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PIECEWISE LINEAR FIBRATIONS

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By an ANR *fibration* we will mean a Hurewicz fibration $p\colon E \to B$, where E is a compact ANR and B is a compact polyhedron. In case E is also a polyhedron and p is a piecewise linear (PL) map, we say that E is a PL fibration. An important special case of this is the notion of a PL manifold bundle, which is a PL locally trivial bundle for which the fibers are compact PL manifolds (with boundary). It is known that any ANR fibration $E \to B$ is "homotopic" to a PL manifold bundle $\mathscr{E} \to B$ in the sense that there exists a path through ANR fibrations from E to $\mathscr{E}$. This takes the form of an ANR fibration over $B \times [0,1]$ whose 0-level is E and whose 1-level is $\mathscr{E}$. The purpose of this paper is to prove that if E is additionally assumed to be a PL fibration, then the ANR fibration over $B \times [0,1]$ can be chosen to be a PL fibration.

This establishes a PL link between the categories {PL fibrations} and {PL manifold bundles}, and it is hoped that this will provide a convenient framework for applying the methods of algebraic K-theory to the study of PL manifold bundles. This was the strategy that was adopted in [**8**], but unfortunately there are gaps in the argument. In particular the appropriate PL link was not established. Our Theorem 3, which is stated below, does establish this PL link. Its proof relies on Theorem 1, which is a PL local connectivity result for spaces of PL maps having contractible point-inverses. The main tool used in establishing Theorem 1 is a stable version of the Fibered Controlled h-Cobordism Theorem of [**5**].

In order to state Theorem 1 we will have to introduce some notation. If X and Y are compact polyhedra, then a PL surjection $r\colon X \to Y$ is said to be a *contractible map* (c-map) if all of the point-inverses are contractible. A *c-homotopy* $r_t\colon X \to Y$ is a fiber-preserving (f.p.) c-map $r = \{r_t\}\colon X \times [0,1] \to Y \times [0,1]$. When we say that a statement is *stably* true regarding X, we actually mean that there is an integer k for which the corresponding statement is true for $X \times I^k$, where I^k is the k-cell $[0,1]^k$. Similarly when we say that a statement is stably true regarding a map $r\colon X \to Y$, we actually mean that there is an integer k for which the corresponding statement is true for the composition $X \times I^k \xrightarrow{\text{proj}} X \xrightarrow{r} Y$. The first result that we establish is the following local connectivity result. In its statement we use Δ^n for the standard n-simplex.

THEOREM 1. *Let $n \geq 0$ and $\varepsilon > 0$ be given. For any compact polyhedron Y there exists a $\delta > 0$ so that if M is a compact PL manifold and r, s: $\Delta^n \times M \to \Delta^n \times Y$ are f.p. c-maps such that the b-levels r_b, s_b: $M \to Y$ satisfy $r_b = s_b$ for all $b \in \partial\Delta$ and $d(r_b, s_b) < \delta$ for all $b \in \Delta$, then (stably) there is a f.p. c-homotopy $r \simeq s$ rel $\partial\Delta \times M$ which is also an ε-homotopy.*

It easily follows that the above statement is also true for f.p. c-maps of E to $B \times Y$, where E is a compact PL manifold bundle over the compact n-dimensional polyhedron B.

Our proof of Theorem 1 proceeds by a double induction, where the primary induction is carried out on n and the secondary induction is carried out on $\dim Y$. There is a significant difference between the case $n = 0$ and the cases $n \geq 1$, so they will be dealt with separately. In §3 we treat the case $n = 0$, and in §4 we establish a key lemma which is needed in §5 to treat the cases $n \geq 1$. In the sequel we will use Theorem 1_n to represent Theorem 1 for all bases Δ^k, $k \leq n$.

Adopting the notation of [**8**] let $\mathscr{S}$ be the classifying space for PL fibrations. It is a semi-simplicial complex for which a typical n-simplex is a PL fibration $E \to \Delta^n$. For any $m \geq 0$ let $\mathscr{B}_m$ be the subcomplex of $\mathscr{S}$ for which a typical n-simplex is a PL manifold bundle $E \to \Delta^n$ of fiber dimension m. Then define $\mathscr{B}_\infty$ to be the direct limit $\varinjlim\{\mathscr{B}_1 \to \mathscr{B}_2 \to \cdots\}$, where the bonding maps are the stabilizations $E \mapsto E \times [0,1]$. These stabilizations give us homotopy commutative triangles

$$\begin{array}{ccc} \mathscr{B}_m & \to & \mathscr{B}_{m+1} \\ & \searrow \quad \swarrow & \\ & \mathscr{S} & \end{array}$$

so there is induced a map $\mathscr{B}_\infty \to \mathscr{S}$. We say that a PL manifold bundle $E \xrightarrow{p} B$ of fiber dimension m is *nice* if E is f.p. PL embedded in $B \times R^m$ so that $\partial E = \bigcup\{\partial p^{-1}(b) \mid b \in B\}$ coincides with the topological boundary of E and it is f.p. PL bicollared. Then define $\mathscr{B}'_m$ to be the subcomplex of $\mathscr{B}_m$ in which only nice bundles are allowed, and similarly define $\mathscr{B}'_\infty$ to be the direct limit $\varinjlim\{\mathscr{B}'_1 \to \mathscr{B}'_2 \to \cdots\}$. The second result that we establish is concerned with the restriction of the map $\mathscr{B}_\infty \to \mathscr{S}$ to $\mathscr{B}'_\infty$. Its proof (given in §6) uses Theorem 1.

THEOREM 2. *$\mathscr{B}'_\infty \to \mathscr{S}$ is a homotopy equivalence.*

An immediate consequence of this theorem is that for any PL fibration $E \to B$ there is a PL fibration over $B \times [0,1]$ whose 0-level is E and whose 1-level lies in some $\mathscr{B}'_m$.

Let K be a compact polyhedron and let $p\colon E \to B$ be a PL fibration for which $B \times K$ is f.p. PL embedded in E. We say that E is a PL *fibration rel* K provided that for each $b \in B$ the inclusion $\{b\} \times K \hookrightarrow p^{-1}(b)$ is a homotopy equivalence. In analogy with the semi-simplicial complex $\mathscr{S}$ we define the subcomplex $\mathscr{S}(K)$, where a typical n-simplex is a PL fibration $E \to \Delta^n$ rel K. Now let N be a compact PL manifold and let $p\colon E \to B$ be a PL fibration rel N. If we additionally assume E to be a PL manifold bundle so that $B \times N$ is f.p. PL collared in E, then we say that E is a PL *manifold bundle rel* N. This occurs, for example, when the pair $(E, B \times N)$ is a fibered h-cobordism. We now form the subcomplex $\mathscr{C}(N)$ of $\mathscr{S}(N)$, in which a typical n-simplex is a PL manifold bundle $E \to \Delta^n$ rel N. Observe that stabilization gives us a map $\mathscr{C}(N) \to \mathscr{C}(N \times [0,1])$, so we can define $\mathscr{C}_\infty(N)$ to be the direct limit

$$\varinjlim\{\mathscr{C}(N) \to \mathscr{C}(N \times [0,1]) \to \cdots\}.$$

As in Theorem 1 the inclusions $\mathscr{C}(N \times [0,1]^k) \hookrightarrow \mathscr{S}(N)$ define a map $\mathscr{C}_\infty(N) \to \mathscr{S}(N)$ (where $N \equiv N \times \{*\} \hookrightarrow N \times [0,1]^k$). Here is our third result whose proof is given in §7.

THEOREM 3. *$\mathscr{C}_\infty(N) \to \mathscr{S}(N)$ is a homotopy equivalence.*

Observe that the concept of niceness, which was needed in Theorem 2, does not appear here.

2. Preliminaries. The purpose of this section is to set up some notation and establish some preliminary results which will be needed in the sequel. For any space X and $A \subset X$, $\mathrm{Bd}(A)$ is the topological boundary of A and $\mathrm{Int}(A)$ is the topological interior of A. Also $\bar{A}$ will denote the closure of A. If M is any topological manifold, then ∂M is the combinatorial boundary of M and $\mathring{M}$ is the combinatorial interior of M. If $f, g\colon X \to Y$ are maps and d is a given metric for Y, then the distance between f and g is

$$d(f,g) = \mathrm{lub}\{d(f(x), g(x)) \mid x \in X\}.$$

If $B \subset Y$, then we say that $f = g$ *over* B if $f^{-1}(B) = g^{-1}(B)$ and $f(x) = g(x)$, for all $x \in f^{-1}(B)$.

If X, Y, and B are spaces, $p\colon Y \to B$ is a map, and B has a given metric, then a homotopy $h_t\colon X \to Y$ is said to be an *ε-controlled homotopy* (with respect to p), or a $p^{-1}(\varepsilon)$-homotopy, if the diameter of $\{ph_t(x) \mid 0 \le t \le 1\}$ is less than ε, for all $x \in X$. B is called the *parameter space*. A map $f\colon X \to Y$ is said to be an *ε-controlled homotopy equivalence* (with respect to p), or a $p^{-1}(\varepsilon)$-equivalence, if there exists a map $g\colon Y \to X$ along with ε-controlled homotopies

$$fg \simeq \mathrm{id}_Y \text{ (with respect to } p),$$
$$gf \simeq \mathrm{id}_X \text{ (with respect to } pf).$$

We call the map $g\colon Y \to X$ an ε-controlled *inverse* of f. Recall that we defined a c-map $f\colon K \to L$ to be a PL surjection of compact polyhedra for which each point-inverse is contractible. Using [**1**] it follows that a c-map is an ε-controlled homotopy equivalence (with respect to id: $L \to L$), for all $\varepsilon > 0$.

Here are two well-known facts concerning maps of polyhedra and manifolds that will be useful. The first asserts that if $f\colon K \to [0,1]$ is a PL surjection of compact polyhedra, then $f^{-1}(t)$ is a PL bicollared subpolyhedron of K, for all but a finite number of t. In fact, it follows from the simplicial structure that if $[0,1]$ is the standard 1-simplex and K is triangulated so that f is simplicial, then $f^{-1}(t)$ is PL bicollared for all $t \in (0,1)$ (cf. [**7**, §9]). The second fact asserts that if (M, K) is a compact polyhedral pair, where M is a PL manifold and K is PL bicollared in M, then K is also a PL manifold. This follows easily from [**10**, Lemma 2.6] and the uniqueness of regular neighborhoods.

If $f\colon K \to L$ is a map, then the *topological mapping cylinder* of f, $M(f)$, is the space formed by sewing $K \times [0,1]$ to L via the map $(x,1) \mapsto f(x)$. The *top* of $M(f)$ is $K \equiv K \times \{0\}$, and the *base* of $M(f)$ is the naturally embedded copy of L. The quotient map $q_T\colon K \times [0,1] \to M(f)$ ($T = \mathrm{TOP}$) satisfies $q_T(x,0) = x$ and $q_T(x,1) = f(x)$, for all $x \in K$, and q_T is a homeomorphism over $M(f) - L$. There is also a natural projection map $\pi_T\colon M(f) \to L \times [0,1]$ which satisfies $\pi_T(x) = (f(x), 0)$, for $x \in K$, $\pi_T(x) = (x,1)$, for $x \in L$, and $\pi_T q_T(x,t) = (f(x), t)$, for all $(x,t) \in K \times [0,1]$. Generally we will not be interested in the topological mapping cylinder except as a point of reference. More specifically, we will need the PL analogue of $M(f)$ as described below.

If $f\colon K \to L$ is a PL map of compact polyhedra, then the *simplicial mapping cylinder*, $C(f)$, is a triangulation of $M(f)$ so that the top K and base L are subpolyhedra. We will not give a definition of $C(f)$, but will instead refer the reader to [**7**] for its definition and proofs of some of its properties which we discuss below. Once subdivisions of K and L have been chosen, there is a natural PL quotient map $q_S\colon K \times [0,1] \to C(f)$ which satisfies $q_S(x,0) = x$, $q_S(x,1) = f(x)$, and q_S is a c-map over

$C(f) - L$. There is also a natural PL projection map π_S: $C(f) \to L \times [0,1]$ which satisfies $\pi_S(x) = (f(x), 0)$, for $x \in K$, $\pi_S(x) = (x, 1)$, for $x \in L$, and $\pi_S q_S(x,t) = (f(x), t)$, for all $(x,t) \in K \times [0,1]$. The following statements summarize some properties of $C(f)$ which we will need.

1. If f is a PL homeomorphism (respectively c-map), then so are q_S and π_S. A proof of this can be found in [7].

2. If $K \xrightarrow{r} B$ and $L \xrightarrow{s} B$ are PL maps for which $sf = r$, and p is the composition

$$C(f) \xrightarrow{\pi_S} L \times [0,1] \xrightarrow{s \times \mathrm{id}} B \times [0,1],$$

then one easily checks that

$$pq_S = r \times \mathrm{id}\colon\ K \times [0,1] \to B \times [0,1].$$

3. If $K \xrightarrow{r} B$ is a PL fibration and f is a c-map, then p: $C(f) \to B \times [0,1]$ is also a PL fibration. To prove this one has to use the characterization of Hurewicz fibrations of [**1**], which asserts that a surjection of compact polyhedra is a Hurewicz fibration if nearby fibers have "small" homotopy equivalences between them.

Here is a result which will be needed in the sequel. As in the above discussion its proof uses material from [7].

PROPOSITION 2.1. *Let p: $E \to \Delta^n$ be a* PL *fibration for which p is simplicial with respect to a given triangulation of E and the standard triangulation of Δ^n. If b is the barycenter of Δ^n, then there exists a f.p. c-map $\Delta^n \times p^{-1}(b) \to E$.*

Proof. The proof is by induction on n, and since the cases $n \geq 1$ are all similar it will suffice to treat only the case $n = 1$. For this $\Delta^1 = [0,1]$ and $b = \frac{1}{2}$, and all we have to do is construct a f.p. c-map $[0, \frac{1}{2}] \times p^{-1}(\frac{1}{2}) \xrightarrow{\pi} p^{-1}([0, \frac{1}{2}])$ which is the identity over $\frac{1}{2}$. In what follows we will use the terminology of [7] and the ideas which occur in §9 of that paper. Choose a simplex in E of the form $\sigma\tau$, where $\sigma < p^{-1}(0)$ and $\tau < p^{-1}(1)$. Any point $z \in p^{-1}(\frac{1}{2}) \cap \sigma\tau$ is of the form $z = \frac{1}{2}x + \frac{1}{2}y$, where $x \in \sigma$ and $y \in \tau$. Define a map

$$\pi\colon [0, \tfrac{1}{2}] \times p^{-1}(\tfrac{1}{2}) \to p^{-1}([0, \tfrac{1}{2}])$$

by $\pi(t, z) = (1 - t)x + ty$. Observe that π is f.p. and the $\frac{1}{2}$-level, $\pi_{\frac{1}{2}}$: $p^{-1}(\frac{1}{2}) \to p^{-1}(\frac{1}{2})$, is the identity. So all we have to do is prove that π is a c-map.

The intersection of the cells of E with $p^{-1}([0, \frac{1}{2}])$ gives a complex cell structure on $p^{-1}([0, \frac{1}{2}])$ which we can triangulate without introducing any new vertices. When this is done we observe that π is simplicial, and in fact it is the standard extension of π_0: $p^{-1}(\frac{1}{2}) \to p^{-1}(0)$. Thus π is a c-map over $p^{-1}([0, \frac{1}{2}])$. To see that π_0: $p^{-1}(\frac{1}{2}) \to p^{-1}(0)$ is a c-map it suffices to show that it is an ε-controlled homotopy equivalence, for all $\varepsilon > 0$ (with respect to id: $p^{-1}(0) \to p^{-1}(0)$). (For then any point-inverse can be homotoped to a point in any neighborhood of the point-inverse.) To show that π_0 is an ε-controlled homotopy equivalence we need an ε-controlled inverse, i.e., a map g: $p^{-1}(0) \to p^{-1}(\frac{1}{2})$ along with controlled homotopies $g\pi_0 \simeq \mathrm{id}$ and $\pi_0 g \simeq \mathrm{id}$. Since $p^{-1}([0, \frac{1}{2}]) \to [0, \frac{1}{2}]$ is a fibration there are maps g_1: $p^{-1}(0) \to p^{-1}(\delta)$, $\delta > 0$, which can be made as close to the identity as we please. Since the map $(0, \frac{1}{2}] \times p^{-1}(\frac{1}{2}) \xrightarrow{\pi} p^{-1}((0, \frac{1}{2}])$ is a c-map it must be a δ-controlled homotopy equivalence, for all $\delta > 0$. Let g_2: $p^{-1}((0, \frac{1}{2}]) \to (0, \frac{1}{2}] \times p^{-1}(\frac{1}{2})$ be an inverse, for some small $\delta > 0$. Finally, let $g_3 = \mathrm{proj}$: $[0, \frac{1}{2}] \times p^{-1}(\frac{1}{2}) \to p^{-1}(\frac{1}{2})$ be the projection map. Then our desired inverse g: $p^{-1}(0) \to p^{-1}(\frac{1}{2})$ is defined by $g = g_3 g_2 g_1$, and we leave it to the reader to check out the controlled homotopies $g\pi_0 \simeq \mathrm{id}$ and $\pi_0 g \simeq \mathrm{id}$. □

For our purposes an *h-cobordism* is a compact PL manifold pair (M, N) such that the inclusion $N \hookrightarrow M$ is a homotopy equivalence and N is PL collared in M. Recall that an ordinary h-cobordism additionally requires $\overline{\partial M - N} \hookrightarrow M$ to be a homotopy equivalence. We do not need this additional assumption because we are working in a stable category. In fact, it is easy to show that if (M, N) is an h-cobordism as defined above, then $(M \times I^3, N \times I^3)$ is an ordinary h-cobordism. To see this one first checks that $\overline{\partial(M \times I^3) - (N \times I^3)} \hookrightarrow M \times I^3$ induces an isomorphism on π_1, and then duality on the universal covering spaces is used.

In the sequel we will be interested in controlled h-cobordisms. More specifically, let (M, N) be a compact PL manifold pair for which N is PL collared in M, and let p: $M \to B$ be a map to some parameter space. We say that (M, N) is an *ε-controlled h-cobordism* (with respect to p) provided that the inclusion $N \hookrightarrow M$ is an ε-controlled homotopy equivalence. (M, N) is an *ε-controlled product* provided that there exists a PL homeomorphism h: $[0,1] \times N \to M$ for which $h(0, x) = x$, for all $x \in N$, and for which the diameter of each $ph([0,1] \times \{x\})$ is less than ε. The homeomorphism h defines an *ε-product structure* on M. It is obvious that each ε-controlled product is an ε-controlled h-cobordism. The following result is concerned with a converse to this statement.

PROPOSITION 2.2 (*Stable Controlled h-Cobordism Theorem*). *Let B be a compact polyhedron and let $\varepsilon > 0$ be given. There exists a $\delta > 0$ such that if $(M, N) \xrightarrow{p} B$ is a δ-controlled h-cobordism and the restriction $p\colon N \to B$ is a c-map, then (stably) (M, N) is an ε-controlled product.*

Proof. We will not give a complete proof of this result because of its similarity with other theorems in the literature. Note that in the above statement δ depends only on ε and B, and is independent of $\dim N$. If we allow δ to also be a function of $\dim N$, then stabilization is no longer necessary, and the result is then a consequence of the (non-stable) Thin h-Cobordism Theorem of **[12]**, or the (non-stable) Controlled s-Cobordism Theorem of **[4]**. The proofs of both of these theorems can be adapted to the stable category to give a proof of our result. Probably the latter is the easiest to modify for our purposes.

Specifically we are referring to the proof of the Controlled s-Cobordism Theorem which appears in §14 of **[4]**. The proof proceeds by induction on $\dim B$, with the heart of the inductive step being a somewhat complicated splitting lemma. The reason for the difficulty in the proof of this splitting lemma is that no stabilization is permitted. Fortunately it is somewhat easier to construct a proof of the stable version of this splitting lemma by using an argument which is similar to the infinite-dimensional splitting of **[3**, §5**]**. When this is done δ can be chosen independent of $\dim N$, and we can therefore complete the inductive step for the proof of our result. □

REMARK. There is a relative version of the above result which asserts that if U is open in B, $C \subset U$ is compact, and (M, N) already has a given δ-product structure over U, then the ε-product structure on (M, N) can be chosen to agree with the given product structure over C. The proof is quite similar.

We will also need a version of Proposition 2.2 in the fibered category. For notation let $(\mathscr{E}, B \times N) \to B$ be a PL manifold bundle rel N, let $E \to B$ be a PL fibration, and let $p\colon \mathscr{E} \to E$ be a f.p. map. We say that $(\mathscr{E}, B \times N)$ is a *fibered ε-controlled h-cobordism* (with respect to p) provided that the inclusion $B \times N \hookrightarrow \mathscr{E}$ is a f.p. ε-controlled homotopy equivalence. This just means that all maps and homotopies involved in the usual definition of an ε-controlled homotopy equivalence are f.p. $(\mathscr{E}, B \times N)$ is a *fibered ε-controlled product* provided that there exists a f.p. PL homeomorphism $h\colon [0,1] \times B \times N \to \mathscr{E}$ for which $h(0, x) = x$, for all $x \in B \times N$, and for which the diameter of each $ph([0,1] \times \{x\})$ is less than ε. The homeomorphism h defines a *fibered ε-product structure* on $\mathscr{E}$.

PROPOSITION 2.3. (*Fibered Stable Controlled h-Cobordism Theorem*). *Let* $E \to B$ *be a* PL *fibration and let* $\varepsilon > 0$ *be given. There exists a* $\delta > 0$ *such that if* $(\mathscr{E}, B \times N) \xrightarrow{p} E$ *is a fibered* δ*-controlled h-cobordism and the restriction* $p\colon B \times N \to E$ *is a c-map, then (stably)* $(\mathscr{E}, B \times N)$ *is a fibered* ε*-controlled product.*

Proof. Just as in the case of Proposition 2.2 we will not give the details of the proof. The reason for this is the existence of the (non-stable) Fibered Controlled h-Cobordism Theorem of **[5]**, which is quite similar to the above statement. The only difference is that the δ of **[5]** is also a function of $\dim N$. If we repeat the same argument in the stable category, then we obtain a proof of our result. □

REMARKS 1. There is a relative version of the above result which asserts that if U is open in E, $C \subset U$ is compact, and $(\mathscr{E}, B \times N)$ already has a given fibered δ-product structure over U, then the ε-product structure on $(\mathscr{E}, B \times N)$ can be chosen to agree with the given product structure over C.

2. There is another relative version of the above result which asserts that if $(\mathscr{E}, B \times N)$ already has a given δ-product structure over a compact subpolyhedron B' of B, then the fibered ε-product structure on $(\mathscr{E}, B \times N)$ can be chosen to agree with the given product structure over B'.

3. Proof of Theorem 1_0. Our proof of Theorem 1_0 is given in Theorem 3.2 below, but first we will have to establish a lemma. Here is some notation for its statement. Let K be a compact polyhedron and assume that $K \times [-1,1]$ is a subpolyhedron of a compact polyhedron Y for which $\mathrm{Bd}(K \times [-1,1]) = K \times \{-1,1\}$.

LEMMA 3.1. *For every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *so that if* M *is a compact* PL *manifold and* $r, s\colon M \to Y$ *are c-maps such that* $r^{-1}(K \times \{0\})$, $s^{-1}(K \times \{0\})$ *are* PL *bicollared and* $d(r,s) < \delta$, *then (stably) there exists a* PL *homeomorphism* $h\colon M \to M$ *which satisfies*

(1) $hs^{-1}(K \times \{0\}) = r^{-1}(K \times \{0\})$,

(2) *h is supported on* $r^{-1}(K \times [-1,1])$,

(3) $d(rh, r) < \varepsilon$.

Proof. In what follows we will assume that the stabilizing I^k-factor is present whenever needed, but for notational convenience we will ignore writing it down. Since $r^{-1}(K \times \{0\})$ and $s^{-1}(K \times \{0\})$ are PL bicollared they must be PL manifolds. Also $s^{-1}(K \times \{t\})$ is a PL bicollared manifold for all but a finite number of t. So without loss of generality we may assume that $s^{-1}(K \times \{\frac{1}{3}\})$ is a PL bicollared manifold. In our proof we will be dealing with controlled homotopies, where the parameter space is K and the controlling map to K is either the composition (proj) r: $r^{-1}(K \times [-1,1]) \to K$, or the composition (proj) s: $s^{-1}(K \times [-1,1]) \to K$ (or appropriate restrictions thereof). These maps will be generally denoted by p_r or p_s, respectively.

Assertion 1. *There exists a PL homeomorphism* φ: $s^{-1}(K \times [0, \frac{1}{3}]) \to s^{-1}(K \times \{0\}) \times [0,1]$ *which satisfies*

(1) $\varphi s^{-1}(K \times \{0\}) = s^{-1}(K \times \{0\}) \times \{0\}$,

(2) $\varphi s^{-1}(K \times \{\frac{1}{3}\}) = s^{-1}(K \times \{0\}) \times \{1\}$,

(3) $d(p_s(\text{proj})\varphi, p_s)$ *is small (as small as we want).*

Proof. Since s is a c-map it is clear that

$$\left(s^{-1}(K \times [0, \tfrac{1}{3}]),\, s^{-1}(K \times \{0\})\right) \xrightarrow{p_s} K$$

is a γ-controlled h-cobordism, for every $\gamma > 0$. It then follows from the Stable Controlled h-Cobordism Theorem of §2 that our desired PL homeomorphism φ exists, and it satisfies properties (1) and (3) above. By performing the following stable modification of φ we can also achieve property (2).

In analogy with φ there exists a PL homeomorphism

$$\varphi' \colon s^{-1}(K \times [0, \tfrac{1}{3}]) \to s^{-1}(K \times \{\tfrac{1}{3}\}) \times [0,1]$$

which satisfies

(1)′ $\varphi' s^{-1}(K \times \{\frac{1}{3}\}) = s^{-1}(K \times \{\frac{1}{3}\}) \times \{0\}$,

(3)′ $d(p_s(\text{proj})\varphi', p_s)$ is small.

Now using φ and φ' it is easy to see that there exists a PL homeomorphism α of $s^{-1}(K \times [0, \frac{1}{3}]) \times [0,1]$ onto itself which satisfies

(1) $\alpha(s^{-1}(K \times \{0\}) \times [0,1]) = s^{-1}(K \times [0, \frac{1}{3}]) \times \{0\}$,

(2) $\alpha(s^{-1}(K \times \{\frac{1}{3}\}) \times [0,1]) = s^{-1}(K \times [0, \frac{1}{3}]) \times \{1\}$,

(3) $d(p_s(\text{proj})\alpha,\ p_s(\text{proj}))$ is small.

Similarly there exists a PL homeomorphism β of $s^{-1}(K \times \{0\}) \times [0,1] \times [0,1]$ onto itself which satisfies

(1) $\beta(s^{-1}(K \times \{0\}) \times [0,1] \times \{0\}) = s^{-1}(K \times \{0\}) \times \{0\} \times [0,1]$,

(2) $\beta(s^{-1}(K \times \{0\}) \times [0,1] \times \{1\}) = s^{-1}(K \times \{0\}) \times \{1\} \times [0,1]$,

(3) $d(p_s(\text{proj})\beta, p_s(\text{proj}))$ is small.

In fact, we can take this distance in (3) to be zero since β is constructed as a "twist" on $[0,1] \times [0,1]$. Finally $\tilde{\varphi} = \beta(\varphi \times \mathrm{id})\alpha$ is a PL homeomorphism of $s^{-1}(K \times [0, \frac{1}{3}]) \times [0,1]$ onto $s^{-1}(K \times \{0\}) \times [0,1] \times [0,1]$ that satisfies properties (1)–(3). □

Now using the homeomorphism φ which was constructed above and the given PL bicollarings it is easy to obtain a PL homeomorphism h_1: $r^{-1}(K \times [-1, 1]) \to r^{-1}(K \times [-1, 1])$ for which $h_1 s^{-1}(K \times \{0\}) = s^{-1}(K \times \{\frac{1}{3}\})$ and $d(p_s h_1, p_s)$ is small. Observe that this requires r and s to be at least close enough so that $s^{-1}(K \times [0, \frac{1}{3}])$ lies in $r^{-1}(K \times (-1,1))$. This completes the first step of our argument.

In the following statement W denotes the intersection of $r^{-1}(K \times [0,1])$ and $s^{-1}(K \times [-1, \frac{1}{3}])$. It is clear that if r and s are sufficiently close, then W is a PL submanifold of M.

Assertion 2. For any $\delta_1 > 0$ we can choose $\delta > 0$ small enough so that there exists a PL homeomorphism ψ: $W \to r^{-1}(K \times \{0\}) \times [0,1]$ which satisfies

(1) $\psi r^{-1}(K \times \{0\}) = r^{-1}(K \times \{0\}) \times \{0\}$,
(2) $\psi s^{-1}(K \times \{\frac{1}{3}\}) = r^{-1}(K \times \{0\}) \times \{1\}$,
(3) $d(p_r(\mathrm{proj})\psi, p_r) < \delta_1$.

Proof. If we can show that

$$\left(W, r^{-1}(K \times \{0\})\right) \xrightarrow{p_r} K \quad \text{and} \quad \left(W, s^{-1}\left(K \times \{\tfrac{1}{3}\}\right)\right) \xrightarrow{p_s} K$$

are γ-controlled h-cobordisms, for γ a number whose size depends on the size of δ, then we can repeat the proof of Assertion 1. Because of the similarity of the two cases we only need consider the former. For convenience we will ignore mentioning any specific size estimates on homotopies, but will instead simply assume that $\delta > 0$ is chosen small enough so that all of our constructions are possible. Since r is a c-map there exists a strong deformation retraction of $r^{-1}(K \times [0, \frac{1}{2}])$ onto $r^{-1}(K \times \{0\})$. Restricting this to W gives a deformation taking place in $r^{-1}(K \times [0, \frac{1}{2}])$. Now compose this deformation with a retraction

$$\tilde{\rho}: W \cup s^{-1}\left(K \times [\tfrac{1}{3}, \tfrac{2}{3}]\right) \to W,$$

where $\tilde{\rho}$ trivially extends a retraction ρ of $s^{-1}(K \times [\frac{1}{2}, \frac{2}{3}])$ onto $s^{-1}(K \times \{\frac{1}{3}\})$. □

Using the homeomorphism ψ and the given PL bicollarings we can easily obtain a PL homeomorphism

$$h_2: r^{-1}\left(K \times [-1,1]\right) \to r^{-1}\left(K \times [-1,1]\right)$$

for which $h_2 r^{-1}(K \times \{0\}) = s^{-1}(K \times \{\frac{1}{3}\})$ and $d(p_r h_2, p_r)$ is small (corresponding to a small choice of δ). By construction the homeomorphisms h_1 and h_2 trivially extend to homeomorphisms $\tilde{h}_1$ and $\tilde{h}_2$ of M to M. Then $h = \tilde{h}_2^{-1}\tilde{h}_1$: $M \to M$ is a PL homeomorphism which is supported on $r^{-1}(K \times [-1,1])$ and which takes $s^{-1}(K \times \{0\})$ to $r^{-1}(K \times \{0\})$. The third requirement, $d(rh, r)$, $< \varepsilon$ has not been achieved, but this is only because $d(0, \frac{2}{3})$ is large. So all we have to do is repeat the above argument with $\varepsilon/2$ in place of $\frac{2}{3}$. □

REMARK. There is a relative version of the above absolute result which we now describe. For this we are additionally given $C \subset U \subset K$, where C is compact and U is open, such that $r = s$ over $U \times [-1,1]$. The conclusion states that the PL homeomorphism h: $M \to M$ which was constructed above can be additionally required to satisfy $h = \mathrm{id}$ on $r^{-1}(C \times [-1,1])$. To see how this is achieved recall that the homeomorphisms $\tilde{h}_1$ and $\tilde{h}_2$ of the above proof were constructed by pushing along the $[0,1]$-intervals obtained from the product structures

$$\left(s^{-1}(K \times [0, \tfrac{1}{3}]), s^{-1}(K \times \{0\})\right) \approx s^{-1}(K \times \{0\}) \times [0,1]$$

and

$$\left(W, r^{-1}(K \times \{0\})\right) \approx r^{-1}(K \times \{0\}) \times [0,1],$$

which resulted from the Stable Controlled h-Cobordism Theorem. If we choose $C \subset \operatorname{Int}(D) \subset D \subset U$, where D is compact, then it follows from the relative version of the Stable Controlled h-Cobordism Theorem that these product structures can be chosen to be identical on $r^{-1}(D \times [0, \frac{1}{3}])$. Thus $\tilde{h}_1 = \tilde{h}_2$ on $r^{-1}(C \times [-1,1])$, and so $h = \tilde{h}_2^{-1}\tilde{h}_1$ is the identity on $r^{-1}(C \times [-1,1])$.

THEOREM 3.2. *For every $\varepsilon > 0$ and compact polyhedron Y there exists a $\delta > 0$ so that if M is a compact PL manifold and r, s: $M \to Y$ are c-maps for which $d(r,s) < \delta$, then (stably) there exists a c-homotopy $r \simeq s$ which is also an ε-homotopy.*

Proof. We will actually establish a relative version of this result in which we are additionally given $C \subset U \subset Y$, where C is compact and U is open, for which $r = s$ over U. The conclusion additionally states that the homotopy $r \simeq s$ can be required to be constant over C. The reason why we establish this relative result is that it appears to be necessary in order to carry out the inductive step in our proof which proceeds by induction on $\dim Y$. The relative result is trivially true for $\dim Y = 0$, so assuming it to be true for all bases Y of dimension $\leq k - 1$ we consider the inductive step where $\dim Y = k$. We will only give explicit details for the absolute version and merely remark that the relative version follows from the same proof. In what follows we will not keep careful track of the sizes of the various homotopies that we encounter. Instead, we will simply assume that δ is chosen small enough so that all of our constructions can be carried out. It will then be clear that our c-homotopy $r \simeq s$ will be small provided that δ is chosen correspondingly small.

We start by choosing a triangulation of Y such that all of the simplices have small diameters. Choose a 0-simplex v and let N be a small regular neighborhood of v, which may be viewed as a cone with vertex v. For notation let $N_t \subset N$ be the t-level of the cone such that $N_0 = \{v\}$ and $N_1 = N$. We have already observed in the proof of Lemma 3.1 that $r^{-1}(\mathrm{Bd}(N_t))$ and $s^{-1}(\mathrm{Bd}(N_t))$ are PL bicollared manifolds, for almost all t. So for simplicity we may assume that $r^{-1}(\mathrm{Bd}(N_{1/2}))$ and $s^{-1}(\mathrm{Bd}(N_{1/2}))$ are PL bicollared manifolds. It follows from Lemma 3.1 that (stably) there exists a PL homeomorphism $h_0\colon M \to M$ which satisfies

(1) $h_0 s^{-1}(\mathrm{Bd}(N_{1/2})) = r^{-1}(\mathrm{Bd}(N_{1/2}))$,
(2) h_0 is supported on $r^{-1}(N)$,
(3) rh_0 is close to r.

The same thing can be done for all 0-simplices simultaneously, so we may assume that the above properties are true for all 0-simplices and their correspondingly small regular neighborhoods.

We now focus our attention on the c-map $rh_0\colon M \to Y$, which will be our link between r and s. By property (2) above we have $rh_0 = r$ over $Y - \bigcup \mathrm{Int}(N)$. So by squeezing slightly larger regular neighborhoods towards the 0-simplices of Y we obtain a c-homotopy $f_t\colon Y \to Y$ for which $f_0 = \mathrm{id}$ and $f_1 r = f_1 rh_0$. Thus we obtain a small c-homotopy $r \simeq rh_0$. So we only need to find a small c-homotopy $rh_0 \simeq s$. By property (3) above rh_0 is close to s, and by property (1) we have $(rh_0)^{-1}(\mathrm{Bd}(N_{1/2})) = s^{-1}(\mathrm{Bd}(N_{1/2}))$, for all N. Using the given PL bicollarings and the absolute version of our inductive hypothesis we obtain small c-homotopies $rh_0 \simeq r_0'$ and $s \simeq s_0'$ so that $r_0' = s_0'$ over $\bigcup \mathrm{Bd}(N_{1/2})$, where the union is taken over all N. We may further assume that $r_0' = s_0'$ over $\bigcup \mathrm{Bd}(N_t)$, for all t close to $\frac{1}{2}$. Once again using the above squeezing trick (towards the 0-simplices) we obtain small c-homotopies $r_0' \simeq r_0$ and $s_0' \simeq s_0$ so that $r_0 = s_0$ over $\bigcup N_{1/2}$. Note that r_0 and s_0 are still close together, where the size of this closeness depends on the size of δ. This completes the first step in which we have obtained small c-homotopies $r \simeq r_0$ and $s \simeq s_0$ so that $r_0 = s_0$ over a neighborhood of the 0-skeleton. The reader should have observed that the above argument is somewhat sketchy. In fact, it is a good exercise to write down a c-homotopy that squeezes a regular neighborhood of the 0-skeleton in a polyhedron to the 0-skeleton.

The next step is to repeat the above process for the 1-skeleton. For convenience let us assume that $r_0 = s_0$ over the star of the 0-skeleton in the second barycentric subdivision of Y. The regular neighborhoods N which were used above now become regular neighborhoods of the barycenters of the 1-simplices. For a typical 1-simplex σ with barycenter b we now define N to be the seventh star of b in the fourth barycentric subdivision of Y. We also choose notation so that the $\frac{1}{2}$-level of N, $N_{1/2}$, is the sixth star of b in this fourth barycentric subdivision of Y. Now use the relative version of Lemma 3.1 to find a PL homeomorphism h_1: $M \to M$ which satisfies

(1) $h_1 s_0^{-1}(\mathrm{Bd}(N_{1/2})) = r_0^{-1}(\mathrm{Bd}(N_{1/2}))$,
(2) h_1 is supported on $r_0^{-1}(N)$,
(3) $r_0 h_1$ is close to r_0,
(4) h_1 is the identity over the third star of the 0-skeleton in the fourth barycentric subdivision of Y.

We can arrange this so that we are working on disjoint pieces, thus we may assume that these properties hold for all N simultaneously.

It is now easy to repeat the above argument which was used for the 0-skeleton to obtain small c-homotopies $r_0 \simeq r_0 h_1 \simeq r_1' \simeq r_1$ and $s_0 \simeq s_1' \simeq s_1$ so that r_1 is close to s_1 and $r_1 = s_1$ over a neighborhood of the 1-skeleton. The only difference in the details for this case is that when the inductive hypothesis is used to construct the c-homotopies $r_0 h_1 \simeq r_1'$ and $s_0 \simeq s_1'$, we must use the relative version so that $r_1' = s_1'$ over a neighborhood of the 0-skeleton which lies between its second and third stars in the fourth barycentric subdivision of Y. This completes the second step of the construction.

So far we have only worked up through the 1-skeleton of Y, but it should be clear by now that we can continue this process to obtain small c-homotopies

$$r \simeq r_0 \simeq r_1 \simeq \cdots \simeq r_k,$$

$$s \simeq s_0 \simeq s_1 \simeq \cdots \simeq s_k$$

so that $r_k = s_k$. The size of these combined homotopies is determined by the mesh of the given triangulation of Y, which in turn determines the size of δ. $\square$

4. A bundle theorem. The purpose of this section is to establish a bundle theorem which is needed in §5 for the proof of Theorem 1_n, $n \geq 1$. The result is established in Theorem 4.6 below, but first we will need some preliminary material. We begin with a batch of lemmas culminating in Proposition 4.5.

LEMMA 4.1. *If p: $X \to Y$ is a simplicial surjection of compact polyhedra (with fixed triangulations) and Y_1 is a first derived subdivision of Y, then there exists a first derived subdivision X_1 of X such that p: $X_1 \to Y_1$ is simplicial.*

Proof. Let $\{y_1, y_2, \ldots, y_n\}$ be the set of starring points which are used to construct the subdivision Y_1 of Y. We may assume that the vertices of Y are among the y_i. If σ is a simplex in X, then $p(\sigma)$ is a simplex in Y and so there exists a unique y_i in $p(\mathring{\sigma})$. Choose any $x_\sigma \in \mathring{\sigma} \cap p^{-1}(y_i)$ to be a starring point for σ. Then using the set $\{x_\sigma\}$ for starring points we can form a first derived subdivision X_1 of X for which p: $X_1 \to Y_1$ is simplicial. □

REMARKS 1. Any subdivision X_1 of X which is constructed in the above manner is said to *cover* Y_1.

2. It is obvious that if X_1 and X_2 are any two first derived subdivisions of a triangulated polyhedron X, then there is a unique simplicial isomorphism g: $X_1 \to X_2$ (which we call the *standard isomorphism*) that takes each simplex of X to itself. It then follows by linearity that if X_1 and X_2 are two first derived subdivisions of X which cover Y_1, then the standard isomorphism g: $X_1 \to X_2$ *covers* id_Y (i.e., $pg = p$).

LEMMA 4.2. *If p: $X \to Y$ is as in Lemma 4.1, Y_1 and Y_2 are first derived subdivisions of Y, and X_i is a first derived subdivision of X which covers Y_i, then the standard isomorphism g: $X_1 \to X_2$ covers the standard isomorphism h: $Y_1 \to Y_2$ (i.e., $pg = hp$).* □

LEMMA 4.3. *If X has first derived subdivisions X_1, X_2 and g: $X_1 \to X_2$ is the standard isomorphism, then g is PL isotopic to the identity.*

Proof. Our first step is to define a triangulation K_1 of the polyhedron $X \times [0,1]$. The starting point is the 0-level $X \times \{0\}$, which is triangulated by $X_1 \times \{0\}$. If τ is a 0-simplex in X we triangulate $\tau \times [0,1]$ so that it has two 0-simplices $\tau \times \{0\}$ and $\tau \times \{1\}$ and one 1-simplex $\tau \times [0,1]$.

Proceeding inductively let τ be an i-simplex and assume that $\tau \times \{0\} \cup \partial\tau \times [0,1]$ has been triangulated. If $x \in \mathring{\tau}$ is the starring point for τ in the construction of X_1, then $\tau \times [0,1]$ is triangulated by starring, $(x,1) \cdot (\tau \times \{0\} \cup \partial\tau \times [0,1])$. This completes the inductive description of K_1. Note that the 1-level $X \times \{1\}$ is also triangulated by $X_1 \times \{1\}$. Similarly let K_2 be the triangulation of $X \times [0,1]$ which is inductively defined by starring, $(x',1) \cdot (\tau \times \{0\} \cup \partial\tau \times [0,1])$, where $x' \in \mathring{\tau}$ is the starring point for τ in the construction of X_2. Observe that the 0-level is triangulated by $X_1 \times \{0\}$, and the 1-level is triangulated by $X_2 \times \{1\}$.

It is now clear from the above inductive constructions of K_1 and K_2 that there exists a unique simplicial isomorphism G: $K_1 \to K_2$ which is the identity on the 0-level and it is g on the 1-level. By linearity G must be level preserving, thus G_t: id $\simeq g$ is our desired isotopy. □

REMARKS. 1. We will refer to the above isotopy as the *standard isotopy* of id to g.

2. Observe that if τ is a simplex in X and $x_1, x_2 \in \mathring{\tau}$ are the starring points for X_1 and X_2, respectively, then g takes x_1 to x_2 and by linearity the isotopy G moves x_1 along the straight line $[x_1, x_2]$.

LEMMA 4.4. *Let p: $X \to Y$ be as in Lemma 4.1, let Y_1 and Y_2 be first derived subdivisions of Y, and let X_i be a first derived subdivision of X which covers Y_i. If g: $X_1 \to X_2$ and h: $Y_1 \to Y_2$ are the standard isomorphisms and G_t:* id $\simeq g$, H_t: id $\simeq h$ *are the standard isotopies, then G_t covers H_t.* □

PROPOSITION 4.5. *Let X and B be compact polyhedra and let p: $X \to B \times [-1,1]$ be a PL surjection. Then for every $b_0 \in B$ there exists a finite set $F_{b_0} \subset [-1,1]$ which satisfies the following property: corresponding to each $t_0 \in [-1,1] - F_{b_0}$ there is a neighborhood $U_0 \subset B$ of b_0 such that $p^{-1}(U_0 \times \{t_0\})$ is f.p. PL bicollared in $p^{-1}(U_0 \times [-1,1])$.*

Proof. Just to make sure that there is no confusion, the above statement means that the bicollaring intervals of $p^{-1}(U_0 \times \{t_0\})$ are sent to points in B under the map proj $\circ\, p$: $X \to B$. Choose triangulations of X and $B \times [-1,1]$ so that p is simplicial and so that $\{b_0\} \times [-1,1]$ is a subcomplex of $B \times [-1,1]$. The vertices of $B \times [-1,1]$ which lie in $\{b_0\} \times [-1,1]$ determine our finite set $F_{b_0} \subset [-1,1]$. We are going to describe a first derived subdivision of $B \times [-1,1]$, so we will need starring points. For a fixed $t_0 \in [-1,1] - F_{b_0}$ let e be the 1-simplex in $\{b_0\} \times [-1,1]$ which contains (b_0, t_0), and let (b_0, t_0) be its starring point. For any simplex σ in $B \times [-1,1]$ which contains e as a proper face choose a starring point in $\mathring{\sigma}$ which is of the form (b, t_0), where b is close to b_0. For all other simplices of $B \times [-1,1]$ select starring points in an arbitrary manner. Let K_0 be the first derived subdivision of $B \times [-1,1]$ obtained by using these starring points.

We are now going to describe another first derived subdivision of $B \times [-1,1]$. Choose $t_1 \neq t_0$ so that $(b_0, t_1) \in \mathring{e}$ and let (b_0, t_1) be a new starring point for e. For any simplex σ in $B \times [-1,1]$ which contains e as a proper face choose a starring point which is of the form (b, t_1), where (b, t_0) was the starring point used above for σ in the formation of K_0. (Observe that if t_0 and t_1 are chosen first, then b can be chosen close enough to b_0 so that this can be done.) For all other simplices of $B \times [-1,1]$ select starring points which are the same as for K_0. Let K_1 be the first derived subdivision of $B \times [-1,1]$ obtained by using these starring points.

Now form first derived subdivisions X_0 and X_1 of X which cover K_0 and K_1, respectively. Let g: $X_0 \to X_1$, h: $K_0 \to K_1$ be the standard isomorphisms and let G_t: id $\simeq g$, H_t: id $\simeq h$ be the standard isotopies so that G covers H. We use A_0 for the union of all simplices of K_0 which are of the form $x_1 x_2 \cdots x_r$, where x_i is the starring point of a simplex which contains e. Similarly let A_1 be the union of all simplices of K_1 which are of the form $x_1 x_2 \cdots x_r$, where x_i is the starring point of a simplex which contains e. It is clear that the projections of A_0 and A_1 to B are equal and define a neighborhood U_0 of b_0 for which $A_0 = U_0 \times \{t_0\}$ and $A_1 = U_1 \times \{t_1\}$. Also the isomorphism h takes A_0 to A_1, and in analogy with Remark 2 following Lemma 4.3 it follows that the isotopy H_t moves $U_0 \times \{t_0\}$ along $U_0 \times [t_0, t_1]$ from $U_0 \times \{t_0\}$ to $U_1 \times \{t_1\}$. Then $H_t | U_0 \times \{t_0\}$ gives us a f.p. PL bicollaring of $U_0 \times \{m\}$ in $U_0 \times [-1,1]$, where m is the midpoint of the segment $[t_0, t_1]$. Since G covers H it follows that G gives us a f.p. PL bicollaring of $p^{-1}(U_0 \times \{m\})$. □

THEOREM 4.6. *Let M be a compact PL manifold, let B be a compact polyhedron, and let p: $B \times M \to B \times [-1,1]$ be a f.p. PL surjection (B = base). For every $b_0 \in B$ there exists a finite set $F_{b_0} \subset [-1,1]$ which satisfies the following property: corresponding to each $t_0 \in [-1,1] - F_{b_0}$ there is a neighborhood $U_0 \subset B$ of b_0 such that* proj: $p^{-1}(U_0 \times \{t_0\}) \to U_0$ *is (stably) a PL manifold bundle which is f.p. PL bicollared in $U_0 \times M$.*

Proof. It follows from Proposition 4.5 that such a U_0 exists for which $p^{-1}(U_0 \times \{t_0\})$ is f.p. PL bicollared in $U_0 \times M$. It follows from the PL bicollaring that the fibers of $p^{-1}(U_0 \times \{t_0\}) \to U_0$ are PL manifolds, so we need a trick to sew them together. The f.p. PL bicollaring of $p^{-1}(U_0 \times \{t_0\})$ easily implies that $p^{-1}(U_0 \times \{t_0\}) \times I^2 \to U_0$ is a PL submersion, so by using the PL Bundle Theorem of [**11**, p. 70] we conclude that $p^{-1}(U_0 \times \{t_0\}) \times I^2 \to U_0$ is a PL manifold bundle. □

5. Proof of Theorem 1. Our proof of Theorem 1_n ($n \geq 1$) is somewhat like the proof of Theorem 1_0 given in §3. It is given in Theorem 5.2 below, but first we will need a lemma which is a f.p. version of Lemma 3.1. For notation, let K be a compact polyhedron and assume that $K \times [-1, 1]$ is a compact subpolyhedron of a compact polyhedron Y for which $\mathrm{Bd}(K \times [-1, 1]) = K \times \{-1, 1\}$.

LEMMA 5.1. *Let $n \geq 1$ and $\varepsilon > 0$ be given. There exists a $\delta > 0$ so that if M is a compact PL manifold and r, s: $\Delta^n \times M \to \Delta^n \times Y$ are f.p. c-maps such that $r^{-1}(\Delta \times K \times \{0\})$, $s^{-1}(\Delta \times K \times \{0\})$, and $s^{-1}(\Delta \times K \times \{\frac{1}{3}\})$ are PL manifold bundles over Δ which are f.p. PL bicollared in $\Delta \times M$ and which satisfy $d(r_b, s_b) < \delta$ for all $b \in \Delta$, then (stably) there exists a f.p. PL homeomorphism h: $\Delta \times M \to \Delta \times M$ which satisfies*

(1) *$hs^{-1}(\Delta \times K \times \{0\}) = r^{-1}(\Delta \times K \times \{0\})$,*

(2) *h is supported on $r^{-1}(\Delta \times K \times [-1, 1])$,*

(3) *the compositions $(\mathrm{proj})r_b h_b$, $(\mathrm{proj})r_b$: $r_b^{-1}(K \times [-1, 1]) \to K$ are less than ε apart, for all b.*

Proof. Our proof is similar to the proof of Lemma 3.1 which treats the case $n = 0$. We will again omit the necessary stabilizing factor I^k, and our controlling maps will be

$$p_r = (\mathrm{proj})r\colon r^{-1}(\Delta \times K \times [-1, 1]) \to \Delta \times K,$$

$$p_s = (\mathrm{proj})s\colon s^{-1}(\Delta \times K \times [-1, 1]) \to \Delta \times K.$$

These are just fibered versions of the controlling maps which were used in Lemma 3.1. Again using [**11**, p. 70] it follows that $s^{-1}(\Delta \times K \times [0, \frac{1}{3}])$ is a PL manifold bundle over Δ, thus

$$\left(s^{-1}(\Delta \times K \times [0, \tfrac{1}{3}]),\, s^{-1}(\Delta \times K \times \{0\})\right) \xrightarrow{p_s} \Delta \times K$$

is a fibered γ-controlled h-cobordism, for every $\gamma > 0$. So using the Fibered Stable Controlled h-Cobordism Theorem of §2 there exists a f.p. PL homeomorphism φ of $s^{-1}(\Delta \times K \times [0, \frac{1}{3}])$ onto $s^{-1}(\Delta \times K \times \{0\}) \times [0, 1]$ which satisfies

(1) $\varphi s^{-1}(\Delta \times K \times \{0\}) = s^{-1}(\Delta \times K \times \{0\}) \times \{0\}$,

(3) $d(p_s(\mathrm{proj})\varphi, p_s)$ is small (as small as we want).

Repeating the argument given in Lemma 3.1 (see Assertion 1) we may additionally require φ to satisfy

(2) $\varphi s^{-1}(\Delta \times K \times \{\frac{1}{3}\}) = s^{-1}(\Delta \times K \times \{0\}) \times \{1\}$.

Using φ we obtain a f.p. PL homeomorphism h_1 of $r^{-1}(\Delta \times K \times [-1,1])$ onto itself for which $h_1 s^{-1}(\Delta \times K \times \{0\}) = s^{-1}(\Delta \times K \times \{\frac{1}{3}\})$ and $d(p_s h_1, p_s)$ is small (as small as we want).

Now let W denote the intersection of $r^{-1}(\Delta \times K \times [0,1])$ and $s^{-1}(\Delta \times K \times [-1, \frac{1}{3}])$. Then

$$\big(W, r^{-1}(\Delta \times K \times \{0\})\big) \xrightarrow{p_r} \Delta \times K$$

and

$$\big(W, s^{-1}(\Delta \times K \times \{\tfrac{1}{3}\})\big) \xrightarrow{p_s} \Delta \times K$$

are fibered γ-controlled h-cobordisms, for γ a number whose size depends on the size of δ. Once again using the Fibered Stable Controlled h-Cobordism Theorem we can repeat the proof of Assertion 2 of Lemma 3.1 to obtain a f.p. PL homeomorphism ψ of W onto $r^{-1}(\Delta \times K \times \{0\}) \times [0,1]$ which satisfies

(1) $\psi r^{-1}(\Delta \times K \times \{0\}) = r^{-1}(\Delta \times K \times \{0\}) \times \{0\}$,

(2) $\psi s^{-1}(\Delta \times K \times \{\frac{1}{3}\}) = r^{-1}(\Delta \times K \times \{0\}) \times \{1\}$,

(3) $d(p_r(\text{proj})\psi, p_r) < \delta_1$.

Using ψ we obtain a f.p. PL homeomorphism h_2 of $r^{-1}(\Delta \times K \times [-1,1])$ onto itself for which $h_2 r^{-1}(\Delta \times K \times \{0\}) = s^{-1}(\Delta \times K \times \{\frac{1}{3}\})$ and $d(p_r h_2, p_r)$ is small (provided that δ is chosen small). The homeomorphisms h_1 and h_2 trivially extend to f.p. PL homeomorphisms $\tilde{h}_1, \tilde{h}_2$: $\Delta \times M \to \Delta \times M$, and $h = \tilde{h}_2^{-1}\tilde{h}_1$: $\Delta \times M \to \Delta \times M$ is our desired homeomorphism. □

REMARKS. 1. There is a relative version of the above result which is a f.p. version of the relative version of Lemma 3.1. For this we are additionally given $C \subset U \subset K$, where C is compact and U is open, such that $r = s$ over $\Delta \times U \times [-1,1]$. The conclusion states that the PL homeomorphism h which was constructed above can be additionally required to satisfy $h = \text{id}$ on $r^{-1}(\Delta \times C \times [-1,1])$. The proof is identical.

2. There is another relative version of Theorem 5.1 in which we are additionally given $r_b = s_b$, for all b lying in a compact subpolyhedron P of Δ. The conclusion states that the PL homeomorphism h: $\Delta \times M \to \Delta \times M$ can be additionally required to satisfy $h \mid P \times M = \text{id}$. This follows easily from the correspnding relative version of the Fibered State Controlled h-Cobordism Theorem.

THEOREM 5.2. *Theorem 1_n is true, for all $n \geq 0$.*

Proof. As in Theorem 3.2 (i.e., the case $n = 0$) we will actually establish a relative version in which we are additionally given $C \subset U \subset Y$, where C is compact and U is open, for which $r = s$ over $\Delta \times U$. The conclusion additionally states that the homotopy $r \simeq s$ can be required to be constant over $\Delta \times C$. Our proof proceeds by induction on n, with the case $n = 0$ having been established in Theorem 3.2. For the inductive step we will assume that Theorem 1_{n-1} is true and use this to establish Theorem 1_n. For this proof of Theorem 1_n we also induct on $\dim Y$. It is trivially true for $\dim Y = 0$, so assuming it to be true for all Y of dimension $\leq k - 1$ we consider the inductive step where $\dim Y = k$. Continuing in the spirit of the proof of Theorem 3.2 we only treat the absolute version, and we will not keep careful track of the sizes of the homotopies that arise.

We start by choosing a triangulation of Y so that all the simplices have small diameters. The first step is to (stably) find small f.p. c-homotopies $r \simeq r_0$ and $s \simeq s_0 \operatorname{rel} \partial\Delta \times M$ such that r_0 is close to s_0 and $r_0 = s_0$ over the product of Δ with a neighborhood of some 0-simplex. Let our 0-simplex be v and let N be a small regular neighborhood of v. As in the proof of Theorem 3.2 let N_t be the t-level of N. Using Theorem 4.6 it follows that for each $b \in \Delta$, and all but a finite number of $t \in [0, 1]$, there exists a neighborhood $U \subset \Delta$ of b such that $r^{-1}(U \times \operatorname{Bd}(N_t)) \overset{\text{proj}}{\to} U$ is a PL manifold bundle which is f.p. PL bicollared in $U \times M$. The size of U depends on b and t. Of course a similar statement is true with s in place of r. Choose a fine triangulation of Δ, and for each barycenter b_σ of a simplex σ in Δ let Δ_σ be the closed star of b_σ in the second barycentric subdivision of Δ, i.e., $\Delta_\sigma = \operatorname{St}(b_\sigma, \Delta'')$. Then Δ_σ is an n-cell, and for fixed i the collection $\{\Delta_\sigma \mid \dim\sigma = i\}$ is a collection of pairwise-disjoint n-cells.

Now choose equally spaced numbers

$$0 < t_0 < t_0' < t_1 < t_1' < \cdots < t_n < t_n' < 1.$$

By the usual compactness argument we may assume that the triangulation of Δ is sufficiently fine so that for each σ there are numbers t, $|t_{\dim\sigma} - t| < 1/(2n + 3)$, for which

$$r^{-1}(\Delta_\sigma \times \operatorname{Bd}(N_t)) \overset{\text{proj}}{\to} \Delta_\sigma$$

is a PL manifold bundle which is f.p. PL bicollared in $\Delta_\sigma \times M$. There is no loss of generality in assuming that this is true for $t = t_{\dim\sigma}$. Similarly we may assume that

$$s^{-1}\left(\Delta_\sigma \times \operatorname{Bd}\left(N_{t_{\dim\sigma}}\right)\right) \to \Delta_\sigma$$

and

$$s^{-1}\big(\Delta_\sigma \times \mathrm{Bd}\big(N_{t'_{\dim\sigma}}\big)\big) \to \Delta_\sigma$$

are PL manifold bundles which are f.p. PL bicollared in $\Delta_\sigma \times M$.

It will be convenient to assume that the above bundles extend to bundles over small neighborhoods of the Δ_σ, with a similar statement about the f.p. bicollarings also being true. For any σ we can use Lemma 5.1 to obtain a f.p. PL homeomorphism h_σ: $\Delta_\sigma \times M \to \Delta_\sigma \times M$ which is supported on $r^{-1}(\Delta_\sigma \times N)$, which takes $s^{-1}(\Delta_\sigma \times \mathrm{Bd}(N_{t_{\dim\sigma}}))$ to $r^{-1}(\Delta_\sigma \times \mathrm{Bd}(N_{t_{\dim\sigma}}))$, and which is the identity on $r^{-1}((\Delta_\sigma \cap \partial\Delta) \times N)$. We can do this so that the h_σ have disjoint supports for different σ. By the construction of the h_σ given in Lemma 5.1 we can isotope them to the identity on slightly larger neighborhoods to obtain f.p. PL homeomorphisms $\tilde{h}_\sigma$: $\Delta \times M \to \Delta \times M$ with disjoint supports. By piecing together the $\tilde{h}_\sigma$ we obtain a f.p. PL homeomorphism h_0: $\Delta \times M \to \Delta \times M$ which satisfies

(1) h_0 is supported on $r^{-1}(\Delta \times N)$,
(2) $h_0 s^{-1}(\Delta_\sigma \times \mathrm{Bd}(N_{t_{\dim\sigma}})) = r^{-1}(\Delta_\sigma \times \mathrm{Bd}(N_{t_{\dim\sigma}}))$, for all σ,
(3) $h_0 \mid \partial\Delta \times M = \mathrm{id}$.

By choosing the numbers t'_i close to t_i it is clear that h_0 can be constructed so that rh_0 is close to r. Of course we can similarly treat all of the 0-simplices in Y simultaneously, but recall that we are working with a fixed 0-simplex v. This restriction considerably simplifies notation.

As in the case of the proof of Theorem 3.2 we now focus our attention on the f.p. c-map rh_0: $\Delta \times M \to \Delta \times M$. By squeezing towards the 0-simplex v of Y it is clear that there is a small f.p. c-homotopy $r \simeq rh_0$. This homotopy must be feathered out near $\partial\Delta \times M$ so that it is rel $\partial\Delta \times M$. Using our inductive hypotheses $(k-1)$ and the given f.p. PL bicollarings we obtain small f.p. c-homotopies $rh_0 \simeq r'_0$ and $s \simeq s'_0$ rel $\partial\Delta \times M$ so that $r'_0 = s'_0$ over $\Delta_\sigma \times \mathrm{Bd}(N_{t_{\dim\sigma}})$, for all σ. Moreover by using the collar structures we can additionally require that $r'_0 = s'_0$ over $\Delta_\sigma \times \mathrm{Bd}(N_t)$, for all t close to $t_{\dim\sigma}$. In analogy with the construction of h_0 above these homotopies are easily constructed by piecing together homotopies whose supports lie in slightly larger neighborhoods of the $s^{-1}(\Delta_\sigma \times \mathrm{Bd}(N_{t_{\dim\sigma}}))$.

Our goal is to construct small f.p. c-homotopies $r'_0 \simeq r_0$ and $s'_0 \simeq s_0$ rel $\partial\Delta \times M$ such that $r_0 = s_0$ over $\Delta \times N_t$, for some $t > 0$. This was accomplished in Theorem 3.2 with a squeeze towards the 0-simplex v. We can use a similar squeeze here, but it will have to be a variable squeeze over Δ. Assuming that $r'_0 = s'_0$ over $\tilde{\Delta}_\sigma \times \mathrm{Bd}(N_t)$, for $\tilde{\Delta}_\sigma$ a slightly larger neighborhood of Δ_σ and for all t close to t_0, we can perform a constant squeeze over $\bigcup\{\Delta_\sigma \mid \dim\sigma = 0\}$ to obtain small f.p. c-homotopies $r_0' \simeq a_0$ and $s_0' \simeq b_0$ rel $\partial\Delta \times M$ so that $a_0 = b_0$ over $\bigcup\{\Delta_\sigma \mid \dim\sigma = 0\} \times N_t$, for all t in a small neighborhood of t_0. We can also require the homotopies to be constant on $r^{-1}(\Delta \times (Y - N_{t_0'+\varepsilon}))$, for some small $\varepsilon > 0$, so in particular it follows that $a_0 = b_0$ over

$$\bigcup\{\Delta_\sigma \times \mathrm{Bd}(N_t) \mid \dim\sigma \geq 1 \text{ and } t \text{ close to } t_{\dim\sigma}\}.$$

For each i let $S_i = \bigcup\{\Delta_\sigma \mid \dim\sigma \leq i\}$. Now define a PL map $\varphi\colon S_1 \to [t_0, t_1]$ so that $\varphi(S_0) = \{t_0\}$ and $\varphi(b) = t_1$, for all b outside of a small neighborhood of S_0. Using the inductive hypothesis $(n-1)$ there exist small f.p. c-homotopies $a_0 \simeq a_1'$ and $b_0 \simeq b_1'$ rel$(\partial\Delta \times M) \cup (S_0 \times M)$ such that $a_1' = b_1'$ over

$$\bigcup\{\{b\} \times \mathrm{Bd}(N_t) \mid b \in S_1 \text{ and } t \text{ close to } \varphi(b)\}.$$

We can now perform a variable squeeze over S_1 to obtain small f.p. c-homotopies $a_1' \simeq a_1$ and $b_1' \simeq b_1$ rel $\partial\Delta \times M$ so that $a_1 = b_1$ over $\bigcup\{\{b\} \times N_{\varphi(b)} \mid b \in S_1\}$. All of these homotopies are constant over $r^{-1}(\Delta \times (Y - N_{t_1'}))$, so we have $a_1 = b_1$ over

$$\bigcup\{\Delta_\sigma \times \mathrm{Bd}(N_t) \mid \dim\sigma \geq 2 \text{ and } t \text{ close to } t_{\dim\sigma}\}.$$

It is now clear that we can continue this process to obtain small f.p. c-homotopies

$$r_0' \simeq a_0 \simeq a_1 \simeq \cdots \simeq a_n = r_0,$$
$$s_0' \simeq b_0 \simeq b_1 \simeq \cdots \simeq b_n = s_0$$

rel $\partial\Delta \times M$ so that $r_0 = s_0$ over $\Delta \times N_t$, for some $t > 0$. This achieves our desired goal. By carrying out the same procedure over each 0-simplex in Y we may assume that $r_0 = s_0$ over the product of Δ with a neighborhood of the 0-skeleton.

We are now in a position to repeat the same argument for the 1-skeleton of Y. The first thing to do is to completely abandon the above triangulation of Δ and repeat the process with a new triangulation (chosen in conjunction with Theorem 4.6). As explained in the corresponding step in the proof of Theorem 3.2, the only new ingredient is the use of the relative version of Lemma 5.1. When this step is carried out we obtain small f.p. c-homotopies $r_0 \simeq r_1$ and $s_0 \simeq s_1$ rel $\partial\Delta \times M$ such that $r_1 = s_1$ over the product of Δ with a neighborhood of the 1-skeleton. After k steps the process terminates. □

6. Proof of Theorem 2. Our proof of Theorem 2 is given below, but first we will have to establish some preliminary results.

LEMMA 6.1. *Let $E_0 \to B$ and $E_1 \to B$ be nice PL manifold bundles, let $E \to B$ be a PL fibration, and let r_0: $E_0 \to E$, r_1: $E_1 \to E$ be f.p. c-maps. Then for every $\varepsilon > 0$ there (stably) exists a f.p. PL homeomorphism h: $E_0 \to E_1$ such that $d(r_1 h, r_0) < \varepsilon$.*

Proof. It will be simpler to first treat the case $B = \{\text{point}\}$. For this we may assume that E_0 and E_1 are nicely embedded in some R^m (i.e., $\dim E_0 = \dim E_1 = m$) and we may also assume that E is a subpolyhedron of R^m. Fix $\delta > 0$ and let us assume that there is a PL embedding u: $E \to \mathring{E}_0$ which is a δ-inverse of r_0. If we choose $m \geq 2 \dim E + 2$, then we can extend u to a PL homeomorphism $\tilde{u}$: $R^m \to R^m$. Since we are allowed to stabilize we may assume that r_1: $E_1 \to E$ is of the form $r_1'(\text{proj})$, where r_1' takes the spine of E_1 to E. If $m \geq 2\dim(\text{spine}) + 2$, then there exists a PL homeomorphism v: $R^m \to R^m$ such that $v \,|\, E_1$ is close to r_1 (as close as we want). If it is close enough, then $\tilde{u}v(E_1) \subset \mathring{E}_0$.

Assertion. If $v \,|\, E_1$ is chosen sufficiently close to r_1, then the inclusion $\tilde{u}v(E_1) \hookrightarrow E_0$ is an $r_0^{-1}(3\delta)$-equivalence.

Proof. Since u: $E \to E_0$ is a δ-inverse of r_0 one can easily check that the inclusion $u(E) \hookrightarrow E_0$ is an $r_0^{-1}(3\delta)$-equivalence with inverse ur_0: $E_0 \to u(E)$. Observe that for any given $\gamma > 0$ we can choose $v \,|\, E_1$ close enough to r_1 so that $r_1 v^{-1}$: $v(E_1) \to E$ is γ-homotopic to the identity. We may therefore assume that

$$u\left(r_1 v^{-1}\right)\tilde{u}^{-1}\colon \tilde{u}v(E_1) \to u(E)$$

is homotopic to the identity via a homotopy that is small and takes place in E_0. So in order to check that $\tilde{u}v(E_1) \hookrightarrow E_0$ is an $r_0^{-1}(3\delta)$-equivalence it suffices to check that the composition

$$\tilde{u}v(E) \xrightarrow{u(r_1 v^{-1})\tilde{u}^{-1}} u(E) \hookrightarrow E_0$$

is an $r_0^{-1}(3\delta)$-equivalence. Since the first map is a c-map and the second map is an $r_0^{-1}(3\delta)$-equivalence, this is straightforward. □

Now consider the cobordism $(E_0 \times [0,1], \tilde{u}v(E_1))$, where $\tilde{u}v(E_1) \equiv \tilde{u}v(E_1) \times \{0\}$. Using $r_0(\text{proj})$: $E_0 \times [0,1] \to E$ as a controlling map we see from the Assertion that $(E_0 \times [0,1], \tilde{u}v(E_1))$ is a 3δ-controlled h-cobordism. Moreover the restriction $r_0(\text{proj})\,|$: $\tilde{u}v(E_1) \to E$ is homotopic to the c-map $r_1 v^{-1}\tilde{u}^{-1}$: $\tilde{u}v(E_1) \to E$ via the following homotopy in E:

$$r_0 \,|\, \tilde{u}v(E_1) \simeq r_0\left(u r_1 v^{-1}\tilde{u}^{-1}\right) |\, \tilde{u}v(E_1) \simeq r_1 v^{-1}\tilde{u}^{-1} \,|\, \tilde{u}v(E_1).$$

The first comes from the proof of the above Assertion, and it can be made as small as we want. The second comes from the fact that $r_0 u \simeq \mathrm{id}$, so it is a δ-homotopy. Thus we have a δ-homotopy of $r_0(\mathrm{proj})$ to a map p: $E_0 \times [0,1] \to E$ for which $p \mid \tilde{u}v(E_1)$ is a c-map. Using p as a controlling map we see that $(E_0 \times [0,1], \tilde{u}v(E_1))$ is a δ'-controlled h-cobordism, where the size of δ' depends on the size of δ. Using the Stable Controlled h-Cobordism Theorem of §2 we conclude that $(E_0 \times [0,1], \tilde{u}v(E_1))$ is (stably) an ε'-product, where the size of ε' depends on the size of δ'. This product structure gives us a PL homeomorphism k: $E_0 \times [0,1] \to \tilde{u}v(E_1) \times [0,1]$ such that $r_0(\mathrm{proj})k$ is close to $r_0(\mathrm{proj})$. Then our desired homeomorphism is

$$h = \left((\tilde{u}v)^{-1} \times \mathrm{id}\right)k\colon\ E_0 \times [0,1] \to E_1 \times [0,1].$$

This completes the proof of the case $n = 0$.

Finally we remark that the proofs of the cases $n \geq 1$ are all similar to the above proof of the case $n = 0$, with the main difference being the use of the Fibered Stable Controlled h-Cobordism Theorem of §2. □

REMARKS 1. It follows from the above proof that if we are additionally given a compact subpolyhedron B' of B for which $r_0 = r_1$ over B', then the homeomorphism h can be constructed to additionally satisfy $h = \mathrm{id}$ over B'.

2. Since $E_0 \to B$ is a nice PL manifold bundle it follows that the product bundle $E_0 \times [0,1] \to B \times [0,1]$ is nice. Using the fact that $E_0 \times [0,1]$ is f.p. PL homeomorphic to $C(h)$ (over $B \times [0,1]$) we conclude that $C(h) \to B \times [0,1]$ is also a nice PL manifold bundle.

LEMMA 6.2. *Let $E_0 \to B$ and $E_1 \to B$ be nice PL manifold bundles, let $E \xrightarrow{p} B$ be a PL fibration, and let r_0: $E_0 \to E$, r_1: $E_1 \to E$ be f.p. c-maps. Then (stably) there exists a nice PL manifold bundle $\mathscr{E} \to B \times [0,1]$ such that $\mathscr{E} \mid B \times \{0\} = E_0$, $\mathscr{E} \mid B \times \{1\} = E_1$, and there exists a f.p. c-map r: $\mathscr{E} \to E \times [0,1]$ for which $r = r_0$ over $B \times \{0\}$ and $r = r_1$ over $B \times \{1\}$.*

Proof. To make sure that there is no confusion in the above statement we are regarding $E \times [0,1]$ as the product fibration $E \times [0,1] \xrightarrow{p \times \mathrm{id}} B \times [0,1]$. Using Lemma 6.1 there exists a f.p. PL homeomorphism h: $E_0 \to E_1$ such that $r_1 h$ is close to r_0. It follows from Theorem 1 that if these c-maps are sufficiently close, then there is a c-homotopy between them. Thus we obtain a f.p. c-map F: $E_0 \times [0, \frac{1}{2}] \to E \times [0, \frac{1}{2}]$ for which $F_0 = r_0$ and

$F_{1/2} = r_1 h$. To define $\mathscr{E} \mid B \times [0, \frac{1}{2}]$ and r over $B \times [0, \frac{1}{2}]$ we let $\mathscr{E} \mid B \times [0, \frac{1}{2}] = E_0 \times [0, \frac{1}{2}]$ and $r = F$.

Recalling the Remarks following Lemma 6.1 we know that $C(h) \to B \times [0,1]$ is a nice PL manifold bundle. If we linearly identify $[0,1]$ with $[\frac{1}{2},1]$ it follows that $C(h) \to B \times [\frac{1}{2},1]$ is a nice PL manifold bundle which equals E_0 over $B \times \{\frac{1}{2}\}$ and equals E_1 over $B \times \{1\}$. Then sew $C(h)$ to $\mathscr{E} \mid B \times [0, \frac{1}{2}]$ by identifying $E_0 \times \{\frac{1}{2}\}$ in $\mathscr{E} \mid B \times [0, \frac{1}{2}]$ with E_0 in $C(h)$. This gives us a nice PL manifold bundle $\mathscr{E} \to B \times [0,1]$ which equals $C(h)$ over $B \times [\frac{1}{2},1]$. To define r over $B \times [\frac{1}{2},1]$ we need a f.p. c-map of $C(h)$ to $E \times [\frac{1}{2},1]$ which agrees with $r_1 h$ on E_0 and which agrees with r_1 on E_1. We observed in §2 that there exists a f.p. c-map of $C(h)$ to $E_1 \times [\frac{1}{2},1]$ which agrees with h on E_0 and which is the identity on E_1. Then compose this map with $r_1 \times \mathrm{id}$: $E_1 \times [\frac{1}{2},1] \to E \times [\frac{1}{2},1]$. □

Here is some notation which will be needed in the statement of our next result. Let $E \xrightarrow{p} B$ be a PL fibration and let $B \xrightarrow{\varphi} B$ be a PL map. The *fibered product* of φ and p is the PL fibration $E' \xrightarrow{p'} B$ which is defined by

$$E' = \{(b,e) \mid \varphi(b) = p(e)\} \subset B \times E$$

and $p'((b,e)) = b$. Here is a fairly straightforward result whose proof is left to the reader.

PROPOSITION 6.3. (1) *If ψ: $B \to B$ is another PL map, then the fibered product of ψ and p' is f.p. PL homeomorphic to the fibered product of $\varphi\psi$ and p.*

(2) *If E is a nice PL manifold bundle, then so is E'.*

(3) *If E is the f.p. c-image of a nice PL manifold bundle, then so is E'.* □

We say that a PL fibration $E \to B$ is *nice* if there exists a nice PL manifold bundle $\mathscr{E} \to B$ and a f.p. c-map $\mathscr{E} \xrightarrow{r} E$. The map r will be referred to as a *resolution* of the fibration.

PROPOSITION 6.4. *Let $E \xrightarrow{p} \Delta^n$ be a nice PL fibration and let $\mathscr{E}_0 \xrightarrow{r_0} E \mid \partial\Delta$ be a resolution. Then (stably) there exist a PL map φ: $\Delta \to \Delta$ such that $\varphi \mid \partial\Delta = \mathrm{id}$ and a resolution $\mathscr{E} \xrightarrow{r} E'$ (E' = fibered product of φ and p) such that $r = r_0$ over $\partial\Delta$.*

Proof. By choosing a collaring of $\partial\Delta$ in Δ we can write $\Delta = \Delta_0 \cup (\partial\Delta \times [0,1])$ such that $\partial\Delta \times \{0\} = \partial\Delta_0$ and $\partial\Delta \times \{1\} = \partial\Delta$. Define φ: $\Delta \to \Delta$ to be any PL map for which $\varphi((x,t)) = (x,1)$, for all $(x,t) \in \partial\Delta \times [0,1]$. Then the fibered product $E' \to \Delta$ is still nice and it agrees with $E\,|\,\partial\Delta$ over each $\partial\Delta \times \{t\}$. We define $\mathscr{E}\,|\,\partial\Delta = \mathscr{E}_0$ and $r = r_0$ over $\partial\Delta$, and we define $\mathscr{E}|\Delta_0 \xrightarrow{r} E'|\Delta_0$ to be any resolution. Using Lemma 6.2 we can find a resolution $\mathscr{E}|(\partial\Delta \times [0,1]) \to E'|(\partial\Delta \times [0,1])$ which agrees with the choices already made over $\partial\Delta \times \{0\}$ and $\partial\Delta \times \{1\}$. These choices all piece together to give our desired resolution $\mathscr{E} \xrightarrow{r} E'$. □

Proof of Theorem 2. Let $E \to \Delta^n$ be a PL fibration which is a nice PL manifold bundle over $\partial\Delta$. Our goal is to (stably) find a PL fibration over $\Delta \times [0,1]$ which agrees with E over $\Delta \times \{0\}$, which agrees with $E\,|\,\partial\Delta$ over each $\partial\Delta \times \{t\}$, and which is a nice PL manifold bundle over $\Delta \times \{1\}$. This will suffice to establish Theorem 2. Our strategy for doing this involves an induction on the integer n, but in order to carry out the inductive step we will need a somewhat stronger result. Specifically we will establish the following statement. In it, $E \xrightarrow{p} B$ is a PL fibration for which E and B have been triangulated so that p takes each simplex of E linearly to a simplex of B.

$\mathbf{P_n}$: *If* $\dim B = n$ *and* B_0 *is a subcomplex of* B *for which* $E\,|\,B_0$ *is a nice PL manifold bundle, then* (*stably*) *there exists a PL map* φ: $B \to B$ *such that* $\varphi\,|\,B_0 = \mathrm{id}$, φ *is simplex-preserving, and the fibered product of* φ *and* p *has a resolution which is the identity over* B_0.

To see how P_n implies Theorem 2 let $E \to \Delta^n$ be a PL fibration which is a nice PL manifold bundle over $\partial\Delta$. Using P_n there exists a PL map φ: $\Delta \to \Delta$, $\varphi\,|\,\partial\Delta = \mathrm{id}$, such that the fibered product E' of φ and p has a resolution $\mathscr{E} \xrightarrow{r} E'$ which is the identity over $\partial\Delta$. Then $E' = E$ over $\partial\Delta$ and the mapping cylinder $C(r)$ is a PL fibration over $\Delta \times [0,1]$ from $\mathscr{E}$ to E' which equals $E\,|\,\partial\Delta$ over each $\partial\Delta \times \{t\}$. Since $\varphi\,|\,\partial\Delta = \mathrm{id}$ there exists a PL homotopy Φ: $\Delta \times I \to \Delta$ from φ to the identity rel $\partial\Delta$. The fibered product of Φ and p gives us a PL fibration over $\Delta \times I$ from E' to E which equals $E\,|\,\partial\Delta$ over each $\partial\Delta \times \{t\}$. These two PL fibrations piece together to give our desired PL fibration over $\Delta \times [0,1]$ whose 0-level is E and whose 1-level is $\mathscr{E}$.

Turning to the proof of P_n it is obvious that P_0 is true, so all we have to do is establish $P_{n-1} \Rightarrow P_n$. For this we are given a PL fibration $E \xrightarrow{p} B$, $\dim B = n$, and a subcomplex B_0 of B for which $E\,|\,B_0$ is a nice PL manifold bundle. Using the inductive hypothesis there exists a PL map φ_0: $B^{n-1} \cup B_0 \to B^{n-1} \cup B_0$ such that $\varphi_0 | B_0 = \mathrm{id}$, φ_0 is simplex-preserving, and the fibered product of φ_0 and $p|(E|B^{n-1} \cup B_0)$ has a resolution which is the identity over B_0. Extend φ_0 to a simplex-preserving PL map φ_1: $B \to B$. The fibered product of φ_1 and p is a PL fibration $E_1 \xrightarrow{p_1} B$ such that $E_1 | B^{n-1} \cup B_0$ has a resolution $\mathscr{E}_1 \to E_1$ which is the identity over B_0. Since $E | \Delta^n \to \Delta^n$ is simplicial it follows from §2 that $E | \Delta^n$ has a resolution, for all n-simplices Δ^n in B. This implies that $E_1 | \Delta^n$ also has a resolution.

Applying Proposition 6.4 to each Δ^n we can find a PL map φ_2: $B \to B$ such that $\varphi_2 | B^{n-1} \cup B_0 = \mathrm{id}$, φ_2 is simplex-preserving, and the fibered product $E_2 \to B$ of φ_2 and p_1 has a resolution $\mathscr{E} \to B$ which extends the resolution $\mathscr{E}_1 \to E_1 | B^{n-1} \cup B_0$. Thus $\varphi = \varphi_1\varphi_2$ is our desired simplex-preserving map and $\mathscr{E}$ is our desired resolution of φ and p. □

7. Proof of Theorem 3. Our proof of Theorem 3 is modeled on the proof of Theorem 2, so we will first have to show how the tools that were used there can be appropriately modified. The first tool, which is established in Lemma 7.1, is a modification of Theorem 1. Then in Lemma 7.2 we establish a modification of Lemma 6.1 which avoids the "niceness" assumption that was needed there.

LEMMA 7.1. *Let $n \geq 0$ and $\varepsilon > 0$ be given. For any compact polyhedral pair (Y, N), where N is a compact PL manifold, there exists a $\delta > 0$ so that if M is a compact PL manifold for which $N \subset \partial M$ is PL collared in M and r, s: $\Delta^n \times M \to \Delta^n \times Y$ are f.p. c-maps such that $r = s$ on $(\partial\Delta \times M) \cup (\Delta \times N)$ and $d(r, s) < \delta$, then (stably) there is a f.p. c-homotopy $r \simeq s$ rel$(\partial\Delta \times M) \cup (\Delta \times N)$ which is also an ε-homotopy.*

Proof. One way to prove this result is to repeat the proof of Theorem 1. Another way to do it is to use the relative version which arose during the course of the proof of Theorem 1. It is this latter approach that we adopt. Define $\tilde{M} = M \cup (N \times [0,1])$ and $\tilde{Y} = Y \cup (N \times [0,1])$, where the sewings identify N with $N \times \{1\}$. We can trivially extend r and s to f.p. c-maps $\tilde{r}, \tilde{s}$: $\Delta \times \tilde{M} \to \Delta \times \tilde{Y}$ which are equal on $(\partial\Delta \times \tilde{M}) \cup (\Delta \times N \times [0,1])$ and which are still δ-close. By the relative version of Theorem 1 there exists a small f.p. c-homotopy $\tilde{r} \simeq \tilde{s}$ rel$(\partial\Delta \times \tilde{M}) \cup (\Delta \times N \times \{0\})$. If we identify $N \times [0,1]$ with a collaring of N in M, then we obtain an identification of $\tilde{M}$ with M so that $N \times \{0\}$ in $\tilde{M}$ is identified with N in M. Thus $\tilde{r}$ and $\tilde{s}$ become maps from $\Delta \times M$ to $\Delta \times \tilde{Y}$.

Let $u\colon \tilde{Y} \to Y$ be the c-map obtained by collapsing-out the $[0,1]$ factor in $N \times [0,1]$ and then consider the maps

$$(\mathrm{id} \times u)\tilde{r}, (\mathrm{id} \times u)\tilde{s}\colon \Delta \times M \to \Delta \times Y.$$

By the above c-homotopy $\tilde{r} \simeq \tilde{s}$, there exists a small f.p. c-homotopy $(\mathrm{id} \times u)\tilde{r} \simeq (\mathrm{id} \times u)\tilde{s}\,\mathrm{rel}(\partial\Delta \times M) \cup (\Delta \times N)$. Finally by using the collaring of N in M it is easy to see that there exist small f.p. c-homotopies $(\mathrm{id} \times u)\tilde{r} \simeq r$ and $(\mathrm{id} \times u)\tilde{s} \simeq s\ \mathrm{rel}(\partial\Delta \times M) \cup (\Delta \times N)$. □

LEMMA 7.2. *Let $(E_0, B \times N) \to B$ and $(E_1, B \times N) \to B$ be PL manifold bundles rel N, let $(E, B \times N) \to B$ be a PL fibration rel N, and let $r_0\colon E_0 \to E$, $r_1\colon E_1 \to E$ be f.p. c-maps such that $r_0 = r_1 = \mathrm{id}$ on $B \times N$. Then for every $\varepsilon > 0$ there (stably) exists a f.p. PL homeomorphism $h\colon E_0 \to E_1$ such that $h = \mathrm{id}$ on $B \times N$ and $d(r_1 h, r_0) < \varepsilon$.*

Proof. Following closely the proof of Lemma 6.1 it will suffice to consider only the case $B = \{\text{point}\}$. Recall that the key to the proof of Lemma 6.1 was the fact that E_0, E_1, and E could all be embedded in R^m so that E_0 and E_1 are nice. Once everything was inside R^m, PL unknotting of polyhedra was used to construct an isotopy that moved E_1 into E_0 so that the Stable Controlled h-Cobordism Theorem could be applied.

Unfortunately, in our situation the given manifolds are not necessarily nice, so we will need a replacement for the ambient space R^m. Since we are allowed to stabilize it follows that (E_0, N) and (E_1, N) are ordinary h-cobordisms. Thus by the invertibiltiy of h-cobordisms [**13**, p. 208] we may assume that $E_0 \subset N \times [0,1]$ and $E_1 \subset N \times [0,1]$, where $N \equiv N \times \{0\}$ in $N \times [0,1]$. Moreover we may assume that $E \subset N \times [0,1]$. Now using the ambient space $N \times [0,1]$ in place of R^m we may proceed as in the proof of Lemma 6.1. □

REMARKS (1) It follows from the above proof that if we are additionally given a compact subpolyhedron B' of B for which $r_0 = r_1$ over B', then the homeomorphism h can be constructed to additionally satisfy $h = \mathrm{id}$ over B'.

(2) It is also easy to see that $C(h) \to B \times [0,1]$ is a PL manifold bundle rel N.

Proof of Theorem 3. Armed with Lemmas 7.1 and 7.2 we are now able to easily modify the proof of Theorem 2 in order to obtain a proof of Theorem 3. □

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