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SOME EXPLICIT UPPER BOUNDS ON THE CLASS NUMBER AND REGULATOR OF A CUBIC FIELD WITH NEGATIVE DISCRIMINANT

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Explicit upper bounds are developed for the class number and the regulator of any cubic field with a negative discriminant. Lower bounds on the class number are also developed for certain special pure cubic fields.

1. Introduction. Let $\mathcal{K}$ be any cubic number field with discriminant $\Delta < 0$ and regulator R. Since either $4 \mid \Delta$ or $\Delta \equiv 1 \pmod 4$, we may assume that $\Delta = df^2$, where d is the discriminant of a quadratic field. Further, since $d < 0$ and either $4 \mid d$ or $d \equiv 1 \pmod 4$, we must have $|d| \geq 3$. Let $\mathcal{O}_{\mathcal{K}}$ be the ring of all algebraic integers of $\mathcal{K}$ and let h be the number of ideal classes of $\mathcal{O}_{\mathcal{K}}$.

From a classical, general result of Landau [11] we know that

$$hR = O\left(\sqrt{|\Delta|}\,(\log|\Delta|)^2\right).$$

More recently Siegel [19] and Lavrik [13] have given general results from which an explicit constant c can be easily determined such that

$$hR < c\sqrt{|\Delta|}\,(\log|\Delta|)^2.$$

However, in the case of a pure cubic field ($d = -3$), Cohn [6] has shown that

$$hR = O\left(\sqrt{|\Delta|}\log|\Delta|\log\log|\Delta|\right).$$

In this paper we will develop an explicit upper bound on hR which depends on d and f ($= \sqrt{\Delta/d}$). In the pure cubic case our results give

$$hR < \frac{\sqrt{|\Delta|}}{6\sqrt{3}}\log|\Delta|.$$

We make use of the well-known fact that

$$\Phi(1) = \lim_{s\to 1}\frac{\zeta_{\mathcal{K}}(s)}{\zeta(s)} = h\kappa,$$

where

$$\kappa = CR \quad\text{and}\quad C = 2\pi/\sqrt{|\Delta|}\,.$$

Now

$$\Phi(s) = \zeta_{\mathscr{K}}(s)/\zeta(s) = \sum_{n=1}^{\infty} \alpha(n) n^{-s},$$

where

$$\alpha(n) = \sum_{j \mid n} \mu(j) F(n/j) \tag{1.1}$$

and $F(k)$ denotes the number of distinct ideals of $\mathcal{O}_{\mathscr{K}}$ with norm k. Also,

$$\Phi(1-s) = C^{-2s+1}(\Gamma(s)/\Gamma(1-s))\Phi(s);$$

hence, by using a result of Barrucand [**1**], we get

$$\Phi(1) = \sum_{j=1}^{\infty} \alpha(j) j^{-1} e^{-jC} + C \sum_{j=1}^{\infty} \alpha(j) E(jC),$$

where

$$E(x) = \int_x^{\infty} e^{-t} t^{-1}\, dt < e^{-x}/x.$$

Thus,

$$\Phi(1) < 2 \sum_{j=1}^{\infty} |\alpha(j)| j^{-1} e^{-jC},$$

and, if we put

$$A(x) = \sum_{j=1}^{\infty} |\alpha(j)| j^{-1} e^{-jx}, \tag{1.2}$$

we get

$$hRC < 2A(C) \tag{1.3}$$

It follows that we can easily bound R once we can obtain an upper bound on $A(C)$.

2. The function $\alpha(k)$. As $\alpha(k)$ is a rather difficult function to work with, we will develop a simpler function $\beta(k)$ such that

$$|\alpha(k)| \leq \beta(k). \tag{2.1}$$

We first note that since $F(k)$ is a multiplicative function and $F(1) = 1$, then $\alpha(k)$ is also a multiplicative function and $\alpha(1) = 1$. We need now only consider the problem of determining $\alpha(p^n)$, where p is any rational prime. By (1.1) we have

$$\alpha(p^n) = F(p^n) - F(p^{n-1}); \tag{2.2}$$

hence, it suffices here to determine $F(p^n)$. In order to do this we will need to know how the ideal (p) splits in $\mathcal{O}_{\mathscr{K}}$. A convenient summary, describing the five different types A, B, C, D, E of possible rational prime factorization in $\mathcal{O}_{\mathcal{K}}$, can be found in Hasse [11] or Barrucand [2]. In Table 1 below we present those results which will be useful in the sequel. As usual we use the symbol (a/b) to denote the Kronecker symbol. We also use the symbols $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{p}''$ to denote prime ideal factors of (p) with norm p and the symbol $\mathfrak{q}$ to denote a prime ideal factor of (p) with norm p^2.

TABLE 1

Type	Factorization of (p)	Quadratic Characters	Remarks
A	$\mathfrak{p}\mathfrak{p}'\mathfrak{p}''$	$(\Delta/p) = 1$	—
B	(p)	$(\Delta/p) = 1$	inert
C	$\mathfrak{p}\mathfrak{q}$	$(\Delta/p) = -1$	—
D	$\mathfrak{p}^2\mathfrak{p}'$	$(d/p) = 0, (f/p) \neq 0$	ramified
E	$\mathfrak{p}^3$	$(f/p) = 0$	ramified

Define

$$\beta^*(k) = \begin{cases} \beta(k) & \text{when } (k, f) = 1, \\ 0 & \text{when } (k, f) > 1, \end{cases}$$

where

$$\beta(k) = \sum_{j \mid k} (d/j). \tag{2.3}$$

If p is of type A, we see that $F(p^n)$ is the number of possible triples of non-negative integers k, j, k such that $i + j + k = n$; that is, $F(p^n) = \binom{n+2}{2}$. By using similar reasoning and (2.2) we get the results listed in Table 2.

TABLE 2

Type	n	$F(p^n)$	$\alpha(p^n)$	$\beta^*(p^n)$
A	any	$(n+2)(n+1)/2$	$n+1$	$n+1$
B	$n \equiv 0 \pmod 3$	1	1	$n+1$
B	$n \equiv 1 \pmod 3$	0	-1	$n+1$
B	$n \equiv 2 \pmod 3$	0	0	$n+1$
C	$n \equiv 0 \pmod 2$	$(n+2)/2$	1	1
C	$n \equiv 1 \pmod 2$	$(n+1)/2$	0	0
D	any	$n+1$	1	1
E	any	1	0	0

Since $\beta(k)$ is multiplicative and $\beta(1) = 1$, we get

$$\beta(k) \geq \beta^*(k) \geq |\alpha(k)| \geq 0.$$

3. An upper bound on CRh. If we put

$$B(x) = \sum_{j=1}^{\infty} \beta(j) j^{-1} e^{-jx}, \tag{3.1}$$

then by (1.2), (1.3), (2.1), and (2.3) we get

$$hRC < 2B(C). \tag{3.2}$$

In this section we will determine an explicit upper bound on $B(C)$. If we take x and c to be positive real numbers, by an inverse Mellin transform

$$\begin{aligned} B(x) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \Gamma(s) \sum_{n=1}^{\infty} \frac{\beta(n)}{n^{s+1}}\, ds \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \Gamma(s) \zeta(s+1) L(s+1)\, ds, \end{aligned}$$

where $L(s)$ is the associated L function

$$L(s) = \sum_{n=1}^{\infty} (d/n) n^{-s}.$$

Now the functions ζ and L satisfy the functional equations

$$\zeta(1-s) = \frac{2}{(2\pi)^s} \cos\frac{\pi s}{2} \Gamma(s) \zeta(s),$$

$$L(1-s) = \frac{2}{(2\pi)^s} |d|^{s-1/2} \sin\frac{\pi s}{2} \Gamma(s) L(s) \qquad (d < 0) \tag{3.3}$$

(see [**8**] Ch. 9); thus, by using the relation

$$\Gamma(s)\Gamma(-s) = -\pi/(s \sin \pi s),$$

we see that the integrand

$$\Lambda(s) = x^{-s} \Gamma(s) \zeta(s+1) L(s+1)$$

satisfies

$$\Lambda(-s) = -\frac{|d|^{s-1/2} x^s}{s(2\pi)^{2s-1}} \Gamma(s) \zeta(s) L(s). \tag{3.4}$$

As $s \to 0$, $\Gamma(s) = s^{-1} - \gamma + O(s)$ and $\zeta(s+1) = s^{-1} + \gamma + O(s)$. ($\gamma$ here is Euler's constant $.577215665\ldots$.) (See [**16**], §§12.1, 13.21.) Thus, $\Lambda(s)$ has a double pole at $s = 0$ and if we write $L(s+1) = a + bs + O(s^2)$ with $a = L(1)$, $b = L'(1)$, we find, by expanding the various functions about $s = 0$,

$$\begin{aligned}\Lambda(s) &= (1 - s\log x + \cdots)(s^{-1} - \gamma + \cdots)\\ &\quad\times(s^{-1} + \gamma + \cdots)(a + bs + \cdots)\\ &= \frac{a}{s^2} + \frac{b - a\log x}{s} + O(1)\end{aligned}$$

as $s \to 0$. From the functional equations for ζ and L we see that $\zeta(s+1)L(s+1)$ has simple zeros at integral values of $s < -1$; hence, $\Lambda(s)$ has no poles except for the double pole at $s = 0$ and the simple pole at $s = -1$. Also, the residue at $s = -1$ is

$$kx = \lim_{s\to -1}(s+1)\Lambda(s) = -\zeta(0)L(0)x.$$

Since $\zeta(0) = -1/2$ and, by (3.3), $L(0) = |d|^{1/2}L(1)/\pi = |d|^{1/2}a/\pi$, we have

$$k = a|d|^{1/2}/2\pi.$$

Let S be a positive real number > 1. By Stirling's formula in the form

$$\Gamma(\sigma + it) = O\left(e^{-\pi|t|/2}|t|^{\sigma - 1/2}\right)$$

as $|t| \to \infty$, and standard estimates for ζ and L (as in **[20]** §13.51),

$$\Lambda(\sigma + it) = O\left(e^{-\pi|t|/2}|t|^{S}\right)$$

as $|t| \to \infty$, uniformly for $-S \le \sigma \le c$ and for each fixed x. We can therefore move the line of integration in the integral for $B(x)$ from $\mathrm{Re}(s) = c$ to $\mathrm{Re}(s) = -S$. This gives

$$(3.5)\quad B(x) = b - a\log x + kx + \frac{1}{2\pi i}\int_{-S-i\infty}^{-S+i\infty}\Lambda(s)\,ds \qquad (S > 1).$$

By (3.4) The integral here is

$$\begin{aligned}T(x) &= \frac{1}{2\pi i}\int_{S-i\infty}^{S+i\infty}\frac{|d|^{s-1/2}x^s}{s(2\pi)^{2s-1}}\Gamma(s)\zeta(s)L(s)\,ds\\ &= \frac{1}{2\pi i}\int_{S-i\infty}^{S+i\infty}\frac{|d|^{s-1/2}x^s\Gamma(s)}{s(2\pi)^{2s-1}}\sum_{n=1}^{\infty}\frac{\beta(n)}{n^s}\,ds\\ &= \frac{2\pi}{\sqrt{|d|}}\sum_{n=1}^{\infty}\beta(n)\left(\frac{1}{2\pi i}\int_{S-i\infty}^{S+i\infty}\left(\frac{4\pi^2 n}{|d|x}\right)^{-s}\frac{\Gamma(s)}{s}\,ds\right).\end{aligned}$$

Thus, by evaluating the Mellin transforms above, we get

$$T(x) = \frac{2\pi}{\sqrt{|d|}} \sum_{n=1}^{\infty} \beta(n) E\left(\frac{4\pi^2 n}{|d| x}\right). \tag{3.6}$$

Since $E(y) < e^{-y}/y$ when $y > 0$, from (3.6) we have

$$T(C) < \frac{1}{f} \sum_{n=1}^{\infty} \frac{\beta(n)}{n} e^{-2\pi f n/\sqrt{|d|}}$$

Put[1] $N = [|d|/4\pi^2 f^2]$, and set

$$G = \frac{1}{f} \sum_{n=1}^{N} \frac{\beta(n)}{n} e^{-2\pi f n/\sqrt{|d|}},$$

$$H = \frac{1}{f} \sum_{n=N+1}^{\infty} \frac{\beta(n)}{n} e^{-2\pi f n/\sqrt{|d|}}.$$

Since $\beta(n) \leq n$, we have

$$fH \leq e^{-2\pi f N/\sqrt{|d|}} \left(e^{2\pi f/\sqrt{|d|}} - 1\right)^{-1} < e^{-2\pi f N/\sqrt{|d|}} \sqrt{|d|}/(2\pi f) < 1.$$

Also,

$$fG < \sum_{n=1}^{N} \delta(n)/n,$$

where $\delta(n)$ is the number of divisors of n. It is well known (see for example Shapiro [**18**]), that there exist constants c_1 and c_2 such that

$$\sum_{n=1}^{N} \delta(n)/n < (\log N)^2/2 + 2\gamma \log N + c_1 + c_2/\sqrt{N}. \tag{3.7}$$

Indeed, (3.7) is true with $c_2 = 0$ and $c_1 = 7.442$. It follows that

$$\begin{aligned} fT(C) &< \left(\log(|d|/4\pi^2 f^2)\right)^2/2 + 2\gamma \log(|d|/4\pi^2 f^2) + 8.442 \\ &< \tfrac{1}{2}\log^2|d| + 2\gamma \log|d| \qquad (|d| > 8), \end{aligned} \tag{3.8a}$$

when $|d| > 4\pi^2 f^2$ and

$$fT(C) < \sqrt{|d|}/2\pi f < 1 \tag{3.8b}$$

when $|d| < 4\pi^2 f^2$.

[1] By $[\alpha]$ we denote that integer such that $\alpha - 1 < [\alpha] \leq \alpha$.

By (3.2) and (3.5) we get

$$Rh < \frac{\sqrt{|\Delta|}}{\pi}\left(\frac{a}{2}\log|\Delta| + b - a\log 2\pi + \frac{a}{f} + T(C)\right). \tag{3.9}$$

By using these results we can derive an explicit upper bound on Rh in terms of $L(1)$ and $L'(1)$. In fact, if we use the formula following (3.8a), we get

$$Rh < \frac{\sqrt{|\Delta|}}{\pi}\left(\frac{a\log|\Delta|}{2} + b + \frac{\log^2|d|}{2f} + \frac{2\gamma\log|d|}{f}\right). \tag{3.10}$$

4. The main results. We need now to discuss bounds on $a = L(1)$ and $b = L'(1)$. It is well known (see, for example, Chandrasekharan [**5**], p. 157) that

$$0 < L(1) < \log|d| + 2; \tag{4.1}$$

indeed, if we use the result of Pintz [**16**] we get

$$L(1) < (\lambda + o(1))\log|d|, \tag{4.2}$$

where $\lambda = 3(1 - e^{-1/2})/4 \simeq .295102$. However, since (4.2) is not an explicit result, we will make use of (4.1) here.

Also, by a simple refinement to the argument given in [**5**], p. 158–159, we can derive

$$|L'(1)| < (\log|d|)^2. \tag{4.3}$$

By using (4.1), (4.3), (3.9) and (3.8b) or (3.10), we get for $|d| > 200$

$$Rh < .453\sqrt{|\Delta|}\log|\Delta|\log|d| \qquad (|d| < 4\pi^2 f^2) \tag{4.4}$$

and

$$Rh < .767\sqrt{|\Delta|}\log|\Delta|\log|d| \qquad (|d| > 200). \tag{4.5}$$

When $|d|$ is small compared to $|\Delta|$, these results are better than the results mentioned in §1.

We can also give a and b as finite sums. It is well known that

$$a = L(1) = -\pi|d|^{-3/2}\sum_{n=1}^{|d|} n(d/n). \tag{4.6}$$

(see [**8**] Ch. 6). When $|d|$ is large, however, it is often easier to compute a by finding h' and using

$$2\pi h' = w\sqrt{|d|}\,L(1), \tag{4.7}$$

where h' is the class number of the quadratic field of discriminant d and w is the number of roots of unity in that field. Buell **[4]** has described how h' can be efficiently computed.

In terms of the Hurwitz Zeta-function

$$\zeta(s,\alpha) = \sum_{n=0}^{\infty} (n+\alpha)^{-s},$$

we have

$$L(s) = |d|^{-s} \sum_{n=1}^{|d|} (d/n)\zeta(s, n/|d|);$$

whence,

$$\begin{aligned} L'(0) &= \sum_{n=1}^{|d|} (d/n)\zeta'(0, n/|d|) - L(0)\log|d| \\ &= \sum_{n=1}^{|d|} (d/n)\log\Gamma(n/|d|) - L(0)\log|d|. \end{aligned}$$

(see **[20]**, §13.21). From the functional equations for L,

$$|d|^{1/2}L(1) = \pi L(0),$$

$$|d|^{1/2}L'(1) = \pi\left[L'(0) + \left(\log(|d|/2\pi) - \gamma\right)L(0)\right].$$

So we obtain

$$\text{(4.8)} \quad b = L'(1) = (\gamma + \log 2\pi)a - \pi|d|^{-1/2} \sum_{n=1}^{|d|} (d/n)\log\Gamma(n/|d|).$$

In the case where $\mathscr{K}$ is a pure cubic field we have $\Delta = -3f^2$, $a = L(1) = \pi/3\sqrt{3}$ and

$$b = L'(1) = \frac{\pi}{3\sqrt{3}}\left(\gamma + \log 2\pi + 3\log\frac{\Gamma(2/3)}{\Gamma(1/3)}\right) \approx .222662987$$

by (4.8). It follows from (3.9) and (3.8b) that

$$\text{(4.9)} \qquad hR < \frac{\sqrt{|\Delta|}}{6\sqrt{3}}\log|\Delta| = (2f\log f + f\log 3)/6.$$

Other results of this type for $|d| < 200$ can be easily derived by using Table 3 below together with the formulas (3.8) and (3.9).

TABLE 3

d	$L(1)$	$L'(1)$	d	$L(1)$	$L'(1)$
−3	0.6045997881	0.2226629870	−103	1.5477516108	−0.8809087714
−4	0.7853981634	0.1929013168	−104	1.8483510282	−1.4168771966
−7	1.1874104117	0.0185659811	−107	0.9111276756	−0.3227283614
−8	1.1107207345	−0.0230045879	−111	2.3854942292	−2.0120281805
−11	0.9472258251	−0.0797737528	−115	0.5859100510	0.0021206331
−15	1.6223114704	−0.4272680579	−116	1.7501373307	−1.3044164518
−19	0.7207307841	−0.0611999045	−119	2.8798932638	−2.6880771121
−20	1.4049629462	−0.4460960312	−120	1.1471474419	−0.5084996029
−23	1.9652020541	−0.8295529542	−123	0.5665357400	0.1051756228
−24	1.2825498302	−0.4226371999	−127	1.3938563455	−0.6756070246
−31	1.6927400922	−0.7636917993	−131	1.3724111229	−1.0129497686
−35	1.0620521591	−0.3841359021	−132	1.0937621702	−0.4421925820
−39	2.0122297265	−1.1251079939	−136	1.0775573904	−0.4920159080
−40	0.9934588266	−0.2795058488	−139	0.7993992331	−0.3215125571
−43	0.4790883882	0.1195240860	−143	2.6271317553	−2.4098111988
−47	2.2912419285	−1.4690506571	−148	0.5164746508	0.3635813641
−51	0.8798219250	−0.2759159416	−151	1.7896142906	−1.2898755068
−52	0.8713210307	−0.1705046261	−152	1.5289008746	−1.0381270761
−55	1.6944490680	−0.9400942441	−155	1.0093551772	−0.4772813436
−56	1.6792519084	−1.0135002063	−159	2.4914450356	−2.3185606656
−59	1.2270015789	−0.6541524535	−163	0.2460685276	0.5335570640
−67	0.3838066289	0.2526843656	−164	1.9625373721	−1.7270709177
−68	1.5238962757	−0.8855692531	−167	2.6741411208	−2.5496223412
−71	2.6098691772	−2.0424190523	−168	0.9695165413	−0.2486118800
−79	1.7672839421	−1.1177717634	−179	1.1740682982	−0.7410094492
−83	1.0345037784	−0.4748405533	−183	1.8578656914	−1.3440359401
−84	1.3711034417	−0.7765396209	−184	0.9264051326	−0.2653014650
−87	2.0208845180	−1.4284849560	−187	0.4594720151	0.1890727660
−88	0.6697898042	0.0872717101	−191	2.9551296636	−3.0461589353
−91	0.6586567884	−0.0879919892	−195	0.8998964910	−0.4200739607
−95	2.5785648429	−2.1505771251	−199	2.0043143873	−1.7042768578

By a result of Cusick [7] we know that

$$R \geq \frac{1}{3}\log(|\Delta|/27);$$

hence we can use this result in (4.4) or (4.5) to get an upper bound on h. In the case of the pure cubic field we can use (4.9) to get

$$h < \frac{\sqrt{|\Delta|}}{2\sqrt{3}} \frac{\log|\Delta|}{\log(|\Delta|/27)} = \frac{f}{2}\left(1 + \frac{\log 27}{\log(f^2/9)}\right) \tag{4.10}$$

Thus, when $f > 9\sqrt{3}$, we have $h < f$. It can be verified by direct computation that $h < f$ also holds for $f < 9\sqrt{3}$. We remark here that if the radicand D of $\mathscr{K}$ satisfies $D \equiv \pm 1 \pmod 9$, then $f \leq D$. Hence $h < D$ in this case and $D \nmid h$. Also $D \nmid h$ if $D \not\equiv \pm 1 \pmod 9$ and the cube free part of D has a non-trivial square factor.

We also point out that in the pure cubic case with $f > 61$ we have

$$\frac{2}{f} + \frac{2T(C)}{a} < .048819144$$

by (3.8b). Hence

$$2\left(\log 2\pi - \frac{b}{a}\right) - \frac{2}{f} - \frac{2T(C)}{a} > \log 18$$

and by (3.9) we get

$$Rh < \frac{\sqrt{|\Delta|}}{6\sqrt{3}} \log(|\Delta|/18) = (2f \log f - f \log 6)/f \qquad (f > 61) \tag{4.11}$$

and

$$h < \frac{f}{2}\left[\frac{1 - \frac{1}{2}\log 6/\log f}{1 - \log 3/\log f}\right] \qquad (f > 61). \tag{4.12}$$

5. A lower bound on the class number. In this section we will derive a lower bound on the class number of $\mathscr{K}$. This, unfortunately, will involve R, and another function $\pi_d(x)$; however, as we will illustrate for the case of a pure cubic field, when $|d|$ is small and R can be bounded from above, we can get some interesting inequalities on h.

Let $\mathfrak{a}$ be any ideal of $\mathcal{O}_{\mathscr{K}}$. Denote by $M(\mathfrak{a})$ the least positive rational integer in $\mathfrak{a}$. We say that $\mathfrak{a}$ is a *reduced* ideal of $\mathcal{O}_{\mathscr{K}}$ if $\mathfrak{a}$ is primitive ($\mathfrak{a}$ has no rational integer divisors) and there does not exist a non-zero element $\alpha \in \mathfrak{a}$ such that all of

$$|\alpha| < M(\mathfrak{a}), \quad |\alpha'| < M(\mathfrak{a}), \quad |\alpha''| < M(\mathfrak{a})$$

hold. Here α' and α'' are the conjugates of α in $\mathscr{K}$. (Of course, because $\Delta < 0$ two of $|\alpha|$, $|\alpha'|$, $|\alpha''|$ are equal.)

If $\mathfrak{b}$ is any ideal of $\mathcal{O}_{\mathscr{K}}$, there always exists a reduced ideal $\mathfrak{a}$ such that $\mathfrak{a} \sim \mathfrak{b}$. Also, if $\mathfrak{a}$ $(= \mathfrak{a}_1)$ is any reduced ideal of $\mathcal{O}_{\mathscr{K}}$ then Voronoi's continued fraction algorithm can be used on a basis of $\mathfrak{a}$, to produce a sequence of bases of ideals

$$\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \ldots, \mathfrak{a}_\rho, \mathfrak{a}_{\rho+1}, \ldots$$

such that $\mathfrak{a}_i \sim \mathfrak{a}_1$ and $\mathfrak{a}_i$ $(i = 1, 2, 3, \ldots, \rho)$ are all *distinct* reduced ideals which belong to the same ideal class. In fact, if we assume that the generator of $\mathscr{K}$ is real, Voronoi's algorithm can be used to produce elements $\theta_g^{(i)}$ (> 1) of $\mathscr{K}$ such that

$$(M(\mathfrak{a}_1)\theta_n)\mathfrak{a}_n = (M(\mathfrak{a}_n))\mathfrak{a}_1,$$

where

$$\theta_n = \prod_{i=1}^{n-1} \theta_g^{(i)}.$$

In this case ($\Delta < 0$) Voronoi's algorithm is completely periodic; that is, $\mathfrak{a}_{\rho+k} = \mathfrak{a}_k$ for all $k \in \mathbf{Z}^+$. It follows that

$$\varepsilon_0 = \prod_{i=1}^{\rho} \theta_g^{(i)},$$

where ε_0 (> 1) is the fundamental unit of $\mathscr{K}$. The value of ρ is the *period length* of Voronoi's continued fraction algorithm expanded on a basis of $\mathfrak{a}_1$ ($= \mathfrak{a}$). For the proofs of the above statements, we refer the reader to Delone and Faddeev **[9]** or Williams **[21]**.

We remark here that by using an earlier (non-explicit) form of our result (4.9), Dubois **[10]** has shown that

$$\rho = O\left(\sqrt{|\Delta|}\log|\Delta|\right) \tag{5.1}$$

when $\mathscr{K}$ is a pure cubic field. More recently Buchmann **[3]** has given the explicit upper bound

$$\rho \leq 4\sqrt{|\Delta|}\log^2|\Delta| \tag{5.2}$$

for any cubic field $\mathscr{K}$ with $\Delta < 0$. This was obtained by using the upper bound on hR given by Siegel **[18]**. Now Williams **[22]** has shown that

$$\varepsilon_0 > \tau^{\rho/2},$$

where

$$\tau = (1 + \sqrt{5})/2; \quad \text{hence}$$

$$\rho < 2R/\log \tau. \tag{5.3}$$

Thus, by using (5.3) with (4.5) we can get an improvement on (5.2). In the pure cubic case we can use (4.9) and (5.3) to get

$$\rho < .4\sqrt{|\Delta|}\log|\Delta|, \tag{5.4}$$

an explicit form of (5.1).

By referring to Table 1, we note that for those primes p such that $(\Delta/p) = (d/p) = -1$, we have $(p) = \mathfrak{p}\mathfrak{q}$ and $N(\mathfrak{q}) = p^2$, $M(\mathfrak{q}) = p$; put $\mathfrak{s} = \mathfrak{q}$ in this case. For those primes p such that $p \mid f$, we have $(p) = \mathfrak{p}^3$; thus, if $\mathfrak{s} = \mathfrak{p}^2$, we get $N(\mathfrak{s}) = p^2$, $M(\mathfrak{s}) = p$. Suppose p is any prime such that $(d/p) = -1$ or $p \mid f$ and suppose further, that $p \leq \sqrt[4]{|\Delta|/27}$.

For the ideal $\mathfrak{s}$ which we have defined above we get

$$M(\mathfrak{s})^4 < \sqrt{|\Delta|/27}\, N(\mathfrak{s}).$$

By a result of Williams [22], we know that $\mathfrak{s}$ must be a reduced ideal of $\mathcal{O}_{\mathscr{K}}$.

Let $\pi_d(x)$ be the number of primes up to x for which d is a quadratic non-residue. If T is the number of all ideals of $\mathcal{O}_{\mathscr{K}}$ which are reduced and ρ_i is the number of reduced ideals belonging to the ith ideal class, we have

$$T = \sum_{i=1}^{h} \rho_i \geq \pi_d\left(\sqrt[4]{|\Delta|/27}\right).$$

Since $\rho_i < 2R/\log\tau$, we get $T < 2Rh/\log\tau$ and

$$h > \frac{(\log\tau)\pi_d\left(\sqrt[4]{|\Delta|/27}\right)}{2R}. \tag{5.5}$$

When $\mathscr{K}$ is a pure cubic field, then $d = -3$ and $(d/p) = -1$ when $p \equiv 2 \pmod 3$; thus,

$$\pi_d(x) = \pi(x;\ 3, 2),$$

where $\pi(x;\ 3, 2)$ denotes the number of primes $p \leq x$ such that $p \equiv 2 \pmod 3$. From a result of McCurley [14], we can easily deduce that

$$\pi(x;\ 3, 2) > .460517x/\log x$$

when $x > 4$. Thus, if $\Delta < -6912$, from (5.5) we get

$$h > .44\sqrt[4]{|\Delta|/27}\,/\big(R\log(|\Delta|/27)\big). \tag{5.6}$$

Hence, in a pure cubic field $\mathscr{K}$ with discriminant $\Delta < -6912$, we have $h > 1$ whenever

$$R < .44\sqrt[4]{|\Delta|/27}\,/\log(|\Delta|/27).$$

When $\mathscr{K}$ is a pure cubic field with radicand D, where $D\ (= \delta^3) = K^3 + k$ and $k \mid 3K^2$, then for $\theta = \delta - K$, we have $\theta < 1$, $N(\theta) = k$. Hence $\theta^3/k \in \mathcal{O}_{\mathscr{K}}$ and $N(\theta^3/k) = 1$. It follows that

$$\varepsilon_0 \leq (\delta^2 + K\delta + K^2)^3/k^2.$$

In fact, in the case where $|k| = 1$, we have $\varepsilon_0 \leq \delta^2 + K\delta + K^2$. When D is cube-free, we can replace these inequalities by equalities, for all but 6 values of D (see Rudman [17]). Also,

$$|\Delta| > 3D \geq 3(K^3 - 3K^2) \geq 3(\delta^2 + K\delta + K^2)$$

when $\delta > 6$. Thus, $|\Delta| > 3\varepsilon_0^{1/3}$ and $R < 3\log(|\Delta|/3)$; by (5.6) we get

$$h > \frac{.14\sqrt[4]{|\Delta|/27}}{\log(|\Delta|/3)\log(|\Delta|/27)}, \tag{5.7}$$

an explicit lower bound for h. We notice here that $h > 1$ for all $|\Delta|$ that are sufficiently large. Also, the bound given in (5.7) is much larger than those obtained by Mollin **[15]** in the analogous case of certain real quadratic fields $\mathcal{Q}(\sqrt{D})$ when $D = K^2 + k$ and $k\,|\,4K$.

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