{\mathrm{ab}} = H_1(F_X).$$

(b) In order to prove the "only if" direction first remark that the exact sequence

$$0 \to \pi_2(X) \to \pi_1(F_X) \xrightarrow{j_\#} N_X \to 0$$

obtained from the homotopy fibration $F_X \xrightarrow{j} W \xrightarrow{i} X$ naturally splits.

Now, if we suppose that $(N_X)_{\mathrm{ab}}$ is a $\mathbf{Z}[\pi_1(X, w_0)]$-module freely generated by the ϕ_i's then $H_1(F_X)$ is isomorphic to $(N_X)_{\mathrm{ab}}$.

Thus $\pi_2(X, w_0) = 0$ and then $\pi_{\geq 2}(X, w_0) = 0$. □

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