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VECTOR-VALUED SINGULAR INTEGRAL OPERATORS ON L^p-SPACES WITH MIXED NORMS AND APPLICATIONS

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We establish L^P and $L^P(l^Q)$ estimates for singular integral operators with variable operator-valued product kernels. Application to the strong maximal function, double Hilbert transform, Littlewood-Paley inequalities and Fourier multipliers for L^P-spaces with mixed norm are given.

Introduction. A classical theorem due to Hardy and Littlewood and the improvement given by Fefferman-Stein [**6**] assert that the maximal function

$$Mf(x) = \sup_I |I|^{-1} \int_I |f(u)|\, du$$

is bounded on L^p, $1 < p < \infty$, and has a vectorial extension $\tilde{M}(f_j) = (Mf_j)$ bounded in $L^p(l^q)$, $1 < p, q < \infty$. On the other hand, another classical theorem, due to M. Riesz, which asserts that the Hilbert transform

$$Hf(x) = \text{p.v.} \int_{-\infty}^{\infty} \frac{f(y)}{x - y}\, dy$$

is bounded on L^p, $1 < p < \infty$, was improved by Burkholder [**5**] for L^p-function with values in Banach spaces with the so called UMD property. In particular, the Hilbert transform has a vectorial extension $\tilde{H}(f_j) = (Hf_j)$ bounded in $L^p(l^q)$, $1 < p, q < \infty$. Recently Rubio de Francia-Ruiz-Torrea [**13**] and [**14**] have shown that the maximal operator and the Hilbert transform are operators of same kind: vector-valued singular integral operators. Actually, they improved a theorem on vector-valued convolution operators due to Benedek-Calderón-Panzone [**2**].

Let us now consider the *rectangular* (strong) maximal function

$$Mf(x, y) = \sup_{I,J} |I \times J|^{-1} \int_J \int_I |f(u, v)|\, du\, dv$$

and the double Hilbert transform

$$Hf(x, y) = \text{p.v.} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x-u, y-v)}{(x-u)(y-v)} \, du \, dv.$$

By iteration, we see at once that rectangular maximal function and the double Hilbert transform are bounded on $L^p(\mathbf{R}^2)$, $1 < p < \infty$. But, this is not the case if we replace the usual L^p spaces by the $L^r(L^s)$ spaces with mixed norm of Benedek-Panzone **[1]**. The boundedness now does not follow from a simple iteration. The $L^r(L^s)$-norm estimate for the strong maximal function was stated by Stöckert **[17]**, but a very nice proof was given by E. Hernández **[9]**; on the other hand, for the double Hilbert transform it goes back to M. Cotlar.

Our concern here is to establish a theory of vector singular integral operators with variable product kernels. This will be done in the mold of Rubio de Francia-Ruiz-Torrea **[13]** and **[14]**, and in such a way to be possible to handle with it the scalar strong maximal function and double Hilbert transform as well as its sequential extensions. As applications we also obtain an inequality of Littlewood-Paley type for $L^r(L^s)$-spaces and derive a multiplier theorem of Marcinkiewicz-Lizorkin type in a simple and natural way.

1. Vector-valued singular integral operators with product kernels. We begin by recalling the Rubio de Francia-Ruiz-Torrea theorem on vector-valued singular integral operators (see **[13]**).

1.1. THEOREM. *Suppose E and F denote Banach spaces. For $\Delta = \{(x, y) \in \mathbf{R}^n \times \mathbf{R}^n,\ x = y\}$, let $k \in L^1_{\text{loc}}(\mathbf{R}^n \times \mathbf{R}^n - \Delta, L(E, F))$ be an operator-valued kernel which satisfies*

$$(1) \qquad \int_{|y'-x| \geq 2|y'-y|} \| k(x, y) - k(x, y') \|_{L(E,F)} \, dx \leq C,$$

and

$$(2) \qquad \int_{|y-x'| \geq 2|x-x'|} \| k(x, y) - k(x', y) \|_{L(E,F)} \, dy \leq C.$$

Let T be a linear bounded operator from $L^r(\mathbf{R}^n, E)$ into $L^r(\mathbf{R}^n, F)$, for some r with $1 < r < \infty$, such that

$$(3) \qquad Tf(x) = \int_{\mathbf{R}^n} k(x, y) f(y) \, dy,$$

for all $f \in L^\infty_c(\mathbf{R}^n, E)$ (the linear space of all E-valued measurable functions which are essentially bounded and have compact support) and $x \notin \operatorname{supp} f$.

Then, for all p with $1 < p < \infty$, *we have*

(4) $$\|Tf\|_{L^p(\mathbf{R}^n, F)} \leq C\|f\|_{L^p(\mathbf{R}^n, E)}, \qquad f \in L^\infty_c(\mathbf{R}^n, E).$$

Moreover, for all q with $1 < q < \infty$, *we also have*

(5) $$\|TF\|_{L^p(\mathbf{R}^n, l^q(F))} \leq C\|F\|_{L^p(\mathbf{R}^n, l^q(E))}, \qquad F = (f_j) \in L^\infty_c(\mathbf{R}^n, l^q(E)).$$

We shall rely on the above theorem to give a version of it for product kernels $k(x,u,y,v) = k_2(y,v)k_1(x,u)$ and the $L^P = L^{p_2}(L^{p_1})$ spaces with mixed norms of Benedek-Panzone [**1**], from where we take notations. The proof we shall give follows an idea by Benedek-Calderón-Panzone [**2**].

1.2. THEOREM. *Suppose E, F and G denote Banach spaces. Let us consider operator-valued kernels* k_1 *and* k_2 *in* $L^1_{\mathrm{loc}}(\mathbf{R}^m \times \mathbf{R}^m - \Delta, L(E,F))$ *and* $L^1_{\mathrm{loc}}(\mathbf{R}^n \times \mathbf{R}^n - \Delta, L(F,G))$, *respectively, which satisfy*

(1) $$\int_{|z-w'|\geq 2|w-w'|} \left\|k_j(z,w) - k_j(z,w')\right\|_{L_j} dz \leq C_j, \qquad j = 1,2,$$

and

(2) $$\int_{|z'-w|\geq 2|z'-z|} \left\|k_j(z,w) - k_j(z',w)\right\|_{L_j} dw \leq C_j, \qquad j = 1,2,$$

where $L_1 = L(E,F)$ *and* $L_2 = L(F,G)$. *Let* T_1 *and T be linear bounded operators from* $L^p(\mathbf{R}^m, E)$ *into* $L^p(\mathbf{R}^m, F)$ *and from* $L^p(\mathbf{R}^n \times \mathbf{R}^n, E)$ *into* $L^p(\mathbf{R}^m \times \mathbf{R}^n, G)$, *for all p with* $1 < p < \infty$, *respectively. Suppose also that* T_1 *and T satisfy*

(3) $$T_1 f(x) = \int_{\mathbf{R}^m} k_1(x,u) f(u)\, du$$

for all $f \in L^\infty_c(\mathbf{R}^m, E)$ *and* $x \notin \operatorname{supp} f$, *and*

(4) $$Tf(x,y) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^m} k_2(y,v)k_1(x,u)f(u,v)\, du\, dv$$

for all $f \in L^\infty_c(\mathbf{R}^m \times \mathbf{R}^n, E)$ *and* $(x,y) \notin \operatorname{supp} f$. *Then, for all* $P = (p_1, p_2)$ *with* $1 < p_1, p_2 < \infty$, *the linear operator T can be extended to all* $L^P(\mathbf{R}^m \times \mathbf{R}^n, E)$ *into* $L^P(\mathbf{R}^m \times \mathbf{R}^n, G)$ *such that*

(5) $$\|Tf\|_{L^P(\mathbf{R}^m \times \mathbf{R}^n, G)} \leq C\|f\|_{L^P(\mathbf{R}^m \times \mathbf{R}^n, E)},$$

for all $f \in L^P(\mathbf{R}^m \times \mathbf{R}^n, E)$.

Proof. Step 1. Let f be in $M(\mathbf{R}^{m+n}, E)$, the set of all E-valued measurable functions on $\mathbf{R}^{m+n}$. For each $y \in \mathbf{R}^n$ and each $f \in M(\mathbf{R}^{m+n}, E)$ we associate the functions $f = f_y \in M(\mathbf{R}^m, E)$ defined by $\pi f = f_y(\cdot) = f(\cdot, y)$. Thus, we shall have $L^p(\mathbf{R}^{m+n}, E) = L^p(\mathbf{R}^n, L^p(\mathbf{R}^m, E))$, $L^P(\mathbf{R}^{m+n}, E) = L^{p_2}(\mathbf{R}^n, L^{p_1}(\mathbf{R}^m, E))$, $P = (p_1, p_2)$ and

$$\|\pi f\|_{L^{p_2}(\mathbf{R}^n, L^{p_1}(\mathbf{R}^m, E))} = \|f\|_{L^P(\mathbf{R}^{m+n}, E)}.$$

Moreover, if $f \in L^\infty_c(\mathbf{R}^{m+n}, E)$ we shall have $f_y \in L^\infty_c(\mathbf{R}^m, E)$.

Step 2. For all $f \in M(\mathbf{R}^{m+n}, E)$ and $\lambda > 0$, we define $H_\lambda f$ by

$$(H_\lambda f)(x, y) = \chi_\lambda(x) f(x, y),$$

where χ_λ is the characteristic function of the set $\{|x| < \lambda\}$. Now, if $y \in \mathbf{R}^n$, we define $K_\lambda(y) \in L(L^p(\mathbf{R}^m, E), L^p(\mathbf{R}^m, G))$ by

$$[K_\lambda(y, v)h](u) = \chi_\lambda k_2(y, v)[T_1(\chi_\lambda h)](u).$$

Then, if $\|\cdot\|$ denotes the operator norm on $L(L^p(\mathbf{R}^m, E), L^p(\mathbf{R}^n, F))$, since the singular integral operator $T_1 f$ is bounded from $L^p(\mathbf{R}^m, E)$ into $L^p(\mathbf{R}^m, F)$, it follows that

$$\begin{aligned}
\|K_\lambda(y, v)\| &= \sup\{\|K_\lambda(y, v)h\|_{L^p(G)};\ \|h\|_{L^p(E)} \le 1\} \\
&\le \sup\{\|k_2(y, v)T_1(\chi_\lambda h)\|_{L^p(G)};\ \|h\|_{L^p(E)} \le 1\} \\
&\le \|k_2(y, v)\|_{L(F,G)} \sup\{\|T_1(\chi_\lambda h)\|_{L^p(G)};\ \|h\|_{L^p(E)} \le 1\} \\
&\le \|k_2(y, v)\|_{L(F,G)} \sup\{C\|\chi_\lambda h\|_{L^p(E)};\ \|h\|_{L^p(E)} \le 1\} \\
&\le C\|k_2(y, v)\|_{L(F,G)},
\end{aligned}$$

which shows that $K_\lambda(y, v) \in L^1_{\text{loc}}(\mathbf{R}^n, L(L^p(E), L^p(G))$. Moreover, we have

$$\begin{aligned}
&\int_{|y-v'|>2|v-v'|} \|K_\lambda(y, v) - K_\lambda(y, v')\|\, dy \\
&\qquad \le C_1 \int_{|y-v'|>2|v-v'|} \|k_2(y, v) - k_2(y, v')\|\, dy \le C_1 C_2
\end{aligned}$$

and

$$\begin{aligned}
&\int_{|v-y'|>2|y-y'|} \|K_\lambda(y, v) - K_\lambda(y', v)\|\, dv \\
&\qquad \le C_1 \int_{|v-y'|>2|y-y'|} \|k_2(y, v) - k_2(y', v)\|\, dv \le C_1 C_2.
\end{aligned}$$

Now, for $F \in L^\infty_c(\mathbf{R}^n, L^p(\mathbf{R}^m, E))$, we get

$$T_\lambda F(y) = \int_{\mathbf{R}^n} K_\lambda(y, v) F(v)\, dv.$$

Since $\pi H_\lambda T H_\lambda f = T_\lambda \pi f$, due to hypothesis (4) we get

$$\begin{aligned}\|T_\lambda \pi f\|_{L^p(\mathbf{R}^n, L^p(\mathbf{R}^m, G))} &= \|\pi H_\lambda T H_\lambda f\|_{L^p(\mathbf{R}^n, L^p(\mathbf{R}^m, G))}\\ &= \|H_\lambda T H_\lambda f\|_{L^p(\mathbf{R}^m \times \mathbf{R}^n; G)}\\ &\le \|T H_\lambda f\|_{L^p(\mathbf{R}^{m+n}, G)} \le C\|H_\lambda f\|_{L^p(\mathbf{R}^{m+n}, E)}\\ &\le C\|f\|_{L^p(\mathbf{R}^{m+n}, E)} = C\|\pi f\|_{L^p(\mathbf{R}^n, L^p(\mathbf{R}^m, E))}.\end{aligned}$$

Consequently, setting $p = p_1$, $A = L^{p_1}(\mathbf{R}^m, E)$ and $B = L^{p_1}(\mathbf{R}^m, G)$, we have

$$\|T_\lambda F\|_{L^{p_1}(\mathbf{R}^n, B)} \le C\|F\|_{L^{p_1}(\mathbf{R}^n, A)},$$

for all F given by $F = \pi f$, $f \in L_c^\infty(\mathbf{R}^m \times \mathbf{R}^n, E)$. Since $L_c^\infty(\mathbf{R}^{m+n}, E) = L_c^\infty(\mathbf{R}^n, L_c^\infty(\mathbf{R}^m, E))$ is dense in $L_c^\infty(\mathbf{R}^n, L^p(\mathbf{R}^m, E)) = L_c^\infty(\mathbf{R}^n, A)$, in the norm of $L^p(\mathbf{R}^n, A)$, we have 1.1(4) for all $F \in L_c^\infty(\mathbf{R}, A)$. Now, the Rubio de Francia-Ruiz-Torrea theorem yields

$$\|T_\lambda F\|_{L^{p_2}(\mathbf{R}^n, B)} \le C\|F\|_{L^{p_2}(\mathbf{R}^n, A)},$$

for all $F \in L^{p_2}(\mathbf{R}^n, A)$ and $1 < p_2 < \infty$.

Step 3. We shall have

$$\begin{aligned}\|H_\lambda T H_\lambda f\|_{L^P(\mathbf{R}^{m+n}, G)} &= \|\pi H_\lambda T H_\lambda f\|_{L^{p_2}(\mathbf{R}^n, L^{p_1}(\mathbf{R}^m, G))}\\ &= \|T_\lambda \pi f\|_{L^{p_2}(\mathbf{R}^n, L^{p_1}(\mathbf{R}^m, G))}\\ &\le C\|\pi f\|_{L^{p_2}(\mathbf{R}^n, L^{p_1}(\mathbf{R}^m, E))} = C\|f\|_{L^P(\mathbf{R}^{m+n}, E)}.\end{aligned}$$

Finally, since $H_\lambda f = f$, for λ large enough, we get

$$\begin{aligned}\|Tf\|_{L^P(\mathbf{R}^{m+n}, G)} &\le \varliminf_{\lambda \to \infty} \|H_\lambda T H_\lambda f\|_{L^P(\mathbf{R}^{m+n}, G)}\\ &\le C\|f\|_{L^P(\mathbf{R}^{m+n}, E)},\end{aligned}$$

for all $f \in L_c^\infty(\mathbf{R}^{m+n}, E)$ and consequently for all $f \in L^P(\mathbf{R}^{m+n}, E)$.

The proof is complete.

1.3. Let us recall that $l^Q(X)$, $Q = (q_1, q_2)$, denotes the linear space of all X-valued double sequences (C_{ij}) such that

$$(1) \quad \|(C_{ij})\|_{l^Q(X)} = \|(C_{ij})\|_{l^{q_2}(l^{q_1}(X))} = \left\{\sum_j \left\{\sum_i |C_{ij}|_X^{q_1}\right\}^{q_2/q_1}\right\}^{1/q_2} < \infty.$$

1.4. COROLLARY. *Suppose E, F and G denote Banach spaces and let us consider operator-valued kernels k_1 and k_2 as in Theorem* 1.2. *We define the kernels $\tilde{k}_1(x) \in L(l^Q(E), l^Q(F))$, and $\tilde{k}_2(y) \in L(l^Q(F), l^Q(G))$ by*

$$\tilde{k}_1(x)(a_{ij})_{ij} = \big(k_1(x)a_{ij}\big)_{ij} \tag{1}$$

and

$$\tilde{k}_2(y)(b_{ij})_{ij} = \big(k_2(y)b_{ij}\big)_{ij}. \tag{2}$$

We define, for $f = (f_{ij}) \in L_c^\infty(\mathbf{R}^m, l^Q(E))$,

$$\tilde{T}_1 f(x) = \tilde{T}_1(f_{ij}) = \int_{\mathbf{R}^m} \tilde{k}_1(x,u)\big(f_{ij}(u)\big)\,du \tag{3}$$

and for $f \in L_c^\infty(\mathbf{R}^{m+n}, l^Q(E))$

$$\tilde{T}f(x,y) = \tilde{T}(f_{ij}) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^m} \tilde{k}_2(y,v)\tilde{k}_1(x,u)\big(f_{ij}(u,v)\big)\,du\,dv. \tag{4}$$

We shall assume that, for all p with $1 < p < \infty$, the operators $\tilde{T}_1$ and $\tilde{T}$ are bounded from $L^p(\mathbf{R}^m, l^Q(E))$ into $L^p(\mathbf{R}^m, l^Q(F))$ and from $L^p(\mathbf{R}^{m+n}, l^Q(E))$ into $L^p(\mathbf{R}^{m+n}, l^Q(G))$, respectively. Then, for all $P = (p_1, p_2)$ with $1 < p_1, p_2 < \infty$, the linear operators T can be extended to all $L^P(\mathbf{R}^{m+n}, l^Q(E))$.

Moreover, we shall have

$$\|Tf\|_{L^P(\mathbf{R}^{m+n}, l^Q(G))} \le C\|f\|_{L^P(\mathbf{R}^{m+n}, l^Q(E))}. \tag{5}$$

Proof. Since, for $\nu = 1,2$,

$$\big\|\tilde{k}_\nu(z,w) - \tilde{k}_\nu(z,w')\big\|_{L(l^Q(E), l^Q(F))} \le \big\|k_\nu(z,w) - k_\nu(z,w')\big\|_{L(E,F)},$$

and

$$\big\|\tilde{k}_\nu(z,w) - \tilde{k}_\nu(z',w)\big\|_{L(l^Q(F), l^Q(G))} \le \big\|k_\nu(z,w) - k_\nu(z',w)\big\|_{L(F,G)},$$

it follows that the hypotheses of Theorem 1.2 are fulfilled with the kernels k_1 and k_2 and the spaces E, F and G replaced by the kernels $\tilde{k}_1$ and $\tilde{k}_2$ and the spaces $l^Q(E)$, $l^Q(F)$ and $l^Q(G)$, respectively. Hence, the corollary follows as desired.

1.5. REMARK. When $k_\nu(z,w) = k_\nu(z-w)$, $\nu = 1,2$, we have the singular integral operators of *convolution type*. This particular type of singular integral operators was studied, also in the product case, by the author in **[8]**.

2. The double Hilbert transform.

2.1. Our concern here is to obtain L^P and $L^P(l^Q)$ estimates for the *double Hilbert transform* which is defined by

$$Hf(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(u,v)}{(x-u)(y-v)}\,du\,dv. \tag{1}$$

This is an integral singular operator of convolution type and the mixed L^P-estimate goes back to M. Cotlar. The mixed $L^P(l^Q)$-estimate seems new.

2.2. THEOREM. *The double Hilbert transform given by* 2.1(1) *is an integral singular operator bounded in the spaces* $L^P(\mathbf{R}^2)$, *where* $P = (p_1, p_2)$ *with* $1 < p_1, p_2 < \infty$, *i.e.*

$$\|Hf\|_{L^P} \le C_P \|f\|_{L^P}, \tag{1}$$

for all $f \in L^P(\mathbf{R}^2) = L^{p_2}(L^{p_1})$.

Proof. Step 1. The kernels $k_j(z,w) = 1/(z-w)$ satisfy conditions 1.2(1)–(2).

Step 2. The integral singular operator T_1 associated with the kernel k_1 is bounded in $L^p(\mathbf{R})$, $1 < p < \infty$.

Step 3. The integral given by 2.1(1) is well defined for all $f \in L_c^\infty(\mathbf{R}^2)$ and $(x,y) \notin \operatorname{supp} f$.

Step 4. By iteration we see that H is bounded in $L^p(\mathbf{R}^2)$, $1 < p < \infty$.
Hence, the conditions of Theorem 1.2 are satisfied and the assertion follows.

2.3. COROLLARY. *In the conditions of Theorem* 2.2, *for all* $F = (f_{ij}) \in L^P(l^Q)$, $1 < Q = (q_1, q_2) < \infty$, *we have*

$$\|(Hf_{ij})\|_{L^P(l^Q)} \le C \|(f_{ij})\|_{L^P(l^Q)}. \tag{1}$$

Proof. It follows from Corollary 1.3.

3. The rectangular maximal function of F. Zó's type. The following version of a theorem due to F. Zó [19] will be needed to obtain the maximal inequality which we are looking for.

3.1. THEOREM. *Suppose* $\varphi \in L^1(\mathbf{R}^m)$ *and* $\psi \in L^1(\mathbf{R}^n)$. *For* $s > 0$ *and* $t > 0$, *let us set* $\varphi_s(x) = s^{-m}\varphi(s^{-1}x)$ *and* $\psi_t(y) = t^{-n}\psi(t^{-1}y)$. *We shall also suppose that*

$$(1) \qquad \int_{|x|>4|u|} \sup_{s>0} |\varphi_s(u-x) - \varphi_s(x)|\, dx \leq C_1, \qquad u \in \mathbf{R}^m$$

and

$$(2) \qquad \int_{|y|>4|v|} \sup_{t>0} |\psi_t(v-y) - \psi_t(y)|\, dy \leq C_2, \qquad v \in \mathbf{R}^n.$$

Now, for $f \in L_c^\infty(\mathbf{R}^{m+n})$, we get

$$(3) \qquad M_{\varphi\psi} f = \sup\{|\psi_t \varphi_s * f|;\ s > 0,\ t > 0\}.$$

Then, for all $P = (p_1, p_2)$ with $1 < p_1,\ p_2 < \infty$, we have

$$(4) \qquad \|M_{\varphi\psi} f\|_{L^P} \leq C \|f\|_{L^P},$$

for all $f \in L^P(\mathbf{R}^{m+n})$.

Proof. *Step* 1. If $f \in L^1 \cap L^\infty$, the mappings $(s, t) \to \psi_t\varphi_s * f(x, y)$ are uniformly continuous. Indeed since the mapping $(s, t) \to \psi_t\varphi_s$ is continuous from $\mathbf{R}^2$, into $L^1(\mathbf{R}^2)$ we have

$$|\psi_{t'}\varphi_{s'} * f(x, y) - \psi_{t''}\varphi_{s''} * f(x, y)| \leq \|f\|_{L^\infty} \|\psi_{t'}\varphi_{s'} - \psi_{t''}\varphi_{s''}\|_{L^1}.$$

Therefore, it is enough to prove (4) taking the supremum on $Q_+ \times Q_+$.

Step 2. Fixing an enumeration of the positive rationals, let us denote by Q_j the set of rationals with indices in $\{1, \ldots, j\}$. For c and d in $\mathbf{N}$, let us set

$$(5) \qquad M_{cd} f = \sup\{|\psi_t\varphi_s * f|;\ (s, t) \in Q_c \times Q_d\}.$$

If l_j^∞ and l_{ij}^∞ stand for the complex euclidean spaces $\mathbf{C}^j$ and $\mathbf{C}^{i+j}$, respectively, equipped with the sup-norm, the non-linear operator M_{cd} can be viewed as a vector-valued linear operator

$$(6) \quad f \in L_c^\infty(\mathbf{R}^{n+m}) \to N_{cd} f = \{\psi_t\varphi_s * f\}_{(s,t)\in Q_c \times Q_d} \in L^\infty(\mathbf{R}^{m+n}, l_{cd}^\infty).$$

The kernel k_{cd} of this operator is a product of the two kernels:

$$k_c^1(x) \in L(\mathbf{C}, l_c^\infty) \simeq l_c^\infty, \qquad x \in \mathbf{R}^m,$$

and

$$k_d^2(y) \in L(l_c^\infty, l_{cd}^\infty), \qquad y \in \mathbf{R}^n,$$

given by

$$k_c^1(x)a = \{\varphi_s(x)a;\ s \in Q_c\}$$

and

$$k_d^2(y)(b_j) = \{\psi_t(y)b_j;\ t \in Q_d, 1 \le j \le c\}.$$

Hence

$$k_{cd}(x, y) = k_d^2(y)k_c^1(x)z = \{\psi_t(y)\varphi_s(x)z;\ s \in Q_c,\ t \in Q_d\}.$$

On the other hand, we have

$$\begin{aligned}\|k_c^1(x)\| &= \sup_{|z|=1} \|k_c^1(x)z\|_{l^\infty}\\ &= \sup_{|z|=1}\ \sup_{s\in Q_c} |\varphi_s(x)z| = \sup_{s\in Q_c} |\varphi_s(x)|\end{aligned}$$

and

$$\begin{aligned}\|k_d^2(y)\| &= \sup_{\|(a_j)\|_\infty=1} \|k_d^2(y)(a_j)_j\|_{l^\infty}\\ &= \sup_{\|(a_j)\|_\infty=1}\ \sup_{t\in Q_d} |\psi_t(y)a_j| = \sup_{t\in Q_d} |\psi_t(y)|.\end{aligned}$$

Therefore k_c^1 and k_d^2 are locally integrable. Moreover

$$\begin{aligned}&\int_{|x|>4|u|} \|k_c^1(u-x) - k_c^1(x)\|\,dx\\ &\qquad = \int_{|x|>4|u|} \sup_z \|[k_c^1(u-x) - k_c^1(x)]z\|_{l_c^\infty}\,dx\\ &\qquad = \int_{|x|>4|u|} \sup_z\ \sup_s |\varphi_s(u-x) - \varphi_s(x)|\,|z|\,dx \le C_1\end{aligned}$$

and

$$\begin{aligned}&\int_{|y|>4|v|} \|k_d^2(v-y) - k_d^2(y)\|\,dy\\ &\qquad = \int_{|y|>4|v|} \sup_{(a_j)} \|[k_d^2(v-y) - k_d^2(y)](a_j)\|_{l^\infty}\,dy\\ &\qquad = \int_{|y|>4|v|} \sup_{(a_j)}\ \sup_{t,j} |\psi_t(v-t) - \psi_t(y)|\,|a_j|\,dy\\ &\qquad = \int_{|y|>4|v|} \sup_t |\psi_t(v-y) - \psi_t(y)|\,dy \le C_2.\end{aligned}$$

Step 3. Due to Zó's result, for all p with $1 < p < \infty$, we have

$$\|M_{cd}f\|_{L^p_{xy}} \leq \|M_{\varphi\psi}f\|_{L^p_{xy}} \leq \|M_\varphi M_\psi f\|_{L^p_{xy}} \leq C\Big\| \|M_\psi f\|_{L^p_x}\Big\|_{L^p_y}$$

$$\leq C\Big\| \|M_\psi f\|_{L^p_y}\Big\|_{L^p_x} \leq C^1\Big\| \|f\|_{L^p_y}\Big\|_{L^p_x} = C^1\|f\|_{L^p_{xy}},$$

where L^p_{xy}, L^p_x and L^p_y have obvious meaning.

Step 4. Since the operator $M_{cd}f = k_{cd} * f$ satisfies the hypotheses of Theorem 1.2 it follows that

$$\|M_{cd}f\|_{L^p(\mathbf{R}^{m+n}, l^\infty_{cd})} \leq C\|f\|_{L^p(\mathbf{R}^{m+n})} \tag{7}$$

for all $f \in L^\infty_c(\mathbf{R}^{m+n})$. But $M_{cd}f$ has also a sense for all $f \in L^p(\mathbf{R}^{m+n})$ and it is not hard to see that the extension $\tilde{M}_{cd}$ of M_{cd} to all $L^p(\mathbf{R}^{m+n})$ coincides with M_{cd}. Thus (7) holds for all $f \in L^p(\mathbf{R}^{m+n})$. Finally, letting $|(c, d)| \to \infty$, the monotone convergence theorem yields (4).

The proof is complete.

3.2. THEOREM. *Let φ_s and ψ_t be as in Theorem* 3.1. *For $f = (f_{ij})$ in $L^\infty_c(\mathbf{R}^{m+n}, l^Q)$, where $Q = (q_1, q_2)$ is given with $1 < q_1, q_2 < \infty$, let us consider the vectorial rectangular maximal function*

$$\tilde{M}_{\varphi\psi}(f_{ij})_{ij} = (M_{\varphi\psi}f_{ij})_{ij} \tag{1}$$

where $M_{\varphi\psi}f_{ij}$ is the maximal function given by 2.1(3). *Then, if $P = (p_1, p_2)$ is given with $1 < p_1, p_2 < \infty$, we have*

$$\|\tilde{M}_{\varphi\psi}(f_{ij})\|_{L^P(\mathbf{R}^{m+n}, l^Q)} \leq C\|(f_{ij})\|_{L^P(\mathbf{R}^{m+n}, l^Q)}, \tag{2}$$

for all $f = (f_{ij}) \in L^P(\mathbf{R}^{m+n}, l^Q)$.

Proof. As before, let us replace the maximal function $M_{\varphi\psi}g$ by $M_{cd}g$ and let us consider the vectorial linear operator

$$\tilde{T}_{cd}\colon (f_{ij}) \in L^\infty_c(\mathbf{R}^{m+n}, l^Q)$$

$$\to \tilde{T}_{cd}(f_{ij}) = \big((\psi_t\varphi_s * f_{ij})_{st}\big)_{ij} \in L^\infty\big(\mathbf{R}^{m+n}, l^Q(l^\infty_{cd})\big).$$

The kernel of this operator is a product kernel $k(x, y) = k_2(y)k_1(x)$, where

$$k_1(x)\colon (a_{ij}) \in l^Q \to k^1_c(a_{ij}) = (\varphi_s(x)a_{ij}) \in l^Q(l^\infty_c)$$

and

$$k_2(y)\colon (b_{ijs}) \in l^Q(l^\infty_c) \to k_2(y)(b_{ijs}) = (\psi_t(y)b_{ijs}) \in l^Q(l^\infty_{st}).$$

If $\|\cdot\|$ denotes the norm on $L(l^Q, l^Q(l_c^\infty))$, we shall have

$$\int_{|x|>4|u|} \left\| k_c^1(u-x) - k_c^1(x) \right\| dx$$

$$= \int_{|x|>4|u|} \sup_{\|(a_{ij})\|_{l^Q}=1} \left\| \left([\varphi_s(u-x) - \varphi_s(x)] a_{ij} \right) \right\|_{l^Q(l_c^\infty)} dx$$

$$\leq \int_{|x|>4|u|} \sup_{s \in Q_s} |\varphi_s(u-x) - \varphi_s(x)| \, dx \leq C_1,$$

and if $\|\cdot\|$ denotes the norm on $L(l^Q(l_c^\infty), l^Q(l_{cd}^\infty))$, we also have

$$\int_{|y|>4|v|} \left\| k_d^2(v-y) - k_d^2(y) \right\| dy$$

$$\leq \int_{|y|>4|v|} \sup_{\|(b_{ijs})\|_{l^Q(l_c^\infty)}=1} \left\| \left([\psi_t(v-y) - \psi_t(y)] b_{ijs} \right) \right\|_{l^Q(l_{cd}^\infty)} dy$$

$$\leq \int_{|y|>4|v|} \sup_{t \in Q_d} \left\| \psi_t(v-y) - \psi_t(y) \right\| \sup_{\|b\|=1} (b_{ijs})_{l^Q(l_c^\infty)} dy$$

$$\leq \int_{|y|>4|v|} \sup_{t \in Q_d} |\psi_t(v-y) - \psi_t(y)| \, dy \leq C_2.$$

Now, it remains to prove that, for all p with $1 < p < \infty$, we have

$$\left\| \tilde{T}_{cd}(f_{ij}) \right\|_{L^p(l^Q(l^\infty))} \leq C \left\| (f_{ij}) \right\|_{L^p(l^Q)}, \tag{3}$$

where C is a constant independent of c, d and p. Let us consider the partial operators $\tilde{M}_c$ and $\tilde{T}_d$ given by

$$\tilde{M}_c(f_{ij}) = (M_c f_{ij}) = \left(\sup_{s \in Q_c} |\varphi_s *_y f_{ij}| \right)_{ij}$$

and

$$\tilde{T}_d(f_{ji}) = (T_d f_{ij}) = (f_{ij} *_y \psi_t).$$

We shall have

$$T_{cd} f_{ij} \leq T_d M_c f_{ij}.$$

Thus, due to Zó's result (and Fubini's theorem) we see that $\tilde{M}_c$ and $\tilde{T}_d$ are bounded operators from $L_{xy}^p(l_{ij}^Q)$ into $L_{xy}^p(l_{ij}^Q)$ and from $L_{xy}^p(l_{ij}^Q(l_c^\infty))$ into $L_{xy}^p(l_{ij}^Q)$, respectively. Consequently, if $(f_{ij}) \in L^p(l^Q)$ we shall have $(g_{ij}) \in L^p(l^Q)$, where

$$g_{ij} = M_c f_{ij} = \sup_{s \in Q_c} |\varphi_s *_x f_{ij}|.$$

Hence

$$\|\tilde{T}_{cd}(f_{ij})\|_{L^P(l^Q(l^\infty))} \le \|(T_d M_c f_{ij})\|_{L^P(l^Q(l^\infty))}$$
$$\le C\|(M_c f_{ij})\|_{L^P(l^Q)} \le C\|(f_{ij})\|_{L^P(l^Q)}.$$

The proof is complete.

4. The rectangular maximal function. We are now ready to state inequalities for the rectangular maximal functions of Hardy-Littlewood and of Fefferman-Stein type for L^P spaces with mixed norms.

4.1. THEOREM. *Suppose $f \in L^1_{\mathrm{loc}}(\mathbf{R}^{m+n})$ and let us consider the rectangular maximal function Mf defined by*

$$(1) \qquad Mf(x,y) = \sup_{I,J} |I \times J|^{-1} \int_J \int_I |f(x-u, y-v)|\, du\, dv,$$

where I and J are (hyper)-cubes centered at the origin of $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. Then, if $f \in L^P(\mathbf{R}^{m+n})$, where $P = (p_1, p_2)$ with $1 < p_1, p_2 < \infty$, $Mf(x,y)$ is finite for a.e. $(x,y) \in \mathbf{R}^{m+n}$. Moreover, there is a constant $C > 0$ such that

$$(2) \qquad \|Mf\|_{L^P(\mathbf{R}^{m+n})} \le C\|f\|_{L^P(\mathbf{R}^{m+n})}$$

for all $f \in L^P(\mathbf{R}^{m+n})$.

Proof. Let I_1 and J_1 be the unit cubes on $\mathbf{R}^m$ and $\mathbf{R}^n$, and let us consider the dilated cubes I_s and I_t with side length s and t, respectively. Now, let $\varphi \in C_c^\infty(\mathbf{R}^m)$ and $\psi \in C_c^\infty(\mathbf{R}^n)$ such that $\varphi(x) = \psi(y) = 1$, for $x \in I_1$ and $y \in J_1$ respectively. Then

$$|I_s \times J_t|^{-1} \chi_{I_s \times J_t} \le \varphi_s \psi_t$$

and

$$Mf(x,y) = \sup_{s,t} \int_{\mathbf{R}^n} \int_{\mathbf{R}^m} |f(x-u, y-v)|\, |I_s \times J_t|^{-1} \chi_{I_s \times J_t}(u,v)\, du\, dv$$
$$\le M_{\varphi\psi} f(x,y).$$

Now, from Theorem 3.1, the maximal inequality (2) follows at once.

4.2. THEOREM. *Suppose $f = (f_{ij}) \in L^1_{\mathrm{loc}}(\mathbf{R}^{m+n}, l^Q)$, where $Q = (q_1, q_2)$ is given with $1 < q_1, q_2 < \infty$. The vectorial rectangular maximal function is given by*

$$(1) \qquad \tilde{M}(f_{ij})_{ij} = (Mf_{ij})_{ij}$$

where Mf_{ij} is the rectangular maximal function. Then, there is a constant $C > 0$ such that for all $P = (p_1, p_2)$ with $1 < p_1, p_2 < \infty$ and $f = (f_{ij}) \in L^P(\mathbf{R}^{m+n}, l^Q)$ we have

$$\|\tilde{M}(f_{ij})_{ij}\|_{L^P(\mathbf{R}^{m+n}, l^Q)} \le C\|(f_{ij})\|_{L^P(\mathbf{R}^{m+n}, l^Q)}. \tag{2}$$

Proof. It follows at once from Theorem 2.2 as in the proof of Theorem 3.1.

4.3. REMARK. The inequality 4.1(2) in the case $m = n = 1$ was stated by Stöckert [17]. But the inequality 4.2(2) seems new and it was proved by the author in [8]. (However see Schmeisser [15] and the references quoted there.)

5. Application to the Littlewood-Paley theorem.

5.1. PROPOSITION. *For $f \in S(\mathbf{R}^2)$ and I and J numerical intervals, the (iterated) partial sum operator is defined by*

$$(S_{I\times J}f)^{\wedge}(s, t) = \chi_I(s)\chi_J(f)\hat{f}(s, t) \tag{1}$$

and we have

$$\|S_{I\times J}f\|_{L^P} \le C\|f\|_{L^P}, \qquad 1 < P = (p_1, p_2) < \infty, \tag{2}$$

for all $f \in S(\mathbf{R}^2)$, with C independent of f. Moreover, $S_{I\times J}$ can be extended to all $L^P(\mathbf{R}^2)$.

Proof. If $I = J = (0, \infty)$, then

$$S_{I\times J}f = (1/4)(f + iH_{10}f + iH_{01} - H_{11}f) \tag{3}$$

where $H_{10}f$, $H_{01}f$ and $H_{11}f$ are the partial and double Hilbert transform. In this case we have obviously (2). The general case follows by modifications of (3) as in the one-dimensional case.

As the partial and double Hilbert transform have an l^Q-extension, Theorem 5.1 has the following extension.

5.2. THEOREM. *Let $(I_i \times J_j)_{i,j\in\mathbf{N}}$ be a double sequence of intervals in $\mathbf{R}^2$ and (f_{ij}) be a double sequence of functions in $S(\mathbf{R}^2, l^Q)$. Then*

$$\|(S_{I_i\times J_j}f_{ij})\|_{L^P(l^Q)} \le C\|(f_{ij})\|_{L^P(l^Q)}, \qquad 1 < P, Q < \infty, \tag{1}$$

where C is independent of $(I_i \times J_j)$ and (f_{ij}). Moreover, the operator $S(f_{ij}) = (S_{I_i\times J_j}f_{ij})$ can be extended continuously to all $L^P(\mathbf{R}^d)$.

We shall reverse inequality 5.2(1) for $Q = (2,2)$ and the family of dyadic intervals, i.e. we shall obtain the Littlewood-Paley inequalities for mixed norms. We shall need some preliminaries.

5.3. LEMMA. *Let $\varphi \in S(\mathbf{R})$ be given with $\hat{\varphi}(0) = 0$ and $\hat{\varphi}(t) = 1$ if $t \in [1/2, 1]$. Setting $\varphi_j(x) = 2^j\varphi(2^jx)$, $j \in \mathbf{Z}$, we have*

(1) $$\sum_{j\in\mathbf{Z}} |\hat{\varphi}_j(t)|^2 \le C;$$

(2) $$\sum_{j\in\mathbf{Z}} |\varphi_j(x)|^2 \le C|x|^{-2};$$

(3) $$\left(\sum_{j\in\mathbf{Z}} |\varphi_j(x-y) - \varphi_j(x)|^2\right)^{1/2} \le C|y|/|x|^2, \quad \text{if } |x| \ge 2|y|.$$

Proof. See **[13]** or **[14]**.

5.4. THEOREM. *Let φ and ψ be given as in Lemma 5.3. Then*

(1) $$\left\|(\varphi_i\psi_j * f)_{ij}\right\|_{L^P(l^2)} \le C\|f\|_{L^P}$$

for all $f \in L^P(\mathbf{R}^2)$, with $1 < P = (p_1, p_2) < \infty$.

Proof. We consider the operator

(2) $$T\colon f \in S(\mathbf{R}^2) \to Tf = (\varphi_i\psi_j * f)_{ij} \in M(\mathbf{R}^2, l^2).$$

We have to show that T is bounded from L^P into $L^P(l^2)$.

Step 1. T is well defined. Indeed, by 5.3(1), we have

$$\int\int \sum_j\sum_i |\varphi_i\psi_j * f|^2\,dx\,dy = \sum_j\sum_i \int\int |\psi_j\varphi_i * f|^2\,dx\,dy$$
$$= \sum_j\sum_i \int\int |\hat{\psi}_j(t)\hat{\varphi}_i(s)\hat{f}(s,t)|^2\,ds\,dt \le C\int\int |\hat{f}(s,t)|^2\,ds\,dt.$$

i.e., we have $\sum_{ij}|\psi_j\varphi_i * f(x,y)|^2 < \infty$, a.e., and $Tf(x,y) \in l^2(\mathbf{Z}^2)$.

Step 3. Tf is measurable. Since $l^2(\mathbf{Z}^2)$ is separable it is enough to show that Tf is weakly measurable. But for all $\alpha = (\alpha_{ij}) \in l^2(\mathbf{Z}^2)$ we have

$$\langle Tf(x,y), \alpha\rangle = \sum_{ij} \alpha_{ij}\varphi_i\psi_j * f(x,y)$$

which is obviously measurable.

Step 4. T has a bounded extension from L^2 into $L^2(l^2)$ because (3) holds.

Step 5. The kernel k_φ defined by

$$k_\varphi(x)\colon \lambda \in \mathbf{C} \to k_\varphi(x) = (\varphi_i(x)\lambda)_i \in l^2(\mathbf{Z})$$

is well defined, belongs to $L^1_{\mathrm{loc}}(\mathbf{R} - \{0\},\ L(\mathbf{C}, l^2))$ and verifies Hörmander's condition.

Step 6. The kernel k_ψ defined by

$$k_\psi(y) = (\alpha_i)_i \in l^2(\mathbf{Z}) \to k_\psi(y)(\alpha_i)_i = (\psi_j(y)\alpha_i)_{ij} \in l^2(\mathbf{Z}^2)$$

is well defined by 5.3(2). On the other hand, the mapping

$$(x, y) \to \sum_j \sum_i \alpha_{ij}\varphi_i(x)\psi_j(y)$$

is measurable for all $\alpha = (\alpha_{ij}) \in l^2(\mathbf{Z}^2)$. Thus, k_ψ is measurable, belongs to $L^1_{\mathrm{loc}}(\mathbf{R} - \{0\},\ L(l^2(\mathbf{Z}), l^2(\mathbf{Z}^2))$, and satisfies Hörmander's condition.

Step 7. The above results clearly also hold for the cut operators T_{mn} and the respective kernels k_φ^m and k_ψ^n. Thus, since $T_{mn}f = (\psi_j\varphi_i * f;\ 1 \le i \le m,\ 1 \le j \le n)$ is a sequence in l^2 we have

$$T_{mn}f(x, y) = \int \int k_\psi^n(y - v)k_\varphi^n(x - u)f(u, v)\, du\, dv.$$

Step 8. The operators T_{mn} are bounded from $L^p(\mathbf{R}^2)$ into $L^p(\mathbf{R}^2, l^2(\mathbf{Z}^2))$, with operator norms bounded by a constant independent of m and n.

Step 9. From Theorem 1.2 and Corollary 1.4 we obtain

$$\|T_{mn}f\|_{L^p(l^2)} \le C\|f\|_{L^p}, \tag{3}$$

with the constant C independent of m and n.

Step 10. The monotone convergence theorem applied to (3) yields (1), as desired.

5.5. Remark. For a related result, but with a different proof, of Theorem 5.4 see Bordin-Fernandez [**3**].

5.6. Let Δ_1 be the set of all dyadic intervals in $\mathbf{R}$, and let $\Delta = \Delta_1 \times \Delta_1$ be the set of dyadic bi-dimensional intervals.

5.7. THEOREM. *If* $f \in L^P(\mathbf{R}^2)$, $1 < P = (p_1, p_2) < \infty$, *then*

$$c_P \| f \|_{L^P} \leq \left\| \left[\sum_{I \in \Delta} |S_I f|^2 \right]^{1/2} \right\|_{L^P} \leq C_P \| f \|_{L^P} \tag{1}$$

where c_P *and* C_P *are independent of* f.

Proof. Let $\varphi \in S(\mathbf{R}^2)$ and let φ and ψ be as in Lemma 5.2. Since $\hat{\varphi}_i(s) = \hat{\varphi}(2^i s)$ and $\hat{\psi}_j(t) = \hat{\psi}(2^j t)$, we have $\hat{\varphi}_i(s) = 1$ if $s \in [2^{i-1}, 2^i]$ and $\hat{\psi}_j(t) = 1$ if $t \in [2^{j-1}, 2^j]$. Hence

$$S_I f = S_{I_1 \times I_2} f = S_{I_1 \times I_2}(\varphi_i \psi_j * f). \tag{2}$$

Now, Theorems 5.2 and 5.4 yield

$$\left\| \left[\sum_{I \in \Delta} |S_I f|^2 \right]^{1/2} \right\|_{L^P} \leq C \| f \|_{L^P}. \tag{3}$$

Finally, to reverse (3) we use polarization and duality as in the well known cases (see **[16]**, **[14]** and **[18]**).

6. Multiplier theorems.

6.1. DEFINITION. A scalar valued measurable function φ, defined in $\mathbf{R} \times \mathbf{R}$, is said to be of *bounded V-variation* if there exists a positive constant M and consequences $(C_{km};\ k \in \mathbf{Z},\ m \in \mathbf{N})$, $(a_{km};\ k \in \mathbf{Z},\ m \in \mathbf{N})$ and $(b_{km};\ k \in \mathbf{Z},\ m \in \mathbf{N})$, which satisfy

$$\lim_{m \to \infty} \sum_{k \in \mathbf{Z}} C_{km} \chi_{(-\infty, a_{km})} \chi_{(-\infty, b_{km})} = \varphi, \qquad \text{a.e.} \tag{1}$$

and

$$\sum_k |C_{km}| \leq M, \quad \text{for all } m. \tag{2}$$

We shall write $V(\varphi)$ for the infimum of such constants M.

6.2. THEOREM. *Let* (φ_{mn}) *be a (double-)sequence of uniformly bounded V-variations, i.e.*

$$V(\varphi_{mn}) \leq M, \quad \textit{for all } m \textit{ and } n. \tag{1}$$

For $g \in S(\mathbf{R}^2)$, *let* T_{mn} *denote the operator defined by*

$$(T_{mn} g)^\wedge = \varphi_{mn} \hat{g}. \tag{2}$$

Then, if (f_{mn}) *is a sequence in* $S(\mathbf{R}^2)$, *we have*

$$\|(T_{mn}f_{mn})\|_{L^p(l^Q)} \le C\|(f_{mn})\|_{L^p(l^Q)}. \tag{3}$$

Proof. Let us suppose $f_{mn} \equiv 0$, for m and/or n large. Let us set $N = (m, n)$, and let

$$h_{mN} = \sum_k C_{kmN}\chi_{I_{kmN}} \quad \left(I_{kmN} = (-\infty, a_{kmi}) \times (-\infty, b_{kmj})\right)$$

be a function which satisfies 6.1(1)–(2) and $h_{mN} \to \varphi_N$, as $m \to \infty$. Next, define $(S_{mN}f_N)^\wedge = h_{mN}\hat{f}_N$. We claim that

$$\|(S_{mN}f_N)\|_{L^p(l^Q)} \le C\|(f_N)\|_{L^p(l^Q)}. \tag{4}$$

In fact, by Hölder's inequality

$$\begin{aligned} |S_{mN}f_N|^{q_1} &= \left|\sum_k C_{kmN}\left(\chi_{I_{kmN}}\hat{f}_N\right)^\vee\right|^{q_1} \\ &\le M\sum_k |C_{kmN}|\left|\left(\chi_{I_{kmN}}\hat{f}_N\right)^\vee\right|^{q_1}; \end{aligned} \tag{5}$$

hence, recalling 5.2(1) and hypothesis (1), we have

$$\begin{aligned} \|(S_{mN}f_N)\|_{L^p(l^Q)} &= \left\|\left[\sum_j\left[\sum_i |S_{mN}f_N|^{q_1}\right]^{q_2/q_1}\right]^{1/q_2}\right\|_{L^p} \qquad (N = (i, j)) \\ &\le C\left\|\left[\sum_j\left[\sum_{isk} |C_{kmN}|\left|\left(\chi_{I_{kmN}}\hat{f}_N\right)^\vee\right|^{q_1}\right]^{q_2/q_1}\right]^{1/q_2}\right\|_{L^p} \\ &\le C\left\|\left[\sum_j\left[\sum_{ik} \left|\left(\chi_{I_{kmN}}|C_{kmn}|^{1/q_1}\hat{f}_N\right)^\vee\right|^{q_1}\right]^{q_2/q_1}\right]^{1/q_2}\right\|_{L^p} \\ &\le C\left\|\left[\sum_j\left[\sum_{ik} |C_{kmN}|\,|f_N|^{q_1}\right]^{q_2/q_1}\right]^{1/q_2}\right\|_{L^p} \\ &\le C\left\|\left[\sum_j\left[\sum_i\left(\sum_k |C_{kmN}|\,|f_N|^{q_1}\right)\right]^{q_2/q_1}\right]^{1/q_2}\right\|_{L^p} \\ &\le C\|(f_N)\|_{L^p(l^Q)}. \end{aligned}$$

Finally, by an application of the Lebesgue dominated theorem and Fatou's lemma, from (4) we get (3) as desired.

The following lemma is well known (see [**5**, Th. 4.2–3 and 5] and will play a major role in the multiplier theorem we shall state.

6.3. LEMMA. *Let m be a bounded measurable function which has continuous derivatives of order (α, β), $\alpha = 0, 1$ and $\beta = 0, 1$, away from the axis, and satisfies*

$$(1) \qquad |x^\alpha y^\beta D^{\alpha\beta} m(x, y)| \leq M, \qquad x \neq 0,\ y \neq 0.$$

Then, the V-variation of the restriction of m to the dyadic intervals are uniformly bounded, i.e.

$$(2) \qquad V(\chi_K m) \leq N,$$

for all dyadic intervals $K = I \times J$ in $\mathbf{R}^2$.

Finally, as a consequence of the foregoing results we obtain a multiplier theorem of Lizorkin type.

6.4. THEOREM. *Let m be a scalar-valued function in $\mathbf{R}^2$ given as in Lemma 6.3. Let T_m be the multiplier operator defined on $\varphi \in S(\mathbf{R}^2)$ by*

$$(1) \qquad (T_m\varphi)^\wedge = m\hat{\varphi}.$$

Then, T_m has an extension to all $L^P(\mathbf{R}^2)$ such that

$$(2) \qquad \|T_m f\|_{L^P} \leq C\|f\|_{L^P},$$

for all $f \in L^P(\mathbf{R}^2)$, where the constant C depends on p only.

Proof. By the Littlewood-Paley inequalities, Theorem 6.2 and Lemma 6.3 we shall have

$$\begin{aligned}
\|T_m f\|_{L^P} &\leq C\left\| \left[\sum_{K\in\Delta_2} |S_K(T_m f)|^2 \right]^{1/2} \right\|_{L^P} \\
&= C\left\| \left[\sum_{K\in\Delta_2} |(\chi_K m \chi_K \hat{f})^\vee|^2 \right]^{1/2} \right\|_{L^P} \\
&\leq C\left\| \left[\sum_{K\in\Delta_2} |(\chi_K \hat{f})^\vee|^2 \right]^{1/2} \right\|_{L^P} \leq C\|f\|_{L^P}.
\end{aligned}$$

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