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ARITHMETIC PROPERTIES OF THIN SETS

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We prove that $\Lambda(p)$ sets do not contain parallelepipeds of arbitrarily large dimension. This fact is used to show that all $\Lambda(p)$ sets satisfy the arithmetic properties which were previously known only for $\Lambda(p)$ sets with $p > 2$. We also obtain new arithmetic properties of $\Lambda(p)$ sets.

1. Introduction. Let G denote a compact abelian group and $\hat{G} = \Gamma$ its necessarily discrete, abelian, dual group. When E is a subset of Γ, an integrable function f on G will be called an *E-function* provided its Fourier transform, $\hat{f}$, vanishes on the complement of E. Similarly, an E-function f will be called an *E-polynomial* if the support of its Fourier transform is finite.

A subset E of Γ is said to be a $\Lambda(p)$ set, $p > 0$, if for some $0 < r < p$ there is a constant $c(p, r, E)$ so that

$$\|f\|_p \leq c(p, r, E)\|f\|_r \tag{1}$$

for all E-polynomials f. An easy application of Holder's inequality shows that if $p < q$ and E is a $\Lambda(q)$ set, then E is a $\Lambda(p)$ set. For standard results on $\Lambda(p)$ sets see **[11]** and [7].

A number of authors (cf. **[11]**, [7], [2], **[10]** and **[1]**) have shown that $\Lambda(p)$ sets with $p > 2$ satisfy certain arithmetic properties. In [9] Miheev was able to extend some of these properties to all $\Lambda(p)$ sets in $\mathbf{Z}$. In §2 we will show that generalizations of the properties attributed to $\Lambda(p)$ sets with $p > 2$ in the papers cited above are satisfied by all $\Lambda(p)$ sets, $p > 0$, in all discrete abelian groups.

One of the important open questions in the study of $\Lambda(p)$ sets is whether there are any $\Lambda(p)$ sets, with $p < 4$, that are not already $\Lambda(4)$. The technique used most often to show that a given set is not a $\Lambda(p)$ set, for some particular value of p, is to show that the set fails to satisfy an arithmetic property which $\Lambda(p)$ sets are known to fulfill. As a consequence of our results, it is impossible to find a $\Lambda(p)$ set with $p < 2$ which does not satisfy all the arithmetic properties of a $\Lambda(2)$ set which are currently known.

The proofs of these results depend upon the following theorem.

DEFINITION 1.1. A subset P of Γ is called a *parallelepiped of dimension* N if $P = \prod_{i=1}^{N}\{\chi_i, \psi_i\}$, where $\chi_i, \psi_i \in \Gamma$ for $i = 1, \ldots, N$, and $|P| = 2^N$.

THEOREM 1.2. *If $E \subset \Gamma$ is a $\Lambda(p)$ set, $p > 0$, then there is an integer N such that E does not contain any parallelepipeds of dimension N.*

We prove this result in §3. The conclusion of this theorem was previously known for $\Lambda(1)$ sets [**4**], and for all $\Lambda(p)$ sets in $\mathbf{Z}$ (for $p = 2$ in [**8**] and for $p > 0$ in [**9**].) In §4 random sequences are considered to show that parallelepipeds are not sufficient to characterize $\Lambda(4)$ sets.

2. Arithmetic properties.

DEFINITION 2.1. A subset P of Γ is called a *pseudo-parallelepiped of dimension* N if $P = \prod_{i=1}^{N}\{\chi_i, \psi_i\}$, where $\chi_i, \psi_i \in \Gamma$ for $i = 1, \ldots, N$.

REMARK. Parallelepipeds and pseudo-parallelepipeds are generalizations of arithmetic progressions, for any arithmetic progression of length 2^N is a parallelepiped of dimension N.

Our results on the arithmetic properties of $\Lambda(p)$ sets will be seen to follow from Theorem 1.2 and

PROPOSITION 2.2. *For each positive integer n, there are constants $c(n)$ and $0 < \varepsilon(n) < 1$, so that if $E \subset \Gamma$ does not contain any parallelepipeds of dimension n, then whenever P_r is a pseudo-parallelepiped of dimension r*

$$|E \cap P_r| \leq c(n)2^{r\varepsilon(n)}.$$

REMARK. This proposition is proved in [**9**] for $E \subset \mathbf{Z}$ and P_r a parallelepiped of dimension r. With appropriate modifications the same proof yields Proposition 2.2.

Combining Theorem 1.2 and Proposition 2.2 we immediately obtain

COROLLARY 2.3. *Let $E \subset \Gamma$ be a $\Lambda(p)$ set for some $p > 0$. There are constants c and $0 < \varepsilon < 1$ so that whenever P_r is a pseudo-parallelepiped of dimension r*

$$|E \cap P_r| \leq c2^{r\varepsilon}.$$

The arithmetic progression of length N, $\{\chi\psi, \dots, \chi\psi^N\}$, is contained in the pseudo-parallelepiped $\chi\psi \cdot \prod_{i=0}^{M-1}\{1, \psi^{2^i}\}$ of dimension M provided $2^M \geq N$. By choosing M with $2^{M-1} < N \leq 2^M$ we have

COROLLARY 2.4 (*see* [**11**, 3.5], [**2**], *or* [**1**] *for* $p > 2$, [**9**] *for* $E \subset \mathbf{Z}$). *Let* $E \subset \Gamma$ *be a* $\Lambda(p)$ *set. There are constants* c *and* $0 < \varepsilon < 1$ *such that if* A *is any arithmetic progression of length* N *then*

$$|E \cap A| \leq 2cN^{\varepsilon}.$$

In particular, if E is a $\Lambda(p)$ set in $\mathbf{Z}$, then any interval of length N contains at most $2cN^{\varepsilon}$ points of E. Thus E has density zero. Moreover, if $E = \{n_k\}$, then $\sum_{n_k \neq 0}(1/|n_k|) < \infty$, so the set of prime numbers is not a $\Lambda(p)$ set for any $p > 0$ [**9**].

DEFINITION 2.5 [7, 6.2]. For positive integers d and N, $\chi_1, \dots, \chi_d \in \Gamma$ and $1 \leq r < \infty$, let

$$A_r(N, \chi_1, \dots, \chi_d) = \left\{ \prod_{j=1}^{d} \chi_j^{n_j} \colon \sum_{j=1}^{d} |n_j|^r \leq N^r \right\}.$$

Let

$$A_\infty(N, \chi_1, \dots, \chi_d) = \left\{ \prod_{j=1}^{d} \chi_j^{n_j} \colon \sup_{1 \leq j \leq d} |n_j| \leq N \right\}.$$

REMARK. These sets may also be viewed as generalized arithmetic progressions. Indeed, if $\Gamma = \mathbf{Z}$ and $b \in \mathbf{Z}$ then

$$A_r(N, b) = \{-Nb, \dots, -b, 0, b, \dots, Nb\}$$

is an arithmetic progression of length $2N + 1$ for any r.

COROLLARY 2.6 (*see* [7, 6.3–6.4], [**1**] *for* $p > 2$ *and* $r < \infty$). *Let* $E \subset \Gamma$ *be a* $\Lambda(p)$ *set. There are constants* c *and* $0 < \varepsilon < 1$ *such that*

$$|A_r(N, \chi_1, \dots, \chi_d) \cap E| \leq c(2N + 1)^{d\varepsilon}$$

for all $\chi_1, \dots, \chi_d \in \Gamma$, $N \in \mathbf{Z}^+$ *and* $1 \leq r \leq \infty$.

Proof. Observe that

$$A_r(N, \chi_1, \dots, \chi_d) \subset A_\infty(N, \chi_1, \dots, \chi_d) = \prod_{i=1}^{d} A_\infty(N, \chi_i).$$

Since $A_\infty(N, \chi_i)$ is an arithmetic progression of length at most $(2N + 1)$, the set $\prod_{i=1}^{d} A_\infty(N, \chi_i)$ is contained in a pseudo-parallelepiped of dimension Md, where $2^M \geq 2N + 1 > 2^{M-1}$. Now apply Proposition 2.2. □

DEFINITION 2.7 ([**11**, 1.6]). For $E \subset \mathbf{Z}$ and $n \in \mathbf{Z}$, let $r_2(E, n)$ be the number of ordered pairs $(m_1, m_2) \in E \times E$ with $m_1 + m_2 = n$.

COROLLARY 2.8 (*see* [**10**] *for* $p > 2$ *and* [**11**, 4.5] *for* $p = 4$). *If* $E \subset \mathbf{Z}^+$ *is a* $\Lambda(p)$ *set there is some* $q < \infty$ *and constant* c *so that if* $1/q + 1/q' = 1$ *then* E *satisfies*

$$\left(\sum_{n=1}^{N} r_2(E, n)^q\right)^{1/q} \leq cN^{1/q'}$$

for all positive integers N.

Proof. If $(m_1, m_2) \in E \times E$ satisfies $m_1 + m_2 = n$ then certainly $m_1, m_2 \in (0, n]$. Thus

$$r_2(E, n) \leq |(0, n] \cap E| \leq cn^\varepsilon$$

for some constants c and $0 < \varepsilon < 1$.

If $q = 2/(1 - \varepsilon)$ then

$$\left(\sum_{n=1}^{N} r_2(E, n)^q\right)^{1/q} \leq \left(\sum_{n=1}^{N} (cn^\varepsilon)^q\right)^{1/q} \leq cN^{\varepsilon + 1/q} \leq cN^{1/q'}. \quad \square$$

DEFINITION. 2.9. Let M be a positive integer. We will say that $A \subset \Gamma$ is a *weak-M-test set* if $|AA^{-1}| \leq M|A|$.

REMARKS. 1. If $A = \{\chi\psi, \ldots, \chi\psi^N\}$ is an arithmetic progression of length N, then $AA^{-1} = \{\psi^k\colon -N + 1 \leq k \leq N - 1\}$, hence A is a weak-2-test set.

2. In [**2**] A is called a *test set of order* M if $|A^2A^{-1}| \leq M|A|$. Since $|AA^{-1}| \leq |A^2A^{-1}|$ any test set of order M is a weak-M-test set.

PROPOSITION 2.10 (*see* [**2**] *for* $p > 2$ *and* A *a test set of order* M). *Let* $E \subset \Gamma$ *be a* $\Lambda(p)$ *set. There are constants* c *and* $0 < \varepsilon < 1$ *so that whenever* M *is a positive integer and* A *is a weak-M-test set, then*

$$|E \cap A| \leq c|A|^\varepsilon.$$

Proof. Let $t = |E \cap A|$ and choose $n \geq 1$ so that E contains no parallelepipeds of dimension $n+1$. We will assume that $t \geq 4(M|A|)^{1-1/2^n}$ and derive a contradiction.

Let $AA^{-1} \setminus \{1\} = \{\chi_1, \ldots, \chi_d\}$ with $\chi_i \neq \chi_j$ if $i \neq j$. Then $d \leq M|A|$. Let $E' = E \cap A$.

For each $i = 1, \ldots, d$ choose a maximal collection $C_{1,i}$ of ordered sets $\{\alpha, \beta\}$ satisfying α, $\beta \in E'$ and $\alpha\beta^{-1} = \chi_i$, and which are pairwise disjoint (as unordered sets). Let $C_1 = \bigcup_{i=1}^{d} C_{1,i}$.

Suppose $\{\alpha, \beta\} \notin C_1$ for $\alpha, \beta \in E'$ with $\alpha \neq \beta$. Since $\alpha\beta^{-1} = \chi_i$ for some i and $\{\alpha, \beta\} \notin C_{1,i}$ it must be that one of $\{\chi, \alpha\}$ or $\{\beta, \chi\} \in C_{1,i}$ for some $\chi \in E'$. Thus

$$|C_1| \geq \frac{1}{3}\left|\{\{\alpha, \beta\} \colon \alpha, \beta \in E', \alpha \neq \beta\}\right| \geq \frac{t(t-1)}{3}$$

and hence

$$\max_{1 \leq i \leq d} |C_{1,i}| \geq \frac{t(t-1)}{3d} \geq \frac{t(t-1)}{3M|A|}.$$

If $t \leq 4$ then $t \leq 4(M|A|)^{1-1/2^n}$ for any $n \geq 1$, thus $t > 4$ and we obtain the inequality

$$|C_{1,i_1}| = \max_i |C_{1,i}| \geq \frac{t^2}{4M|A|}.$$

Let D_1 denote the set of left hand terms of C_{1,i_1}. Observe that if $\psi_1, \ldots, \psi_k \in D_1$ with $\psi_i \neq \psi_j$ for $i \neq j$, then $\{\psi_j, \psi_j\chi_{i_1}^{-1}\}$, $j = 1, \ldots, k$, are distinct pairs in C_{1,i_1}, and so by the disjointness condition all the terms of $\{\psi_1, \ldots, \psi_k\} \cdot \{1, \chi_{i_1}^{-1}\}$ are distinct.

Further, if $|C_{1,i_1}| > 1$ then C_{1,i_1} contains two distinct pairs, $\{\alpha_j, \beta_j\}$, $j = 1, 2$. Since $\alpha_j\beta_j^{-1} = \chi_{i_1}$ these four elements of E form a parallelepiped of dimension 2, namely $\{\alpha_1, \alpha_2\} \cdot \{1, \chi_{i_1}^{-1}\}$. Hence if E contains no parallelepipeds of dimension 2 then $t \leq (4M|A|)^{1/2}$ proving the proposition for $n = 1$.

We proceed inductively to obtain for $k = 2, \ldots, m-1$, $k \leq n$, sets C_{k,i_k} and D_k satisfying:

(i) C_{k,i_k} consists of pairwise disjoint two element sets $\{\alpha, \beta\}$ with $\alpha\beta^{-1} = \chi_{i_k}$, α, $\beta \in D_{k-1}$;

(ii) D_k consists of the left hand terms of C_{k,i_k};

(iii) $|C_{k,i_k}| = |D_k| \geq t^{2^k}/(4M|A|)^{2^k-1}$; and

(iv) If $\{\psi_1, \ldots, \psi_r\}$ are distinct members of D_k then all the terms of the set $\{\psi_1, \ldots, \psi_r\} \cdot \prod_{j=1}^{k} \{1, \chi_{i_j}^{-1}\}$ belong to E and are distinct.

In particular, (iv) implies that if ψ_1, ψ_2 are distinct members of D_k, then E contains the $k+1$ dimensional parallelepiped $\{\psi_1, \psi_2\} \cdot \prod_{j=1}^{k}\{1, \chi_{i_j}^{-1}\}$.

For $i = 1, \dots, d$, let $C_{m,i}$ be a maximal set of pairwise disjoint two element sets $\{\alpha, \beta\}$ with $\alpha, \beta \in D_{m-1}$ and $\alpha\beta^{-1} = \chi_i$. In the same manner as before we see that

$$|C_{m,i_m}| = \max_{1 \le i \le d} |C_{m,i}| \ge \frac{1}{3d} |D_{m-1}|\left(|D_{m-1}| - 1\right)$$
$$\ge \frac{1}{3M|A|}\left(\frac{t^{2^{m-1}}}{(4M|A|)^{2^{m-1}-1}}\right)\left(\frac{t^{2^{m-1}}}{(4M|A|)^{2^{m-1}-1}} - 1\right)$$

and since we are assuming

$$\frac{t^{2^{m-1}}}{(4M|A|)^{2^{m-1}-1}} \ge 4,$$

we have

$$|C_{m,i_m}| \ge \frac{t^{2^m}}{(4M|A|)^{2^m-1}}.$$

Let D_m be the left hand terms of C_{m,i_m} and suppose $\psi_1, \dots, \psi_r$ are distinct terms of D_m. Then $\{\psi_j, \psi_j\chi_{i_m}^{-1}\}$ are pairwise disjoint sets in C_{m,i_m}, so $B = \{\psi_1, \dots, \psi_r, \psi_1\chi_{i_m}^{-1}, \dots, \psi_r\chi_{i_m}^{-1}\}$ is a collection of distinct terms of D_{m-1}. By (iv) the terms of

$$\{\psi_1, \dots, \psi_r\} \cdot \prod_{j=1}^{m}\left\{1, \chi_{i_j}^{-1}\right\} = B \cdot \prod_{j=1}^{m-1}\left\{1, \chi_{i_j}^{-1}\right\}$$

are distinct members of E. This completes the induction step.

Since E contains no parallelepipeds of dimension $n+1$, $|D_n|$ must be at most one. This contradicts our initial assumption. □

The union problem for $\Lambda(p)$ sets with $p \le 2$ is open. However we do have

Proposition 2.11 (*see* [9] *for* $E \subset \mathbf{Z}$). *Let* E_1 *and* E_2 *be* $\Lambda(p)$ *sets. Then* $E_1 \cup E_2$ *does not contain parallelpipeds of arbitrarily large dimension.*

Proof. Choose constants c and $0 < \varepsilon < 1$ so that whenever P_n is a parallelpiped of dimension n, $|E_i \cap P_n| \le c2^{n\varepsilon}$ for $i = 1, 2$. Then

$$|(E_1 \cup E_2) \cap P_n| \le 2c2^{n\varepsilon} < 2^n = |P_n|$$

for n sufficiently large. □

Observe that all these results hold for sets which do not contain parallelepipeds of arbitrarily large dimension. In **[6]** we discuss additional properties of such sets.

3. Proof of Main Theorem. We turn now to proving Theorem 1.2.

Since any $\Lambda(p)$ set with $p \geq 1$ is a $\Lambda(s)$ set for any $s < 1$, we may without loss of generality assume $p < 1$.

We will show in fact that N depends only on $c(p, p/2, E)$, as defined by (1). Since a translate of a $\Lambda(p)$ set is a $\Lambda(p)$ set with the same constant, it suffices to show that $\Lambda(p)$ sets do not contain parallelepipeds of the form $P = \prod_{i=1}^{M}\{1, \chi_i\}$, $|P| = 2^M$, for $M > N$.

The proof will result by establishing a number of lemmas. The main idea in the proof of the principal result in **[9]** is used in Lemma 3.4.

Let us say that $\{\chi_1, \dots, \chi_N\} \subset \Gamma$ is *quasi-dissociate* if

$$\prod_{i=1}^{N} \chi_i^{\varepsilon_i} = 1 \quad \text{for } \varepsilon_i = 0, \pm 1,\ i = 1, \dots, N,$$

implies $\varepsilon_i = 0$ for all $i = 1, \dots, N$.

LEMMA 3.1. *Fix a positive integer N_0 and let $N_1 = 3^{N_0} + 1$. Any subset of Γ of cardinality N_1 contains a quasi-dissociate subset of cardinality N_0.*

Proof. This is essentially an application of the Pigeon Hole Principle.

Consider the subset $\{\chi_i\}_{i=1}^{N_1} \subset \Gamma$. Choose $\psi_1 \in \{\chi_1, \chi_2\}$ so that $\psi_1 \neq 1$. If $A_1 = \{\psi_1^{\varepsilon_1}\colon \varepsilon_1 = 0, \pm 1\}$ then $|A_1| \leq 3$ so it is possible to choose $\psi_2 \in \{\chi_i\}_{i=1}^{4}$ with $\psi_2 \notin A_1$.

Now proceed inductively. Assume $\psi_1, \dots, \psi_n$ have been chosen. Let

$$A_n = \{\psi_1^{\varepsilon_1}\psi_2^{\varepsilon_2} \cdots \psi_n^{\varepsilon_n}\colon \varepsilon_i = 0, \pm 1, i = 1, \dots, n\}.$$

Since $|A_n| \leq 3^n$ we may choose $\psi_{n+1} \in \{\chi_i\}_{i=1}^{3^n+1}$ with $\psi_{n+1} \notin A_n$.

We may choose $\{\psi_i\}_{i=1}^{N_0} \subset \{\chi_i\}_{i=1}^{N_1}$ in this way since $N_1 = 3^{N_0} + 1$.

Now suppose $\prod_{i=1}^{N_0} \psi_i^{\varepsilon_i} = 1$ with $\varepsilon_i = 0, \pm 1, i = 1, \dots, N_0$. Let k be the largest integer with $\varepsilon_k \neq 0$. We cannot have $k = 1$ for then $\psi_1^{\varepsilon_1} = 1$ and hence $\psi_1 = 1$. If $k > 1$ then without loss of generality, $\varepsilon_k = 1$, so $\psi_k = \prod_{i=1}^{k-1} \psi_i^{-\varepsilon_i}$. But this implies $\psi_k \in A_{k-1}$, contradicting its selection. Thus $\varepsilon_i = 0$ for all $i = 1, 2, \dots, N_0$ and hence $\{\psi_i\}_{i=1}^{N_0}$ is a quasi-dissociate set. $\square$

Let us say that the parallelepiped $P_N = \prod_{i=1}^{N}\{1, \chi_i\}$ is

(i) *of order* 2 if $\chi_i^2 = 1$ for $i = 1, \ldots, N$;

(ii) *dissociate* if $\prod_{i=1}^{N} \chi_i^{\varepsilon_i} = 1$ with $\varepsilon_i = 0, \pm 1, \pm 2$, implies $\varepsilon_i = 0$ for all $i = 1, \ldots, N$; and

(iii) *quasi-dissociate* if $\prod_{i=1}^{N} \chi_i^{\varepsilon_i} = 1$ with $\varepsilon_i = 0, \pm 1$ implies $\varepsilon_i = 0$ for all $i = 1, \ldots, N$.

With this notation an immediate corollary of the previous lemma is

COROLLARY 3.2. *If E contains $P = \prod_{i=1}^{N_1}\{1, \chi_i\}$, a parallelepiped of dimension $N_1 = 3^{N_0} + 1$, then E contains a quasi-dissociate, N_0-dimensional parallelepiped.*

Next we will prove

LEMMA 3.3. *Let E be a $\Lambda(p)$ set, $0 < p < 1$, with constant $c(p, p/2, E)$. There is an integer N_1 depending on $c(p, p/2, E)$ such that E does not contain any parallelepipeds of order 2 with dimension greater than N_1.*

Proof. Choose an integer N_0 so that

$$2^{N_0/p} = \frac{2^{(1-1/p)N_0}}{2^{(1-2/p)N_0}} > c(p, p/2, E)$$

and set $N_1 = 3^{N_0} + 1$. By Corollary 3.2 if E contains a parallelepiped of order 2 with dimension N_1 then E contains a quasi-dissociate parallelepiped of order 2 with dimension N_0, say $\prod_{i=1}^{N_0}\{1, \chi_i\}$. Being quasi-dissociate and of order 2 the set $\{\chi_i\}_{i=1}^{N_0}$ is probabilistically independent. Hence

$$\left(\int \prod_{i=1}^{N_0} |1 + \chi_i|^p\right)^{1/p} = \left(\prod_{i=1}^{N_0} \int |1 + \chi_i|^p\right)^{1/p} = 2^{(1-1/p)N_0}.$$

Similarly

$$\left(\int \prod_{i=1}^{N_0} |1 + \chi_i|^{p/2}\right)^{2/p} = 2^{(1-2/p)N_0}.$$

Thus if $f(x) = \prod_{i=1}^{N_0}(1 + \chi_i(x))$, then $f \in \mathrm{Trig}_E(G)$ and

$$\|f\|_p = 2^{(1-1/p)N_0} > c(p, p/2, E)2^{(1-2/p)N_0} = c(p, p/2, E)\|f\|_{p/2}$$

contradicting the fact that E is a $\Lambda(p)$ set with constant $c(p, p/2, E)$. □

LEMMA 3.4. *Let E be a $\Lambda(p)$ set, $0 < p < 1$, with constant $c(p, p/2, E)$. There is an integer N depending on $c(p, p/2, E)$ such that E does not contain any dissociate parallelepipeds of dimension N.*

Proof. It is shown in [9] that for any fixed $r \in (0,1)$ with $r/(1-r)^3 < p^2/256$,

$$A = \left(1 - \frac{(p/2)(1-p/2)r^2}{4} - \left(\frac{r}{1-r}\right)^3\right)^{1/p}$$

$$> \left(1 - \frac{(p/4)(1-p/4)r^2}{4} + \left(\frac{r}{1-r}\right)^3\right)^{2/p} = B.$$

Choose N so that $A^N > c(p, p/2, E)B^N$, and suppose E contains the dissociate parallelepiped $\prod_{i=1}^N \{1, \chi_i\}$. Let R be the least solution of $r = 2R/(1+R^2)$.

Let $f = \prod_{i=1}^N (1 + R\chi_i)$. Then $f \in \mathrm{Trig}_E(G)$, and

$$\|f\|_p = \left(\int \prod_{i=1}^N \left(|1 + R\chi_i|^2\right)^{p/2}\right)^{1/p} \tag{2}$$

$$= (1+R^2)^{N/2}\left(\int \prod_{i=1}^N \left(1 + r\left(\frac{\chi_i + \overline{\chi_i}}{2}\right)\right)^{p/2}\right)^{1/p}.$$

An application of MacLaurin's formula shows that for any $\alpha \in (0,1)$

$$(1+x)^\alpha = 1 + \alpha x - \frac{\alpha(1-\alpha)x^2}{2} + \mathrm{Rem}(x)$$

where $|\mathrm{Rem}(x)| \leq (r/(1-r))^3$ provided $x \in [-r, r]$ and $r \in (0,1)$.

Now $-r \leq r((\chi_i x + \overline{\chi_i(x)})/2) \leq r$ so applying MacLaurin's formula to (2) with $\alpha = p/2$ we obtain

$$\|f\|_p \geq (1+R^2)^{N/2}\left(\int \prod_{i=1}^N \left(1 + \frac{p}{2}r\left(\frac{\chi_i + \overline{\chi_i}}{2}\right) - \frac{(p/2)(1-p/2)}{2}r^2\left(\frac{\chi_i + \overline{\chi_i}}{2}\right)^2 - \left(\frac{r}{1-r}\right)^3\right)\right)^{1/p}$$

$$= (1+R^2)^{N/2}\left(\int \prod_{i=1}^N \left(1 - \frac{(p/2)(1-p/2)r^2}{4} - \left(\frac{r}{1-r}\right)^3 + \frac{p}{2}r\left(\frac{\chi_i + \overline{\chi_i}}{2}\right) - \frac{(p/2)(1-p/2)r^2}{2}\left(\frac{\chi_i^2 + \overline{\chi_i^2}}{4}\right)\right)\right)^{1/p}$$

$$= (1+R^2)^{N/2}\prod_{i=1}^N \left(1 - \frac{(p/2)(1-p/2)r^2}{4} - \left(\frac{r}{1-r}\right)^3\right)^{1/p}$$

because of the dissociateness assumption.

Similarly

$$\|f\|_{p/2} \le (1+R^2)^{N/2} \prod_{i=1}^{N} \left(1 - \frac{(p/4)(1-p/4)r^2}{4} + \left(\frac{r}{1-r}\right)^3\right)^{2/p}.$$

Thus

$$\begin{aligned}\|f\|_p &\ge (1+R^2)^{N/2} A^N > (1+R^2)^{N/2} c(p, p/2, E) B^N \\ &\ge c(p, p/2, E)\|f\|_{p/2}\end{aligned}$$

contradicting the fact that E is a $\Lambda(p)$ set with constant $c(p, p/2, E)$. □

LEMMA 3.5. *For each positive integer N_0 there is an integer $N_2 = N_2(N_0)$ so that if $P = \prod_{i=1}^{N_2}\{1, \chi_i\}$ is a parallelepiped of dimension N_2 with the property that for each $i = 1, 2, \ldots, N_2$ the set $\{j \ne i\colon \chi_j^2 = \chi_i^2\}$ is empty, then P contains a dissociate parallelepiped of dimension N_0.*

Proof. This is another application of the Pigeon Hole Principle similar to Lemma 3.1. □

LEMMA 3.6. *For each positive integer N_0 there is an integer $N = N(N_0)$ so that if E contains a parallelepiped of dimension N, then a translate of E contains either a dissociate parallelepiped or a parallelepiped of order* 2, *with dimension N_0.*

Proof. Fix N_0. Put $N = 2N_0N_2$ with $N_2 = N_2(N_0)$ as in Lemma 3.5. Assume that a translate of E contains $P = \prod_{i=1}^{N}\{1, \chi_i\}$, a parallelepiped of dimension N.

We will say that $\chi_i \sim \chi_j$ if $\chi_i^2 = \chi_j^2$. Let S_i be the equivalence class containing χ_i. We consider two cases.

Case 1. For some $i \in \{1, 2, \ldots, N\}$, $|S_i| \ge 2N_0$. Without loss of generality $i = 1$ and $\{\chi_1, \chi_2, \ldots, \chi_{2N_0}\} \subset S_1$, i.e., $\chi_k^2 = \chi_1^2$ for $k = 1, 2, \ldots, 2N_0$. Then $\chi_1\chi_k^{-1} \equiv \varphi_k$ satisfies $\varphi_k^2 = 1$ for $k = 1, \ldots, 2N_0$.

Certainly $\prod_{j=1}^{N_0}\{\chi_1\varphi_{2j-1}, \chi_1\varphi_{2j}\} \subset P$ and hence is a parallelepiped of dimension N_0 contained in E. A further translate of E contains the N_0-dimensional parallelepiped $\prod_{j=1}^{N_0}\{1, \varphi_{2j}\varphi_{2j-1}^{-1}\}$ of order two.

Case 2. Otherwise $|S_i| \le 2N_0$ for all $i = 1, 2, \ldots, N$. In this case there must be at least N_2 distinct equivalence classes, say $S_1, \ldots, S_{N_2}$. Lemma 3.5 may be applied to $\prod_{i=1}^{N_2}\{1, \chi_i\}$ to obtain a dissociate parallelepiped of dimension N_0 in the original translate of E. □

Proof of Theorem 1.2. Put together Lemmas 3.3, 3.4 and 3.6. □

4. Random sequences. If E does not contain any parallelepipeds of dimension 2 then a modification of [**11**, 4.5] can be used to show that E is a $\Lambda(4)$ set. Parallelepipeds are not sufficient to characterize $\Lambda(p)$ sets however. In this section we will use a method of Erdös and Rényi [3] to show that for each $p > 8/3$ there is a set $E(p)$ which does not contain parallelepipeds of arbitrarily large dimension and yet is not a $\Lambda(p)$ set.

Let $0 < \alpha < 1$ and let $\{\xi_n\}_{n=1}^{\infty}$ be a sequence of independent random variables such that $P(\xi_n = 1) = p_n = 1/n^{\alpha}$ and $P(\xi_n = 0) = 1 - p_n$. Let $\{\nu_k\}$ denote the values of n (in increasing order) with $\xi_n = 1$. Thus p_n is the probability that n is contained in $\{\nu_k\}$.

If $\{\nu_k\}$ contains a parallelepiped of dimension d then there are integers $n, m, k_1, \ldots, k_{2^{d-2}}$, such that $\{\nu_k\}$ contains

$$X(k_1, \ldots, k_{2^{d-2}}, n, m)$$
$$\equiv \left\{k_i, k_i + n, k_i + m, k_i + m + n : i = 1, \ldots, 2^{d-2}\right\}$$

where

$$|X(k_1, \ldots, k_{2^{d-2}}, n, m)| = 2^d.$$

Without loss of generality we may assume $1 \le k_i < k_i + n < k_i + m < k_i + m + n$, so $\{k_1, \ldots, k_{2^{d-2}}, n, m\} \subset \mathbf{Z}^+$. Since $\{\xi_n\}_{n=1}^{\infty}$ are independent random variables the probability that $\{\nu_k\}$ contains $X(k_1, \ldots, k_{2^{d-2}}, n, m)$ is

$$P(X(k_1, \ldots, k_{2^{d-2}}, n, m) \subset \{\nu_k\})$$
$$= \prod_{i=1}^{2^{d-2}} \left(\frac{1}{k_i(k_i + n)(k_i + m)(k_i + m + n)}\right)^{\alpha}.$$

Thus if $\sum'_{n,m,k_1,\ldots,k_{2^{d-2}}}$ denotes the sum over those positive integers $n, m, k_1, \ldots, k_{2^{d-2}}$ such that $|X(k_1, \ldots, k_{2^{d-2}}, n, m)| = 2^d$, then

$$S \equiv \sum_{n,m,k_1,\ldots,k_{2^{d-2}}}' P(X(k_1, \ldots, k_{2^{d-2}}, n, m) \subset \{\nu_k\})$$
$$\le \sum_{n,m,k_1,\ldots,k_{2^{d-2}} \in \mathbf{Z}^+} \prod_{i=1}^{2^{d-2}} \left(\frac{1}{k_i(k_i + n)(k_i + m)(k_i + m + n)}\right)^{\alpha}$$
$$= \sum_{n,m} \left(\sum_k \left(\frac{1}{k(k + n)(k + m)(k + m + n)}\right)^{\alpha}\right)^{2^{d-2}}.$$

Let $t = 2^{d-2}$. By using the inequality

$$\frac{1}{k+n} \le \left(\frac{1}{k}\right)^{\sigma}\left(\frac{1}{n}\right)^{1-\sigma}$$

for $0 < \sigma < 1$, we obtain

$$S \leq \sum_{n,m} \left(\frac{1}{nm}\right)^{(1-\sigma)t\alpha} \left(\sum_k \left(\frac{1}{k}\right)^{2(1+\sigma)\alpha}\right)^t.$$

If we choose t, α and σ so that $(1-\sigma)\alpha t > 1$ and $2(1+\sigma)\alpha > 1$, then $S < \infty$. An application of the Borel-Cantelli Lemma shows that in this case $\{\nu_k\}$ contains only finitely many d dimensional parallelepipeds a.s.

If $\alpha > 1/4$ and $t > 1/2(\alpha - 1/4)$ we see that the inequalities $(1-\sigma)\alpha t > 1$ and $2(1+\sigma)\alpha > 1$, can be simultaneously satisfied for any $\sigma \in (0,1)$ with

$$\frac{1}{2\alpha} - 1 < \sigma < 1 - \frac{1}{t\alpha}.$$

Since

$$\sum_n \frac{p_n(1-p_n)}{(p_1 + \cdots + p_n)^2} \leq \sum_n \frac{1}{n^\alpha n^{2(1-\alpha)}} < \infty,$$

by the Strong Law of Large Numbers

$$\lim_{k\to\infty} \frac{\sum_{i \leq \nu_k} p_i}{k} = 1 \quad \text{a.s.}$$

Thus

$$\lim_{k\to\infty} \frac{\nu_k^{1-\alpha}}{(1-\alpha)k} = 1 \quad \text{a.s.}$$

and so there is a $c > 0$ such that for all N sufficiently large,

$$|\{\nu_k\} \cap [1,N]| \geq cN^{1-\alpha} \quad \text{a.s.}$$

PROPOSITION 4.1. *For each $p > 8/3$ there is an integer $d = d(p)$ and a set $E = E(d,p)$ which contains no parallelepipeds of dimension d but is not a $\Lambda(p)$ set.*

Proof. For $p > 8/3$, say $p = 8/(3-4\varepsilon)$ with $\varepsilon > 0$, let $\alpha = 1/4 + \varepsilon/2$ and let d be any integer satisfying $t = 2^{d-2} > 1/\varepsilon$. Choose $\{\nu_k\}$ as described above so that $\{\nu_k\}$ contains only finitely many parallelepipeds of dimension d and

$$|\{\nu_k\} \cap [1,N]| \sim cN^{3/4-\varepsilon/2}.$$

Let E be the set $\{\nu_k\}$ with the finitely many integers which form parallelepipeds of dimension d deleted. If E was a $\Lambda(p)$ set then by [**11**, 3.5]

$$|E \cap [1, N]| \leq cN^{2/p}.$$

But E and $\{\nu_k\}$ have the same asymptotic density and $2/p < 3/4 - \varepsilon/2$, thus E cannot be a $\Lambda(p)$ set. □

Thus the notion of parallelepipeds is not strong enough to characterize $\Lambda(p)$ sets for $p > 8/3$. The question as to whether or not parallelepipeds characterize $\Lambda(p)$ sets for $p \leq 8/3$ remains open.

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